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Mathematical modeling and optimization of wave energy converters and arrays

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UCD student number: 12251898

The thesis is submitted to University College Dublin in fulfilment of the requirements for the degree of Doctor of Philosophy

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August 2015
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Abstract

The aim of this work is to develop methodologies and understand the dynamics of wave energy converters (WECs) in some problems of practical interest. The focus is on a well known WEC - the Oscillating Wave Surge Converter (OWSC). In the first work, a mathematical model is described to analyze the interactions in a wave energy farm comprising of OWSCs. The semi-analytical method uses Green’s integral equation formulation and Green’s function, yielding hyper-singular integrals which are later solved using the Chebyshev polynomial of the second kind. A new methodology for the optimization of large wavefarms is then presented and the approach includes a statistical emulator, an active learning approach (Gaussian Process Upper Confidence Bound with Pure Exploration) and a genetic algorithm. The modular concept of the OWSC, which has emerged to address some of the shortcomings in the original design of the OWSC, is also described and investigated using a semi analytical approach for cylindrical modules. In another work, the dynamics of the OWSC near a straight coast is analyzed and for a particular case, a significant enhancement in the performance of the OWSC is observed. This interesting result motivated the following study, where it is investigated if a breakwater can artificially enhance the performance of the OWSC. Lastly, a new approach is presented to analyze the interactions between two different kind of WECs (an OWSC and a Heaving Wave Energy Converter), performing different modes of motion.
Declaration

I hereby certify that the submitted work is my own work, was completed while registered as a candidate for the degree stated on the Title Page, and I have not obtained a degree elsewhere on the basis of the research presented in this submitted work. Where collaborative research was involved, every effort has been made to indicate this clearly. A list of the publications arising from the research has been provided.

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List of Publications

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• Sarkar, D., Contal, E., Vayatis, N., and Dias, F. “Prediction and optimization of a wave farm using a machine learning approach" 2015. *Submitted*

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1.1 Background

The history of wave energy conversion is quite old with the first invention dating back to 1799 by Girard and Son in France [14]. However, serious research started only after 1970s with the emergence of an oil crisis when people realised the need to reduce the dependence on conventional energy resources and explore alternate sources. This crisis which escalated the oil prices stimulated the development of wind, solar, geothermal and other renewable sources of energy. Ocean waves also appeared to be one of such sources from which energy can be extracted by various possible mechanisms. In fact among all the sources of renewable energy, the latter has the highest energy density [14]. It was at this time of the energy crisis that strong progress in theoretical work was made by Falnes and Budal in Trondheim, Evans at Bristol and by Newman and Mei at MIT. In one of the seminal works,
Budal showed that a device can capture more energy than that incident on its geometrical width with the assumption that the later is much smaller than the wavelength of the incident wave field, just as a signal from a radio aerial doesn’t depend on the wire diameter [9]. This innovation led to the belief that “small is beautiful” and these “point absorbers” devices were considered as very attractive WECs. One of the initial breakthroughs that made wave energy look feasible was the invention of the Salter Duck [61] at the University of Edinburgh. It essentially comprises of a series of cam-shaped floating buoys, which are moored parallel to the incoming wave-fronts. The exciting force of the waves causes the tip of the cam-bob to move up and down and consequently the Duck rotates about a horizontal spine which is connected to fixed moorings at the two ends of the system. This relative motion is utilised to capture power. However, lack of funding never gave the opportunity to test the device in the real ocean.

At the same time, Evans [18] had shown theoretically that a symmetrical two dimensional wave energy converter is limited to capture a maximum of 50 percent of the energy that is incident on it while performing motion with a single degree of freedom. The untapped power comprises of 25 percent of the incident energy being reflected back and the rest being transmitted. However a symmetrical two dimensional device can completely absorb the incident wave energy while moving with two degrees of freedom in which case, the waves produced by the motion in one mode are cancelled by the movement in the other mode, resulting in no radiated waves. Nevertheless, these limitations strictly apply to symmetrical two dimensional devices with primary dimension much larger than the wavelength of the incident wave. For example, the Salter Duck can be classified as a non-symmetrical, two-dimensional device and is therefore not constrained to a maximum efficiency of 50 percent which a symmetrical two dimensional system is limited to. In a later work, Evans along with Srokosz [77] showed that it is possible for two independently oscillating bodies to capture all the incident wave energy at a given frequency. In reality, most of the WECs have principal dimensions comparable to incident wavelength, and as such they require appropriate modelling as neither the two dimensional nor the point absorber approximation is suitable in such cases.

Abundant number of wave energy conversion technologies are now known/available,
1.1 Background

with the prototypes of a few already tested. Especially, in the last decade, advances in the fledgeling wave energy industry, have led to the accumulation of a lot of practical knowledge, giving better understanding of the complex and dynamic interaction of the WECs with waves. WECs should be designed so as to not just be efficient, but be economical and be able to survive the harsh marine environment. Over the years various inventions have been proposed for the purpose of wave energy conversion. However, unlike wind energy turbines, there has been no general consensus on any particular design of WEC. This is primarily because of the variety of possible mechanisms by which energy can be extracted from water waves. Good review articles are now available in the literature, describing the various technologies and principles for wave power conversion [see e.g. 14, 22]. Most widely studied among the WECs are the Oscillating Water Columns and the heaving buoys. Pitching devices like that of Pelamis (see [53]) and McCabe Wave Pump [44] comprise of multiple connected floating bodies (3 barges in case of McCabe Wave Pump) and extract energy by virtue of the relative rotational motion between the neighbouring bodies. A unique pitching device is the SEAREV - a fully integrated WEC in which the floating body encompasses a heavy horizontal axis wheel acting as an internal pendulum with its center of gravity being off-centered (see [5]). In overtopping devices like the WaveDragon (see [35]), the incident wave is focussed with reflectors to overtop a ramp, fill a water reservoir at a level higher than the surrounding sea and then the potential energy of the stored water is converted into useful energy. Bottom-hinged WECs extract energy by performing pitching motion about a horizontal hinge and devices like the Oyster (surface-piercing) and the WaveRoller (submerged) [Wav] are examples of it. In recent years, new concepts of flexible wave energy converters have received a lot of attention from the research community as they are expected to have better survivability characteristics. The Oyster WEC developed by Aquamarine Power is currently considered as one of frontrunners in the challenge for harnessing energy from the ocean and will be the focus of this work.

1.1.1 Array Analysis

In order to make wave energy commercially viable, a large number of devices needs to be installed at designated locations for the purpose of wave power extraction. In such
1.1 Background

circumstances, the behavior of any particular WEC is influenced by the presence of other WECs in the array. Numerous research studies have tried to understand the effect of the interactions between the WECs in arrays. Various methodologies are known in literature which enables investigation into such problems. The initial works on WEC arrays were largely based on the point absorber approximation, which restricts the analysis to problems in which the characteristic length of the WEC is small compared to the wavelength of the incident wave field [see e.g 19, 23]. In another approximate approach known as the plane wave approximation, the interactions between the WECs takes place only through the progressive wave mode [see e.g. 45, 46, 75]. The interactions between the WECs in such cases can be modelled using the multiple scattering approach (see [42], [43]). In this method, the interaction between the bodies are computed by incorporating successive order of scattering of waves by each of the bodies. In order to compute a particular order of the scattered wave from a particular body, the scattered wave (of the previous order) by all the other bodies in the array are considered incident on the former body, and this transformation is achieved using the addition theorem of Bessel functions. The well known interaction theory of Kagemoto and Yue [34] combines the direct matrix method of [75] and [76], and the multiple scattering approach of [79] and [52].

And in recent years, boundary element methods are widely used in modelling WECs, thanks to commercial codes like WAMIT, Aquaplus. The biggest challenge to such analysis are the high computational cost associated with the evaluation of an array configuration, which increases rapidly as the array size is increased. Recently [8] developed a fast multipole algorithm to accelerate the computation of sparse array of WECs using boundary element method. The effect of the interactions on the array is quantified in terms of the $q$ factor which is defined as the ratio of the power captured by the array to the case if the same number of WECs are not interacting. It has been observed that the $q$ factor has sharp spikes with wide troughs in its variation [9], and a general suggestion is that a good layout of WECs should minimize the effect of the destructive interactions.
1.1.2 Classification

Wave energy devices are traditionally classified into three broad categories: Point Absorbers (PA), Terminator and Attenuator. The initial theories on wave energy devices are mostly based on the point absorber concept. PAs are defined as systems in which the horizontal dimensions of the WECs are much small compared to the wavelength of the incident wave field. Theoretically, in the framework of linear potential flow theory it has been shown that the PAs have the potential to absorb much more energy than what is incident across its primary horizontal dimension [9]. However, such theoretical upper limits in power capture are not permissible in practice as the motions of the WECs are limited by restrictions in the PTO system, viscous dissipation, non-linear behavior and other effects. On the other hand, terminators are devices in which the primary horizontal dimension is much larger than the wavelength of the incident wave. The terminators are oriented in a manner such that the crests of the incident wave field are parallel to the dominant physical dimension of the body. Because of its orientation and large dimensions, its dynamics is similar to that of two dimensional symmetrical systems, and the power capture is limited to 50 percent of that incident across its dominant horizontal dimension for one mode of motion and 100 percent for two modes of motion. The attenuators are similar to the terminators except that their orientation is perpendicular to that of the latter. They attenuate waves as they pass by, from where their name is coined.

Another alternative way of classifying WECs is based on their location - Offshore, Nearshore or Onshore. The Offshore devices are those located at larger depths which in the context of wave energy conversion is referred to as a region where the wave power levels are not significantly affected by the energy loss mechanisms associated with the wave interaction with the sea bed [31]. While Nearshore converters are those located in shallow waters, the terminology ‘Onshore’ is used to classify those devices which are generally located in water depths of up to 10m. In general, the energy density near the coastline is lower than that is available in offshore, because of energy dissipation by wave breaking, whitecapping, bottom friction, etc. However, natural phenomena like refraction and reflection can compensate such effects and lead to energy concentration [14].
1.1 Background

1.1.3 Wave Energy Resource

Ocean waves are created by winds which are in turn produced because of the difference of pressure created by the unequal solar heating of the atmosphere. Assessment of wave energy resource is essential for the purpose of identification of locations for wave energy projects. The highest levels of annual wave energy in the Northern Hemisphere are observed along the west coast of the Northern Europe (UK, Ireland, Iceland and Norway), with somewhat lower energy levels found in the western coastlines of Northern America adjoining the Pacific [6]. This huge wave energy potential is largely untapped at this moment. Therefore, it is no surprise that developed countries like Ireland, UK, Norway, France have recognised the significance of promoting research and industry linked to wave energy.

Apart from estimation of resources, it is also important to understand the interaction between an incoming wave field and a WEC, and how it finally affects the performance of the latter. Of particular significance is the directional spreading of the spectrum at a particular location as a large number of wave energy converters are sensitive to the direction of the incident waves (e.g. the Oyster, the WaveRoller). Globally there are many attractive locations which are appropriate for locating a wave farm. However, the challenge is to identify locations which not only possess the adequate wave energy resource but also provide the necessary conditions required to ensure continuous and reliable operations of the WECs [6].

1.1.4 Theoretical Background

Let us consider a device performing oscillations under the influence of incident monochromatic waves. For a WEC with single degree of freedom (heave or surge or sway), its equation of motion (suppose $x$ is the coordinate of the WEC’s motion) can be written as

$$M \dddot{x} + C \dot{x} = F_r + F_s + F_e \quad (1.1)$$

where, $M$ is the mass of the WEC, $C$ is the restoring coefficient, $F_r$ is the force acting on the body when it is forced to oscillate in calm waters, $F_s$ is the force acting on the
1.1 Background

body when it is held fixed in the presence of incident waves, \( F_e \) is the externally applied force on the body and the variable \( t \) indicates time. Note, in (1.1), the subscript with comma indicates double differentiation of \( x \) with respect to time. Assuming the oscillations to be simple harmonic in nature, the time dependence can be separated out as follows

\[
(x, F_r, F_s, F_e) = Re[(x_0, F_r, F_s, F_e)e^{-i\omega t}].
\]

Now, the radiation force can be decomposed as follows

\[
F_r = \omega^2 \mu x_0 + i\omega \nu x_0,
\]

where \( \mu \) and \( \nu \) are respectively, the added mass and radiation damping coefficients of the oscillating body. Substituting (1.2) into the equation of motion yields

\[
[-\omega^2 (M + \mu) - i\omega \nu + C]x_0 = F_s + F_e.
\]

Let the velocity of the body be denoted by \( V = Re[V_0 e^{-i\omega t}] \) where \( V_0 = -i\omega x_0 \). Now, the average rate of work done by the waves on the device is given by

\[
P_{waves} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} Re[(F_r + F_s)e^{-i\omega t}] Re[V_0 e^{-i\omega t}] dt
\]

\[
= \frac{1}{2} Re[(F_r + F_s)V_0]
\]

\[
= \frac{1}{2} Re[F_r V_0] + \frac{1}{2} Re[(\omega^2 \mu x_0 + i\omega \nu x_0)i\omega V_0]
\]

\[
= \frac{1}{2} Re[F_r V_0] - \frac{1}{2} \nu \omega^2 |x_0|^2
\]

\[
= \frac{1}{4} (F_r V_0 + F_s V_0) - \frac{1}{2} \nu V_0^2 V_0
\]

\[
= -\frac{1}{2} \nu \left[ V_0 \overline{V_0} - V_0 \left( \frac{F_r}{2\nu} - V_0 \left( \frac{F_s}{2\nu} \right) \right) \right]
\]

\[
= -\frac{1}{2} \left( V_0 - \frac{F_r}{2\nu} \right) (V_0 - F_s) + \frac{|F_s|^2}{8\nu}
\]

\[
= -\frac{1}{2} \nu \left| V_0 - \frac{F_s}{2\nu} \right|^2 + \frac{|F_s|^2}{8\nu}
\]

So, the maximum of the average power that can be obtained from the waves is

\[
P_{waves}^{max} = \frac{|F_s|^2}{8\nu},
\]
1.1 Background

which happens when \( V_0 = F_s^2/2\nu \). In two dimensions, it is well known that the radiation and scattering problems can be related as

\[
\frac{|F_s|^2}{8\nu} = \gamma P_{\text{incident}},
\]

(1.6)

[see e.g. 51] where \( W_{\text{incident}} \) is the incident wave power per unit crest width and

\[
\gamma = \frac{|A_-|^2}{|A_+|^2 + |A_-|^2}
\]

(1.7)

where \( A_- \) and \( A_+ \) are the complex wave amplitudes of the waves radiated to \( x = -\infty \) and \( x = \infty \) respectively by the forced oscillations of the body in the absence of incident waves.

Note, the term \( \gamma \) is only dependent on the geometry of the body. For a symmetrical body, waves are equally radiated in both directions (i.e. \( |A_+| = |A_-| \)) and therefore \( \gamma = 1/2 \). The maximum efficiency in two dimensions is given by

\[
E_{\text{max}} = \frac{P_{\text{waves}}}{P_{\text{incident}}} = \frac{|F_s|^2}{8\nu} \frac{P_{\text{waves}}}{P_{\text{incident}}} = \gamma \frac{P_{\text{incident}}}{P_{\text{incident}}}
\]

(1.8)

For non-symmetrical shaped bodies, it is possible to produce \( |A_+| \ll |A_-| \), in which case \( \gamma \) is close to 1 resulting in \( E \approx 1 \). This is the reason why one can obtain such high efficiency \( (E \approx 0.9) \) with a Salters Duck.

For three dimensional bodies, the performance of a system is quantified in terms of the capture width \((l)\) which is defined as

\[
l = \frac{P_{\text{waves}}}{P_{\text{incident}}},
\]

(1.9)

For axi-symmetric bodies performing vertical heave motions, it is well known that

\[
\frac{\pi |F_s|^2}{4\lambda} = \nu P_{\text{incident}}
\]

(1.10)
1.1 Background

[see e.g. 51], where $\lambda$ is the wavelength of the incident wave field. The maximum capture width ($l_{\text{max}}$) then becomes

$$l_{\text{max}} = \frac{\left(\frac{|F_s|^2}{8\nu}\right)}{\left(\frac{\pi |F_s|^2}{4\lambda \nu}\right)} = \frac{\lambda}{2\pi}.$$  \hspace{1cm} (1.11)

Note, $l_{\text{max}} = \lambda / \pi$ for surging or pitching axi-symmetric bodies. The same results for $E_{\text{max}}$ and $l_{\text{max}}$ can also be derived using another approach. The complex force externally applied by the power take-off (PTO) system on a WEC can be expressed as

$$F_e = -\nu_{\text{pto}}V_0 - C_{\text{pto}}x_0 = i\omega \nu_{\text{pto}}x_0 - C_{\text{pto}}x_0,$$  \hspace{1cm} (1.12)

and the equation of motion (1.3) can then be written as

$$[-\omega^2 (M + \mu) - i\omega (v + \nu_{\text{pto}}) + (C + C_{\text{pto}})]x_0 = F_s.$$  \hspace{1cm} (1.13)

The average power absorbed by the PTO system becomes

$$P_{\text{absorbed}} = \frac{1}{2} \text{Re}[F_e V_0] = \frac{1}{2} \text{Re}[(i\omega \nu_{\text{pto}}x_0 - C_{\text{pto}}x_0)(i\omega x_0)]$$

$$= \frac{1}{2} \text{Re}[\omega^2 v_{\text{pto}}|x_0|^2 - i\omega C_{\text{pto}}|x_0|^2]$$

$$= -\frac{1}{2} \omega^2 v_{\text{pto}}|x_0|^2 = -\frac{1}{2} v_{\text{pto}}|V_0|^2.$$  \hspace{1cm} (1.14)

Note, the negative sign of $P_{\text{absorbed}}$ indicates that energy is taken away from the system. The efficiency in the two dimensional case, then becomes

$$E = \frac{P_{\text{absorbed}}}{P_{\text{incident}}} = \frac{\frac{1}{2} v_{\text{pto}}|V_0|^2}{\left(\frac{|F_s|^2}{8\nu \gamma}\right)} = \frac{4v_{\text{pto}} \gamma |V_0|^2}{|F_s|^2}$$

$$= \frac{4v_{\text{pto}} \gamma \omega^2 |x_0|^2}{\left(\omega^2 (M + \mu) + C_{\text{pto}} + C\right)^2 + \omega^2 \left(v + v_{\text{pto}}\right)^2} |x_0|^2.$$  \hspace{1cm} (1.15)
For $E$ to be maximum, the denominator of the RHS of (1.15) must be minimum which requires

$$-\omega^2(M + \mu) + C_{pto} + C = 0$$

$$\Rightarrow C_{pto} = \omega^2(M + \mu) - C \quad (1.16)$$

In addition, in order to obtain the maximum value of $E$ with respect to the variations of $v_{pto}$, we need to find its corresponding value for $\partial E/\partial v_{pto} = 0$, which yields $v_{pto} = v$. Therefore, finally we have

$$E_{max} = \gamma \frac{4v_{pto} v \omega^2}{\omega^2(v + v_{pto})^2} = \gamma. \quad (1.17)$$

The capture width in three dimensions is expressed as

$$l = \frac{P_{absorbed}}{P_{incident}} = \left( \frac{1}{2} v_{pto} |V_0|^2 \right) = \frac{2v\lambda v_{pto} |V_0|^2}{\pi |F_s|^2}$$

$$= \frac{2v\lambda v_{pto} |V_0|^2}{\pi \left[ -\omega^2(M + \mu) + C_{pto} + C \right]^2 + \omega^2 \left( v + v_{pto} \right)^2 |x_0|^2}. \quad (1.18)$$

Using the same arguments as used to obtain the maximum of $E$ (see equation 1.15) for the two dimensional case, the maximum of the capture width $l$ is obtained as

$$l_{max} = \frac{\lambda}{2\pi} \quad (1.19)$$

for a heaving axi-symmetric wave energy converter.

### 1.2 Oscillating Wave Surge Converters

The Oscillating Wave Surge Converter (OWSC) is a novel concept which has evolved from the analysis of Oscillating Water Columns at Queens University Belfast. It is a shallow
1.2 Oscillating Wave Surge Converters

water WEC, ideally located in water depths of 10-15m. The device performs pitching motion about a horizontal hinge axis which is located at some distance above the sea-bed. Performing back and forth motion due to action of the waves, it pumps high pressure water on to the shore where hydro-electric turbines drive a generator to convert mechanical into electrical energy. The device exploits the natural phenomenon of amplified horizontal surge motions of the water particles in shallow waters from where its name OWSC is conceived [81]. Some of the key features of this device are:

- The characteristic length scale of the OWSC (i.e. its width) is comparable to the wavelength of the incident waves and hence the point absorber approximation is not suitable for application. The dynamics of the OWSC is strongly governed by the diffraction phenomenon, and as such proper modelling of the scattered waves needs to be done.

- The performance of the device is strongly sensitive to the direction of the incoming waves. Ideally, the device doesn’t capture any power for grazing wave incidence, while performing the best for normal incidence. However, the device is located in shallow waters where the waves are strongly directional i.e. due to refraction, wave crests align with depth contours.

- Has a wide bandwidth of high efficiency. This feature has enabled to establish the device as frontrunner among the other WECs.

- Is a non-resonant WEC. Most of the WECs that are known in the literature operate on the resonance principle. Such devices usually have a short efficiency bandwidth, while performing large amplitudes of oscillation at the resonance period. One of the pitfalls is that the large extent of the motion may be incompatible with the associated mechanical systems e.g. the power take-off system can have motion constraints. The natural period of the Oyster is above 20s, far away from the operating range of the device.

- The particular OWSC -Oyster800 (analyzed in this work) can be ballasted with water, so as to sink it in extreme wave conditions and when required for maintenance.
The hydrodynamics of a single wide OWSC has been studied extensively since the initial works of [26, 27, 32, 80], and is well understood now. The earlier works on the device were largely developed at Queens University Belfast, where it has been conceptualized. The existing analysis are experimental and/or numerical in nature, performed on a single OWSC [see 31, 33]. Mathematical modelling of the OWSCs is challenging and needs specific treatment, different from those applicable to heaving wave energy converters or oscillating water columns. The first theoretical work on the OWSC was the semi-analytical model of [56], describing the physics of this particular device. It was the first time a three-dimensional mathematical model, specific to the OWSC, was presented, and it provided valuable insights into the behavior of the converter. The mathematical model is based on thin-rigid plate hypothesis which assumes that the thickness of the flap is much small compared to its width [39]. The work was later extended to analyze the hydrodynamics of infinite in-line arrays [58], of an isolated device in the open ocean [57], and a finite array.
1.2 Oscillating Wave Surge Converters

of inline converters [55].

### 1.2.1 Non-linear governing equations

The focus of this work will be on the flap-type wave energy conversion device - the Oscillating Wave Surge Converter (OWSC) and in this section we will first show the non-linear equations governing its dynamics [see 56], and will then derive a weakly non-linear form of the equations. The methodology is similar to that adopted for the analysis of Venice gates [62]. The OWSC is considered to be located at a water depth of \( h' \). The thickness of the device is taken to be \( 2a' \) while the horizontal hinge axis about which the flap performs a pitching motion is located at a distance of \( c' \) above the sea bed (see figure 1.1). The fluid is considered to be inviscid, incompressible and the flow irrotational. Therefore, we introduce the velocity potential \( \Phi' \) which satisfies the Laplace equation in the fluid domain

\[
\nabla'^2 \Phi' = 0,
\]

where \( \nabla' \) is the spatial nabla operator. Utilizing the Bernoulli’s equation on the free surface \( z' = \zeta'(x',y',t') \) and taking pressure to be zero on it, yields

\[
g \zeta' + \Phi' + \frac{1}{2} |\nabla' \Phi'|^2 = 0, \quad z' = \zeta',
\]

which is the dynamic boundary condition on the free surface. The kinematic boundary condition requires

\[
\frac{D}{Dt} [z' - \zeta'(x',y',t')] = 0,
\]

which when combined with dynamic boundary condition (1.21) gives

\[
\Phi'_{x'} + g \Phi'_{z'} + |\nabla' \Phi'|^2 + \frac{1}{2} \nabla' \Phi' \nabla' |\nabla' \Phi'|^2 = 0, \quad z' = \zeta',
\]

known as the free surface mixed kinematic-dynamic boundary condition. The no-flux boundary condition on the sea bottom gives

\[
\Phi'_{z'} = 0, \quad z' = -h'.
\]
1.2 Oscillating Wave Surge Converters

While in motion, let us assume that the front face of the flap is represented as \( x' = \xi'^+ \) while the back face as \( x' = \xi'^- \). Therefore, the kinematic boundary condition on the flap gives

\[
\frac{D}{Dt}[x' - \xi'^\pm(z', t')] = 0, \quad x' = \xi'^\pm, -w'/2 < y' < w'/2, \tag{1.25}
\]

which on simplification yields

\[
\Phi'_x = \Phi'_\xi + \Phi'_\xi \xi', \quad x' = \xi'^\pm, -w'/2 < y' < w'/2, \tag{1.26}
\]

and that along the lateral edges requires

\[
\Phi'_y = 0, \quad \xi'^- < x' < \xi'^+, \quad y' = \pm w'/2. \tag{1.27}
\]

Now, suppose the flap rotates by an angle \( \theta' \), the horizontal displacement of the two sides about the bottom foundation can be expressed as

\[
\xi'^\pm = -(z' + h' - c') \tan \theta' \pm \frac{d'}{\cos \theta'}. \tag{1.28}
\]

Substitution of (1.28) into (1.26) gives

\[
\Phi'_x = \left\{ \frac{-(z' + h' - c') \pm d' \sin \theta'}{\cos^2 \theta'} \Phi'_\xi \tan \theta' \right\} \theta'_p + \Phi'_\xi \tan \theta', \quad H(z' + h' - c'), \quad x' = \xi'^\pm, -w'/2 < y' < w'/2. \tag{1.29}
\]

The equation of motion of the OWSC is given by

\[
I' \frac{d^2 \theta'}{dt'^2} = T'_p(t') + T'_g(t') + T'_{pto}(t'), \tag{1.30}
\]

where, \( I' \) is the second moment of inertia, \( T'_p \) is torque due to the fluid pressure, \( T'_g \) is torque due to its own inertia and \( T'_{pto} \) is the torque acting due to the power take-off mechanism. For computational convenience, the torque \( T'_p \) is decomposed into two components \( T'_{p+} \) and \( T'_{p-} \) which are respectively, the torque acting acting on the front and back face of the flap.
1.2 Oscillating Wave Surge Converters

The total torque due to the fluid pressure is evaluated to be

\[ T_p' = T_p'^+ + T_p'^- = \int_{-w'/2}^{w'/2} \int_{-h'+c'}^{h'+c'} p'(x', y', z', t') \frac{z' + h' - c' + d' \sin \theta'}{\cos^2 \theta'} dz' dy' \]

(1.31)

and

\[ T_g' = S' g \sin(\theta'), \quad (1.32) \]

where \( S' = M' h' \) with \( M' \) being the mass of the flap and \( h' \) the distance from the hinge axis to the center of gravity. The non-dimensional system of variables is chosen as

\[ (x, y, z, r) = (x', y', z', r') / w', t = \sqrt{\frac{g}{w'}} t', \Phi = \frac{\Phi'}{\sqrt{gw'A_i'}}, \theta = \frac{\theta'}{\epsilon}, \quad (1.33) \]

where \( \epsilon = A_i' / w' \) is the small parameter of the system where \( A_i' \) is the amplitude of the incident wave.

**Weakly non-linear analysis**

After non-dimensionalisation and performing series expansion of the terms \( \sin(\epsilon \theta) \) and \( \cos(\epsilon \theta) \), the non-dimensional torque due to the inertia of the OWSC is given by

\[ T_g = S \sin(\epsilon \theta) = S(\epsilon \theta - \frac{\epsilon^3 \theta^3}{6}), \quad (1.34) \]

that due to the fluid pressure is

\[ T_p = \]

\[ - \int_{-w'/2}^{w'/2} \int_{-h'+c'}^{h'+c'} \left( z + \epsilon \Phi_t + \epsilon^2 \frac{1}{2} |\nabla \Phi|^2 \right) \left[ (z + h - c) - a \epsilon \theta + (z + h - c) \epsilon^2 \theta^2 - \frac{5}{6} a \epsilon^3 \theta^3 \right] dz dy \]

\[ + \int_{-w'/2}^{w'/2} \int_{-h'+c'}^{h'+c'} \left( z + \epsilon \Phi_t + \epsilon^2 \frac{1}{2} |\nabla \Phi|^2 \right) \left[ (z + h - c) + a \epsilon \theta + (z + h - c) \epsilon^2 \theta^2 + \frac{5}{6} a \epsilon^3 \theta^3 \right] dz dy \]

(1.35)
1.3 Outline of the work

and the non-dimensional torque due to the external power take-off system becomes

\[ T_{pto} = -\mu_{pto} \epsilon \theta_{tt} - \nu_{pto} \epsilon \theta_t - c_{pto} \epsilon \theta. \]  

(1.36)

The expansion of terms inside (1.35) is similar to that performed in [62], with the limits of the integration being from \( -h + c \) to the free surface in the vertical \( z \) direction, from \(-w/2\) to \( w/2\) in the \( y \) direction and the horizontal hinge axis located at a distance \( c \) above the sea bottom (details can be found in Appendix A). On substitution of (A.18), (1.34) and (1.36) into the non-dimensional equation of motion

\[ I \epsilon \theta_{tt} = T_g + T_p + T_{pto}, \]  

(1.37)

finally yields the non-linear equation of motion of the flap

\[
(I + \mu_{pto}) \theta_{tt} + \nu_{pto} \theta_t + \left( aw(h-c)^2 + c_{pto} - S \right) \theta + \epsilon^2 \frac{1}{6} \left( S + 5aw(h-c)^2 \right) \theta^3
\]

\[
= - \int_{-w/2}^{w/2} \int_{-h+c}^{0} \Delta \Phi_t (z+h-c) dz dy + \epsilon \int_{-w/2}^{w/2} \int_{-h+c}^{0} \left\{ 2a \theta \Phi_t + (z + h - c)^2 \theta \Delta \Phi_{tx} + \frac{1}{2} \Delta |\nabla \Phi|^2 (z + h - c) \right\} dz dy - \epsilon \int_{-w/2}^{w/2} \left\{ \frac{(h - c)}{2} \Delta \xi^2 - \frac{1}{2} (z + h - c)^2 \theta \Delta \nabla \Phi_{tx}^2 - a \theta |\nabla \Phi|^2 \right\} dz dy - \epsilon \int_{-w/2}^{w/2} \left\{ \frac{(h - c)}{2} \Delta \xi^2 - a \theta |\nabla \Phi|^2 - \frac{1}{2} \theta \Delta (\xi \xi_x) + \frac{1}{2} \Delta_0 (\Phi_1 \xi_x^2) - \theta \Delta_0 (\xi \Phi_1) + \frac{(h - c)}{2} \Delta_0 (|\nabla \Phi|^2 \xi) \right\} dz dy + O(\epsilon^3)
\]

(1.38)

1.3 Outline of the work

This work primarily deals with the development of mathematical models for the hydrodynamic analysis of wave energy converters (WECs). With the aid of these models, we will try to address some of the relevant practical issues. The focus will mainly be on the device - Oscillating Wave Surge Converter (OWSC), which is currently seen as a frontrunner in the race for harnessing energy from the ocean.
1.3 Outline of the work

In the first work of this thesis, a mathematical model is described to analyze the hydrodynamic behavior of a wave energy farm consisting of OWSCs. The method is a highly efficient semi-analytical approach based on linear potential flow theory. Wave farms with a large number of such devices are studied for various configurations. For an inline configuration with normally incident waves, the occurrence of a near resonant behavior, already known for small arrays, is confirmed. A strong wave focusing effect is observed in special configurations comprising of a large number of devices. The effects of the arrangement and of the distance of separation between the flaps are also studied extensively. In general, the flaps lying on the front of the wave farm are found to exhibit an enhanced performance behavior in average, due to the mutual interactions arising within the array. A random sea analysis shows that a slightly staggered arrangement can be an ideal layout for a wave farm of this device. The hydrodynamics of two flaps that oscillate back to back is also discussed.

Although the semi-analytical approach developed in this work can analyze arbitrary number and layout of the OWSCs, determination of the optimal arrangement of the OWSCs is still a challenging problem. Mathematical tools are widely available for the hydrodynamic analysis and performance estimation of arrays of WECs, however the high computational costs are a hindrance to the identification of optimal arrangements. In addition, the number of computational evaluations required for the optimisation increases almost exponentially with increase in the number of WECs. In Chapter 3, a new methodology involving multiple optimization strategies is presented to arrive at the solution to the complex problem. The approach includes a statistical emulator to predict the performance of the WECs in arrays, followed by an innovative active learning strategy to simultaneously explore and focus in regions of interest of the problem, and finally a genetic algorithm to obtain the optimal layouts of WECs. The method is extremely fast and easily scalable to arrays of any size. Case studies are performed on a wavefarm comprising of 40 WECs subject to arbitrary bathymetry and space constraints.

One of the shortcomings in the design of the rigid OWSC is the large wave loads acting on the common foundation at the bottom, especially in extreme wave conditions. One possible mechanism of reducing the wave loads is to divide the rigid OWSC into smaller
components. A new concept that has emerged from the above philosophy is the modular concept of the OWSC. In Chapter 4, the hydrodynamics of this new concept is discussed. A mathematical model is presented to analyze the effect of the interactions of the system. The analysis is performed with a modular system comprising of six identical cylindrical modules of total combined width 24m, reminiscent of the Oyster800 device developed by Aquamarine Power. Various design strategies are explored. It is shown that such a closely packed system of modules results in multiple resonances which can potentially be exploited to capture more power. The behavior is found to be similar to that of the rigid flap when the applied power take-off damping per unit width is similar in both cases. It is also observed that the modules lying at the center of the system capture more energy than those lying at the edges. An optimization of power take-off system shows that at lower wave periods it is possible to capture the levels of power similar to those of an equivalent size rigid flap while at higher periods, the modular system has the potential to capture more energy due to the occurrence of multiple resonances.

In chapter 5, the effect of a straight coast on the hydrodynamics and performance of an OWSC is analyzed using the hypersingular integral equation approach, with a Green’s function for a semi-infinite fluid domain. Extremes in the hydrodynamic characteristics of the system are shown to occur at certain wave periods when the device is located at specific distances from the coast. This dynamics can have either detrimental or favourable effects on the performance of the converter, depending on the system parameters. Surprisingly, when the device is located very close to the coast, the qualitative behavior of the system resembles that of a single device in the open ocean. In addition, the analysis shows that under such circumstances, the device consistently achieves much higher levels of efficiency than it would achieve in an open ocean.

The favourable performance of an OWSC near a straight coast motivated the investigation in Chapter 6 where it is analyzed if a breakwater of finite width can reproduce such beneficial performance features. A three dimensional mathematical model is described to analyze the hydrodynamics of the system based on thin-plate approximation and the study is performed within the framework of linear potential flow theory with oblique wave incidence. Results show that a breakwater is in fact capable of reproducing the performance
1.3 Outline of the work

characteristics of an Oscillating Wave Surge Converter (OWSC) near a straight coast and the system comprising of a breakwater and OWSC can emerge as a new hybrid model for the nearshore wave energy converter. It is also found that the hydrodynamics of the system is strongly influenced by the width of the breakwater. A random sea analysis reveals that the new hybrid model of the WEC can be highly effective in locations with high occurrence of low sea states.

In Chapter 7, the mutual hydrodynamic interactions between two different kinds of devices, the OWSC and a generic Heaving Wave Energy Converter modelled as a vertical truncated cylinder, is investigated. A three-dimensional mathematical model is developed based on linear potential flow theory where it is assumed that the effect of the evanescent modes of the disturbed potential due to one of the converters is negligible in the vicinity of the other, considering a reasonable distance of separation between the two devices. The approach combines the diffraction/radiation characteristics of the isolated bodies, and requires the velocity potential to satisfy the matching and boundary conditions on the two converters. A numerical collocation scheme and the method of matched eigenfunctions are used to obtain the solution to the unknown coefficients. Good agreement is observed in the mutual interaction terms arising out of the two radiation problems. The influence of location, oblique wave incidence and other parameters on the power absorption characteristics of the two bodies is analyzed. The results may be useful for the design of practical wave farms which can be constituted of multiple kinds of devices.
Wavefarm modelling of Oscillating Wave Surge Converters

2.1 Introduction

Wave farms comprising of a large number of WECs are planned at sites which have already been identified for the purpose of energy extraction (e.g. Lewis Wave Project, see [3]). The arrangement of the devices in such a farm can follow several possible configurations. The present study analyzes the interaction of waves with an array of OWSCs and the performance of such systems. The OWSC considered here is a bottom hinged flap-type WEC which extracts energy by virtue of its pitching motion and resembles the Oyster® developed by Aquamarine Power Ltd.

Collaborators: Emiliano Renzi, Frederic Dias
2.1 Introduction

Wave power absorption in an array has already been studied in the literature, starting with the seminal work of [9]. However, many of the investigations deal with the hydrodynamics of point absorbers (see e.g. [19],[23],[25]), which is based on the assumption that the body dimensions are much smaller than the wavelength of the incident wave field. Recent studies have shown that for OWSCs like Oyster, the point absorber limit is no longer applicable and hence better and more accurate modelling of the device needs to be undertaken [60]. Some recent investigations also dealt with a detailed analysis of multiple WECs, but most of them did not go beyond three or four of such devices ([4], [16], [12], [83]). Indeed, in the literature there have been very few attempts to understand the dynamics of large finite arrays. The analytical modelling of large and complex systems becomes difficult, while numerical approaches on the other hand are computationally expensive and performing such an analysis experimentally is quite challenging. Recently, [7] used a fast multipole accelerated linearised boundary element method to study large arrays of sparsely distributed generic WECs in deep water. However, despite the recent effort of [55], who devised a new method to investigate the hydrodynamics of a small inline array of OWSCs, to date there is still a need for an unifying theory of large arrays of OWSCs in any configuration and in oblique waves. The analysis in this paper extends the semi-analytical work of [55] to investigate a large farm of OWSCs in any configuration under oblique incident waves.

A mathematical model is developed here within the framework of linear potential theory. The theory allows the analysis of arbitrary configurations of an array of OWSCs, the only constraint being that all the converters have parallel pitching axes (see figure 2.1). The problem is formulated as a boundary value problem for the radiation and scattering potentials. The use of Green’s integral theorem yields hypersingular integrals (HIs) in terms of the jump in potential across the sides of each flap, which are solved using a numerical approach in terms of the Chebyshev polynomial of the second kind. The derivation of the mathematical model is quite general: one can solve for the unknown jump in potential across each flap for arbitrary configurations of the array. A wave farm consisting of various layouts of a finite array of OWSCs is then studied considering complete hydrodynamic interaction among all the devices.
2.1 Introduction

Fig. 2.1 A computer generated 3D graphical view of a portion of a wave farm comprising of five OWSCs.

Fig. 2.2 Geometry of the physical model of a portion of an OWSC array: (a) top view; (b) cross-section of the \( m \)th flap shown with the physical parameters.

The first theoretical model based on HIs was developed for an OWSC in a channel [56] and was then extended to study the hydrodynamics of an infinite array of WECs [58], a single device in the open ocean [57] and a finite array of inline converters [55]. Recently, the same method was also used to analyze the hydrodynamics of a flap-type device near a straight coast [71]. Following the same approach, in this study we develop a mathematical model to investigate the hydrodynamic behavior for the most generalised case consisting of a large number of OWSCs in any configuration with oblique wave incidence.

The generalised mathematical model is derived in the first part of the thesis (§2.2 & §2.3). In §2.4, the effect of the separation distance is studied in detail using three flaps. This is followed by an analysis of both a wave farm comprising of thirteen flaps in various possible arrangements and a wave farm of 40 inline flaps. Finally the semi-analytical model is employed to study the hydrodynamics of two devices located back to back - a configuration which has intrigued many (see [77]).
2.2 Mathematical Model

2.2.1 Governing Equations

A wave farm of $M$ OWSCs is considered to be located in an ocean of constant water depth $h'$. Waves are incident from the right making an angle $\psi$ with the $x'$-axis as shown in figure 2.2. The origin is located on the mean free surface with $y'$ the pitching axis of the flaps and $z'$ directed upwards. Primes in this mathematical model are used to denote the physical variables. With the assumption of irrotational flow and inviscid, incompressible fluid, the velocity potential $\Phi'$ satisfies the Laplace equation

$$\nabla'^2 \Phi' = 0,$$  \hspace{1cm} (2.1)

in the fluid domain, where $\nabla' f' = (f'_{x'}, f'_{y'}, f'_{z'})$ is the nabla operator; subscripts with commas denote differentiation with respect to relevant variables. The linearised kinematic-dynamic boundary condition on the free surface gives

$$\Phi'_{t't'} + g\Phi'_{z'} = 0, \quad z' = 0,$$  \hspace{1cm} (2.2)

where $g$ is the acceleration due to gravity while the no-flux boundary condition at the seabed yields

$$\Phi'_{z'} = 0, \quad z' = -h'.$$  \hspace{1cm} (2.3)

Each device is equipped with an oscillating flap hinged to a rigid foundation at a distance $c'$ above the seabed (see again figure 2.2). The WECs are modelled using a thin-rigid plate approximation (see [39]) and the kinematic boundary condition on their surface is then expressed as

$$\Phi'_{x'} = -\theta_{m'\epsilon'} (z' + h' - c'_\beta) H(z' + h' - c'_\beta), \quad x' = x'_m \pm \epsilon', \epsilon' \rightarrow 0,$$

$$y'_{A'} \leq y' < y'_{B'}, \quad m = 1, \ldots, M,$$  \hspace{1cm} (2.4)
where $x'_m$ is the $x'$ coordinate of the center of the $m$th flap, $y'_m$ and $y''_m$ are the $y'$ coordinates corresponding to the two edges of the device and $H$ is the Heaviside step function.

Like in previous work ([55–58, 60, 71]), a non-dimensional system of variables is chosen as

$$(x, y, z) = (x', y', z')/w', t = \sqrt{g/w'} t', \Phi = \Phi'/\sqrt{gw'A'_I}, \theta_m = (w'/A'_I) \theta'_m,$$

(2.5)

where $w'$ is the length scale of the system (e.g. the width of the largest flap) and $A'_I$ is the amplitude of the incident wave. Assuming the oscillation of the flaps to be simple harmonic in nature, the time dependence of the variables can be separated out as

$$\theta_m = \text{Re}\{\Theta_m \exp^{-i\omega t}\}, \quad \Phi = \text{Re}\{\phi(x, y, z) \exp^{-i\omega t}\},$$

(2.6)

where $\omega = \omega'/\sqrt{w'/g}$ and $\Theta_m$ are respectively, the angular frequency and amplitude of oscillation of the $m$th flap, while $\phi(x, y, z)$ is the complex spatial velocity potential. The spatial potential can in turn be resolved into

$$\phi = \phi^S + \phi^R = \phi^I + \phi^D + \sum_{\beta=1}^{M} V_{\beta} \phi^{(\beta)}$$

(2.7)

where

$$\phi^I = \frac{i A_I \cosh(k(z + h))}{\omega \cosh kh} \exp^{-ik \cos \psi + ik \sin \psi}$$

(2.8)

is the incident wave potential. In (2.7), $\phi^D$ is the diffracted wave potential, $\phi^{(\beta)}$ is the radiation potential induced by the motion of the $\beta$-th flap while the other flaps are held fixed and $V_{\beta} = i\omega \Theta_\beta$ is the complex angular velocity of the moving flap. Also note, in (2.8), $k$ is the solution to the dispersion relation $\omega^2 = k \tanh kh$. On substitution of the factorisation (2.6) and (2.7) in the governing equations (2.1)–(2.4), we obtain a boundary value problem in terms of the spatial radiation and scattering potentials. These potentials satisfy the Laplace equation

$$\nabla^2 \phi^{(\beta, D)} = 0,$$

(2.9)

where the notation $\phi^{(\beta, D)}$ denotes either potential, the linearised free-surface boundary
2.2 Mathematical Model

condition

\[-\omega^2 \phi^{(\beta,D)} + \phi^{(\beta,D)}_{z z} = 0, \quad z = 0, \quad (2.10)\]

the no-flux boundary conditions at the sea bed

\[\phi^{(\beta,D)}_{z} = 0, \quad z = -h, \quad (2.11)\]

and the kinematic conditions on the lateral surfaces of the flaps

\[\begin{aligned}
\phi^{(\beta)}_{x} &= (z+h - c_\beta)H(z+h - c_\beta)\delta_{\beta m}, \quad x = x_m \pm \varepsilon, \varepsilon \to 0, \quad y^A_m \leq y \leq y^B_m, \\
\phi^{D}_{x} &= -\phi^{I}_{x}
\end{aligned} \quad (2.12)\]

for \(\beta = 1, 2, \cdots, M\), \(\delta_{nm}\) being the Kronecker delta. Finally, both \(\phi^{(\beta)}\) and \(\phi^{D}\) obey the radiation condition [49]. The vertical dependence can now be isolated out of the three dimensional governing system (2.9)–(2.13) by using the separation (see [49, 55, 57]):

\[\phi^{(\beta,D)}(x,y,z) = \sum_{n=0}^{\infty} \phi^{(\beta,D)}_{n}(x,y)Z_n(z), \quad (2.14)\]

where

\[Z_n(z) = \frac{\sqrt{2} \cosh \kappa_n(z+h)}{(h + \omega^{-2} \sinh^2 \kappa_n h)^{1/2}}, \quad n = 0, \cdots, \infty, \quad (2.15)\]

are the normalised vertical eigenmodes satisfying the orthogonality relation

\[\int_{-h}^{0} Z_n(z)Z_m(z)dz = \delta_{nm}. \quad (2.16)\]

In (2.15), \(\kappa_0 = k\) and \(\kappa_n = ik_n\) are the solutions of the dispersion relation

\[\begin{aligned}
\omega^2 &= k \tanh kh, \quad \omega^2 = -k_n \tanh k_n h, \quad n = 1, 2, \cdots, \quad (2.17)
\end{aligned}\]

respectively. Using the decomposition (2.14) and the orthogonality relation (2.16) yields a two-dimensional governing system for \(\phi^{(\beta,D)}_n\), where the Laplace equation (2.9) becomes
2.2 Mathematical Model

the Helmholtz equation

$$(\nabla^2 + \kappa_n^2) \begin{pmatrix} \phi_n^\beta \\ \phi_L^n \end{pmatrix} = 0, \quad (2.18)$$

and the kinematic conditions on the flap (2.12) become

$$\begin{pmatrix} \phi_n^{\beta} \\ \phi_n^D \end{pmatrix} = \begin{pmatrix} f_n B \delta_{\beta m} \\ A_1 d_n^m e^{ik \sin \psi} \end{pmatrix} \quad x = x_m \pm \varepsilon, \varepsilon \to 0, \quad y_A^m < |y| < y_B^m, \quad (2.19)$$

where

$$f_n B = \frac{\sqrt{2} [\kappa_n (h - c_\beta) \sinh (\kappa_n h) + \cosh (\kappa_n c_\beta) - \cosh (\kappa_n h)]}{\kappa_n^2 (h + \omega - \sinh^2 (\kappa_n h))^{1/2}} \quad (2.20)$$

and

$$d_n^m = \frac{k \cos \psi (h + \omega - \sinh^2 kh)^{1/2}}{\sqrt{2} \omega \cosh kh} \left[ \cos (k x_m \cos \psi) - i \sin (k x_m \cos \psi) \right] \delta_{\beta m} \quad (2.21)$$

depend on the geometry of the system. Finally the $\phi_n^{(\beta,D)}$ must be outgoing disturbances for $r \to \infty$ (obey a radiation condition). The boundary value problem (2.18) – (2.19) is solved with the application of Green’s integral theorem, similar to the procedure followed in [56]. The formulation yields HI s which are solved using the Chebyshev polynomials of the second kind (see Appendix B for details). Finally the solution for the $\beta$-th mode radiation potential is obtained as

$$\phi^{(\beta)}(x, y, z) = -i \frac{8}{\delta} \sum_{n=0}^{+\infty} \kappa_n Z_n (z) \sum_{m=1}^{M} w_m (x - x_m) \sum_{p=0}^{P} \alpha_{pnm}^{(\beta)} \int_{-1}^{1} (1 - u^2)^{1/2} U_p (u)$$

$$\times \left[ \frac{H_1^{(1)} (\kappa_n \sqrt{(x - x_m)^2 + (y - \frac{uw_m + 2y_m^c}{2})^2})}{\sqrt{(x - x_m)^2 + (y - \frac{uw_m + 2y_m^c}{2})^2}} \right] du, \quad (2.22)$$

where $w_m = y_B^m - y_A^m$ is the non-dimensional width of the $m$th flap, $U_p$ is the Chebyshev polynomial of the second kind and order $p$, $p = 0, 1, \cdots, P \in \mathbb{N}$, $H_1^{(1)}$ is the Hankel function of the first kind and first order, $y_m^c$ the $y$ coordinate of the center of flap $m$ while $\alpha_{pnm}^{(\beta)}$ are the complex solutions obtained using a numerical collocation scheme (see Appendix B).
The solution to the spatial diffraction potential is expressed as

\[
\phi^D(x, y, z) = -\frac{i}{8} A_k Z_0(z) \sum_{m=1}^{M} w_m(x-x_m) \sum_{p=0}^{P} b_{p0m} \int_{-1}^{1} (1-u^2)^{1/2} U_p(u) \times H_1^{(1)} \left( k \sqrt{(x-x_m)^2 + (y - \frac{uw_m + 2y\zeta_m}{2})^2} \right) du,
\]

(2.23)

where the \(b_{p0m}\) are the complex solutions of a system of equations, again solved numerically. Note that in \(\phi^D\) (2.23) only the 0th order vertical mode is present, the flaps being walled structures in the scattering problem (i.e \(\phi^D_n = 0\) for \(n > 0\)). Using the above expressions (2.22) and (2.23), the velocity potential is known in the whole fluid domain. It gives access to the flaps’ hydrodynamic coefficients, which enables solving the equation of motion for the flaps.

### 2.2.2 Hydrodynamic Parameters

The solution for the velocity potential is then used to solve the equation of motion of each individual flap in the frequency domain. Suppose for the \(\alpha\)-th flap, \(I_\alpha = I'_\alpha/\rho w'_\alpha^5\) is the second moment of inertia and \(C_\alpha = C'_\alpha/\rho g w'_\alpha^4\) is the coefficient of the flap restoring buoyancy torque. Then its non-dimensional equation of motion can be expressed as shown in [55]

\[
[-\omega^2(I_\alpha + \mu_{\alpha\alpha}) + C_\alpha - i\omega(v_{\alpha\alpha} + v_{\alpha\alpha}^{\text{pwr}})]\Theta_\alpha - \sum_{\beta=1}^{\infty} [\omega^2 \mu_{\beta\alpha} + i\omega v_{\beta\alpha}]\Theta_\beta = F_\alpha.
\]

(2.24)

In the latter,

\[
\mu_{\beta\alpha} = \frac{\pi w_\alpha}{4} Re \left\{ \sum_{n=0}^{\infty} f_{n\alpha}(\beta) \right\}
\]

(2.25)

is the added moment of inertia,

\[
v_{\beta\alpha} = \frac{\pi \omega w_\alpha}{4} Im \left\{ \sum_{n=0}^{\infty} f_{n\alpha}(\beta) \right\}
\]

(2.26)
2.2 Mathematical Model

is the radiation damping and

\[ F_\alpha = -\frac{\pi \omega w_\alpha}{4} i A_1 b_{00\alpha} f_0 \alpha \]  

(2.27)

is the excitation torque (see [55] for details). Note that in (2.25)-(2.26), the \( a_{0\alpha}^{(b)} \) are linked to the jump in potential across each flap (see B.9), which in turn depends on the geometry of the whole system. Also in (2.24), \( v_\alpha^{\text{pto}} = v_\alpha^{\text{pto}'} / (\rho w_\alpha^{\text{i}} \sqrt{g/w_\alpha}) \) is the power take-off (PTO) damping coefficient of the \( \alpha \)-th flap and following [4], is set equal to the optimal PTO damping for the same OWSC isolated in the open ocean:

\[ v_\alpha^{\text{pto}} = \sqrt{\frac{[C_\alpha - (I_\alpha + \mu_\alpha^{\text{open}}) \omega^2] + (v_\alpha^{\text{open}})^2}{\omega^2}} \]  

(2.28)

where \( \mu_\alpha^{\text{open}} \) and \( v_\alpha^{\text{open}} \) are respectively the added moment of inertia and radiation damping of the \( \alpha \)-th OWSC isolated in the open ocean. According to the theory of damped oscillating systems (see chapter 1, [24]), the average extracted power by the wave farm over a wave period is

\[ P = \frac{1}{2} \omega^2 \sum_{i=1}^{M} v_i^{\text{pto}} |\Theta_i|^2. \]  

(2.29)

The performance of the system is measured with the interaction factor \( q \), defined as the ratio of total power captured by an array of \( M \) flaps to the power captured by an isolated WEC of the same type multiplied by \( M \):

\[ q = \frac{P}{MP_{\text{single}}}. \]  

(2.30)

A value of \( q > 1 \) implies that there is a gain in the net power output from an array because of constructive interaction amongst the flaps. On the other hand, \( q < 1 \) indicates that mutual interactions have a cumulative destructive influence on the array efficiency. However, the interaction factor \( q \) (2.30) doesn’t quantify the performance of individual array elements. In order to understand the performance dynamics of each WEC in an array, [4] defined a
Fig. 2.3 (a) Plan view of two OWSCs in a staggered configuration as analyzed in [55]; (b) Comparison of excitation torque on two flaps versus incident wave period. The solid line and the dotted line represent the variation of the excitation torque $F'$ of flap 1 and 2 respectively, obtained using the semi-analytical approach presented here, while the squares and the diamonds are from the numerical analysis presented in [55]. Results are shown for $A'_f = 1\text{m}$, $h'_f = 13\text{m}$, $w' = 26\text{m}$ and $c' = 4\text{m}$.

The term $q_{\text{mod},m}$ given by

$$q_{\text{mod},m} = \frac{P_m - P_{\text{single}}}{\max(P_{\text{single}})}$$

where $P_m$ is the power captured by the $m$th flap while $\max(P_{\text{single}})$ is the maximum value of $P_{\text{single}}$ in the considered range of incident wave periods. The parameter $q_{\text{mod},m}$ represents the array induced performance modification of each individual WEC, with $q_{\text{mod},m} > 0$ implying a beneficial influence and $q_{\text{mod},m} < 0$ a negative interaction effect. The two terms $q$ and $q_{\text{mod},m}$ together can reasonably describe the global and single scale performance behavior of an array configuration.

### 2.3 Algorithm implementation and computational cost

An algorithm based on the mathematical model described here has been implemented through a code written in Mathematica® 8. The algorithm and the code have been made as general as possible and can handle a large number of flaps in any staggered configuration. The code requires no modification if the number of flaps or their configuration/positions...
are changed. Only the coordinates of the flap centers need to be changed. The other required inputs to the code are the flap width, distance from the sea bottom to the hinge, water depth, incident wave amplitude, range and number of incident wave periods, angle of oblique wave incidence, moment of inertia and buoyancy torque of the flap and total number of vertical eigenmodes, order of Chebyshev polynomials and terms in the remainder of the Hankel function (see (B.8)). A relative error of \( O(10^{-3}) \) is obtained with the first three vertical eigenmodes and sixth-order Chebyshev polynomials \((P = 6)\). From a computational point of view, the semi-analytical approach described here is extremely efficient compared with a full numerical approach. The latter has been used in [55] to model a three-flap inline and a two-flap staggered configurations. The computational expense associated with the full numerical approach was on an average 1 hour for a single wave period evaluation performed on a computer equipped with an i7 2.67 GHz CPU and 12GB RAM. Computations with the semi-analytical model presented here were performed with an i7 3.40 GHz CPU and 16GB RAM equipped computer. For the assessment of a system of thirteen flaps only six minutes are required in average for each wave period.
Fig. 2.5 Variation of $q^{\text{mod,2}}$ versus incident wave period for the central flap for the symmetrical three-flap cluster configuration shown in figure 2.4(a). Each of the sub-plots shows the $q^{\text{mod,2}}$ variation for a particular value of $b'$ while the distance $d'$ is varied.
2.4 Results

The computations are performed for several configurations of OWSCs, each one closely resembling the Oyster800 WEC developed by Aquamarine Power Ltd. The parameters of the system are reported in Table 7.1.

<table>
<thead>
<tr>
<th>A′</th>
<th>w′</th>
<th>h′</th>
<th>c′</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3m</td>
<td>26m</td>
<td>13m</td>
<td>4m</td>
</tr>
</tbody>
</table>

Table 2.1 Dimension of the physical variables

In the following, we shall validate the computational model with available theoretical and numerical results. Then we will discuss the interactions arising in a simple three-flap cluster and further show the potential of the model in handling more complex and populated arrays.

2.4.1 Validation

The solution obtained for an inline array \((x_m = 0, \ m = 1, 2, \ldots, M)\) of flaps and normal incidence \((\psi = 0^\circ)\) is exactly the same as shown in [55] and consequently the same results are obtained for the two-flap inline and three-flap inline cases as presented in [55]. For staggered configurations, results for only two flaps are available in the literature and have been obtained with a numerical tool [55]. Figure 2.3 shows the variation of the excitation torque versus time period of the incident wave for the two-flap staggered case of [55]. A fairly consistent agreement is observed in the results obtained by the current model and the numerical approach of [55]. Small discrepancies can be observed at around 7s which are likely due to the thickness-induced effect. The latter is modelled in the numerical model but not in the semi-analytical solution presented here (see [55] for details).

2.4.2 Three-flap cluster

In order to understand the effects of the mutual interactions arising in a wave farm, we first consider a basic array cluster comprising of only three flaps and we focus our attention on
2.4 Results

the performance of the flap positioned centrally amongst them. This central flap in a way represents an OWSC located well within an array, where the hydrodynamic influences of only its two neighbouring devices are assumed to be dominant. We consider both symmetrical and non-symmetrical configurations of the three flaps with essentially uniform spacing between them in normally incident waves.

Let us first consider the case of the symmetrical configuration shown in figure 2.4(a). Here the distance $d'$, measured from the central flap, is positive in the positive $x'$—direction. Therefore $d' > 0$ m represents the case when the central flap is located behind the two lateral flaps, while $d' < 0$ m indicates otherwise. Figure 2.5 plots the $q_{\text{mod},2}$ of the central OWSC for various distances of separation. Each of the sub-plots shows the behavior for a particular value of the lateral distance $b'$ while varying $d'$. It can be observed that $d' > 0$ m is associated with a strong destructive influence on the central flap’s performance across the entire operating range of periods. On the other hand, for $d' < 0$ m, positive interaction effects dominate and significantly enhance the performance of the central flap, suggesting that an OWSC will have better power absorption characteristics when located at the front of the cluster. The most important thing to note is that for the situations considered here, the qualitative behavior of the $q_{\text{mod}}$ variation is determined by $d'$, while $b'$ primarily dictates the extent of the peaks (see again figure 2.5). In general, as the distance $b'$ is increased there is a shift in the $q_{\text{mod}}$ variations towards higher periods, accompanied by a reduction in the magnitude of the peaks, which means a decrease in the interaction amongst the flaps. It can be inferred that as $b'$ is further increased, there would be a larger number of local maxima and minima of reduced magnitudes and so on average, there would be no distinctive positive or negative interaction effect on the device performance for any value of $d'$.

Now we consider the case where the layout of the flaps with respect to the centerline of the middle OWSC is non-symmetrical, as shown in figure 2.4(b). The noticeable difference with the previous arrangement is that the pitching axes of the extreme flaps are now separated from that of the central flap in opposite directions. The $q_{\text{mod},2}$ variation of the central flap for the various cases is plotted in figure 2.6. Almost ubiquitously for the range of distances considered, such a configuration has a negative influence on the WECs perfor-
Fig. 2.6 Variation of $q_{\text{mod},2}$ versus the incident wave period of the central flap for the non-symmetrical 3-flap cluster configuration shown in figure 2.4 (b). Each of the sub-plots shows the $q_{\text{mod},2}$ variation for a particular value of $b'$ while the distance $d'$ is varied.

mance. This is likely due to the opposite interaction effects on the central OWSC by the two lateral flaps.
2.4 Results

Fig. 2.7 Five possible layouts of a 13 OWSC wave farm are shown here. The magnitude of the spacings between neighbouring flaps in all the cases is fixed at 20m in the $y$—direction and 15m in the $x$—direction.

### 2.4.3 Wave farm of 13 OWSCs

A wave farm consisting of 13 flaps in various configurations is shown in figure 2.7. Typically even larger arrays could be studied using the same computational infrastructure mentioned previously within a reasonable time. The spacing between the flaps is chosen similar to the one planned for the proposed wave farm at the Isle of Lewis in Scotland [3]. For the purpose of identifying each individual converter, the flaps are numbered in an increasing order from right to left of the array with the OWSC located on the extreme right considered as flap 1. The distance between the edges of the neighbouring flaps is 20m in the $x'$—direction for all the configurations shown in figure 2.7, while the pitching axes of the neighbouring flaps in the staggered configurations are separated by a distance of 15m. The wave farms considered in the analysis are symmetrical about the central flap (flap 7), so for normal wave incidence the hydrodynamic behavior is symmetric with respect to the $x'$—axis passing through the center of the central flap.

*Inline:* The inline case corresponds to the configuration in which the pitching axes of all the
2.4 Results

Fig. 2.8 Variation of $\eta^{mod}$ of the individual OWSCs for the five layouts shown in figure 2.7. Since normal wave incidence is considered here, the configuration is symmetrical about the central flap and the results are plotted for only seven flaps. Flap 1 is the OWSC located in the extreme right of the array shown in figure 2.7 while flap 7 is the central flap.

flaps are oriented along the same $x'$ coordinate. As first described by [55], a near resonant behavior is observed in this case which is similar to the resonance of an infinite array of inline OWSCs [58] or a single OWSC in an open channel [56] (see figure 2.8a). At the near resonant period, the performance of every individual OWSC is higher than in the isolated case and $\eta^{mod}$ has a peak for all the flaps. However such a behavior is also accompanied by destructive influences at higher periods. Amongst all the flaps, the outermost OWSC has a slightly distinguishable behavior from the others. This is due to the fact that while

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all the other OWSCs have neighboring flaps on both sides which generate the maximum influence, the outermost flap only experiences the hydrodynamic influence of the converters located on one side. Let us now consider a case of oblique wave incidence on inline OWSCs. As expected, the behavior of the wave farm is no longer symmetrical about its innermost flap. Figures 2.9(a) and (b) show the $q^{\text{mod}}$ of all the 13 flaps when $\psi = 30^\circ$. A similar near resonant behavior is observed in this case as well. However, the strongest near-resonant behavior occurs for flap 1 and the magnitude of the peaks reduces as one moves towards the other end of the array, with flap 13 showing a distinctively different behavior.

Fig. 2.9 Variation of $q^{\text{mod}}$ of 13 flaps in an inline configuration for oblique wave incidence $\psi = 30^\circ$: (a) Flap 1 to flap 7; (b) Flap 7 to flap 13.

S1: In such a configuration, the OWSCs are placed in a zigzag manner with the array comprising of two rows of devices. The flaps located in the same row have similar hydrodynamic behavior, as seen in figure 2.8(b). Flaps 3, 5 & 7, which are positioned in the front, have almost the same $q^{\text{mod}}$ variation and similar are the behaviors of flaps 2, 4 & 6. However, the performance characteristics of the flaps in the two rows are in striking contrast, with the maxima in $q^{\text{mod}}$ of the OWSCs in the front row corresponding to the minima of the OWSCs in the back row and vice versa. This happens since a flap in the front row experiences the maximum constructive interaction, as already anticipated in the cluster analysis of §2.4. Figure 2.10 plots the response amplitude operator (RAO) of the free surface elevation ($|\zeta'/A'|$) for an incident wave period of 5 seconds in the region surrounding the wave farm. There is hardly any noticeable change in the wave field in front

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2.4 Results

Fig. 2.10 Response amplitude operator (RAO) of the free surface elevation in a wave farm of 13 flaps in the staggered S1 configuration shown in figure 2.7 for an incident wave period $T' = 5\text{sec}$.

of the array, but immediately behind the first row there is a reduction in the wave elevation, meaning a less energetic wave field available for extraction by the second row. At the back of the second row, the energy reduction is stronger, but limited in extent to the first 15 meters. At further distance, the reduction is as significant as in the back of the first row.

S2: Here the devices are again placed in a zigzag distribution but now there are three rows in this configuration. Flaps 3 and 7 are located in front of the array and experience a beneficial influence due to constructive interactions leading to relatively high values of $q^{\text{mod}}$ (see figure 2.8c). One can again note the similarity in the behavior of the OWSCs in the second row (flaps 2, 4 and 6). Finally, flap 5, the only non-external flap to be located on the last row, has a predominantly negative $q^{\text{mod}}$ factor. The behavior is indeed similar to that obtained from the corresponding configurations of the three-flap cluster of §2.42.4.2.

S3: The layout of this array resembles an inverted 'V' shape, pointing away from the coast. Figure 2.8(d) shows the $q^{\text{mod}}$ variation of the flaps in such a configuration. The most striking behavior is of the foremost WEC (flap 7). Indeed, one could have expected it to have a positive $q^{\text{mod}}$ factor based on the behavior observed in the cluster model of §2.42.4.2.
2.4 Results

However, the magnification of the $q^{\text{mod}}$ factor in this case is further enhanced by what we believe to be a strong focussing effect. In the S3 configuration (see again figure 2.7) all the flaps behind the central one reflect back some amount of incident wave energy. As a consequence, more energy is available for extraction by the foremost device (flap 7), resulting in the peak of the relevant $q^{\text{mod}}$ in figure 2.8(d). A further insight into such dynamics is offered by figure 2.12, which shows the excitation torque on the flaps in the S3 configuration. The variation of the excitation torque is similar to that of the $q^{\text{mod}}$ factor of figure 2.8(d) and one can notice a sharp increase in $|F'|$ for flap 7 at the same peak period ($T' \sim 7.2s$). Such a behavior again corroborates the well known fact that the dynamics of the OWSCs like Oyster is primarily torque-driven (see [56, 57, 60]).

![Figure 2.11 Variation of the performance parameter $q$ for the five different layouts of 13 flaps as shown in figure 2.7 and an inline layout of a wave farm comprising of 40 flaps.](image)

**S4:** Here again the outermost flaps, which are located in the front, record the highest peak in the $q^{\text{mod}}$ factor (see figure 2.8e). However, although the configuration mirrors to the previous one, there is no such equivalent constructive focussing effect on the central flap (flap 7).

An overview of the general behaviors of all the systems is provided in figure 2.11. Here the variation of the global performance parameter $q$ (2.30) is plotted against the period of
2.4 Results

Fig. 2.12 Behavior of the excitation torque $|F'|$ on individual flaps versus incident wave period for a 13-flap array in the staggered S3 configuration.

the incident wave. Overall, the strongest constructive interaction is achieved in the inline system, while the staggered systems S1 and S2 show the least constructive interference between the flaps, mainly due to the poor performance of the back row because of the sheltering effect of the front row (e.g. see figure 2.10). This confirms the earlier findings of [55] for a smaller system. Finally, the configurations S3 and S4, for which the net power output is the same, show a smaller peak than the inline configuration, but an overall better performance according to the $q$ indicator. It is worth mentioning that our analysis is based purely on the hydrodynamic performance of the system. Other aspects (environmental impact, site bathymetry, etc.) could of course orient the designer towards a less effective configuration from the hydrodynamic viewpoint. Nevertheless, such a hydrodynamic analysis is a first step towards the effective design of such a costly system.
2.4 Results

2.4.4 40 flaps inline

The proposed 40MW wave farm off the north-west coast of Lewis, Scotland is expected to have a deployment of around 40 to 50 Oyster devices on an approximate 3.2 kilometre stretch of coast. In order to check the reproducibility of the results obtained from the small wave farm cases in such large configurations, a simulation of 40 OWSCs in a simple inline configuration is performed. The general geometry is considered to be the same as in the 13 flap configuration. In figure 2.11 the variation of the \( q \) factor for the 40 flap configuration is plotted. The behavior is indeed similar to that of the 13 flap inline case and the near-resonant behavior is again confirmed with a slightly larger spike tending towards that of an infinite number of OWSCs (see again [55] and [58]). It can be reasonably inferred that the general behavior in other configurations would be similar to that in the smaller wave farm case with sharper spikes and troughs.

2.4.5 Random Seas

A random sea analysis is performed in this section for the most probable sea-state at the Isle of Lewis with the significant wave period \( T_{1/3} = 8.24 \text{s} \) and the significant wave height \( H_{1/3} = 1 \text{m} \) (obtained through private communication with Aquamarine Power Limited) with normal wave incidence. The standard Bretschneider spectrum described in [29], is utilized to model the wave climate at the location. Computations are performed for five possible layouts of a 13 OWSC wave farm as shown in figure 2.7. Table 2.2 shows the \( q \)-factor for the various array configurations. The \( q \)-factor for all the layouts are found to be less than 1 which indicates that the effect of the interactions on the net power output from the array in random seas, are destructive in nature for the spectrum considered. The inline configuration which records the highest peak in the \( q \)-factor in monochromatic seas (see figure 2.11) has the lowest values in irregular waves, while \( S1 \), the least staggered configuration of all achieves the best performance in random seas. Note, \( S1 \) has the smallest spikes in \( q \), but at the same time, the destructive influences on its cumulative performance are the least as well (see again figure 2.11). The other staggered layouts \( S2, S3 \) and \( S4 \) have lower values of \( q \) than that of \( S1 \). As already noted earlier, the net power output from the \( S3 \) and \( S4 \) configurations are the same, which consequently results in identical values of
### 2.4 Results

the $q$-factor in random seas as well. It is worth recognising that the performance in random

Table 2.2 $q$-factor for various layouts of a 13 OWSC wave farm in random seas

<table>
<thead>
<tr>
<th>Layout</th>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inline</td>
<td>0.908</td>
</tr>
<tr>
<td>S1</td>
<td>0.982</td>
</tr>
<tr>
<td>S2</td>
<td>0.972</td>
</tr>
<tr>
<td>S3</td>
<td>0.968</td>
</tr>
<tr>
<td>S4</td>
<td>0.968</td>
</tr>
</tbody>
</table>

seas is strongly dependent on the description of the incident wave spectrum. For the sea-
state considered in this analysis, its peak period $T_p = 8.65\text{s}$ (Note: $T_p = 1.05T_{1/3}$, see [29]),
didn’t coincide with any peak of the interaction factor $q$. Although it is not a thumb-rule,
such a co-occurrence can help at attaining $q$-factors greater than 1. For example, a $q$-factor
of 1.027 is observed for the inline layout in a sea-state with $T_p = 5.7\text{s}$ and $H_{1/3} = 1\text{m}$, which
concurred with the peak in the interaction factor as well. In general, the peak period of the
spectrum can vary significantly throughout the year because of the seasonal variations and
therefore for practical purposes, $S1$ would be the ideal layout.

#### 2.4.6 Two flaps back to back

Two flaps with their centers along the same $y'$ coordinate are studied here (see figure 2.13).
It is expected that such a configuration would result in strong hydrodynamic interaction
between the two devices. [77] were the first to analyze the behavior of two top hinged
independently oscillating rolling plates in deep water as a WEC. The novel concept mo-
tivated a few other studies [72], where one of the major drawbacks of such a system was
identified to be its strong directional sensitivity to wave incidence and the concept was
thereafter shelved. Surprisingly, the idea was not pursued in shallow waters where the
waves are predominantly directional. In this study we are going to explore whether it is
wise to place two OWSCs back to back.

Figure 2.14 plots the behavior of the excitation torque ($|F'|$), radiation damping ($\nu'$),
added inertia ($\mu'$) and the performance indicator $q^{mod}$ versus the non-dimensional parameter $kd$ for $d' = 50\text{m}$. The qualitative variation of the hydrodynamic parameters of the
2.4 Results

Fig. 2.13 Geometry of the physical model of two back to back OWSCs: (a) side view; (b) top view. The distance of separation between the flaps is denoted by $d'$ in this case.

The front flap resembles that observed in the case of an OWSC in front of a straight coast [71]. In the latter, periodic occurrences of extremes are observed in the variation of the excitation torque, with the minima occurring at integer values of $kd/\pi$. Also, sharp spikes are observed in the variation of the radiation parameters at values a little less than $kd = (m + 1/2)\pi, m = 1, 2, \cdots$. In the case of the two flaps analyzed here, the hydrodynamic behavior of the front OWSC is similar to that of a flap in front of a straight coast, with however, reduced peaks and a shift where the extremes occur.

As far as the performance of the devices is concerned, the average value of $q^{mod}$ of flap 1 is higher than that of flap 2 (see figure 2.14(b)). The constructive interference effects on flap 1 are very strong at $kd \approx 5$ where $q^{mod}$ almost reaches a value of 0.5. Flap 2 (back OWSC) always captures less power than a single isolated OWSC, which means that the interaction effects are always destructive on its performance. The primary reason for such a behavior is that the back flap lies in the hind side of the front flap where the wave energy is reduced. Figure 2.14 shows the variation of $q$ for various values of the distance $d'$. For $d' = 25\text{m}$, the destructive interaction effects are quite significant and $q \approx 0.5$ at an incident wave period of about 6 seconds. This means that the total power captured by the two devices combined at that frequency is equivalent to the energy extracted by an isolated single device. As the distance $d'$ is increased, the occurrence of the humps in the variation of $q$ increases but the magnitude of such deviations reduces as well. The most important thing to note is that the constructive interference effects are much weaker compared to destructive influences and
2.4 Results

Fig. 2.14 Behavior for the case of two back to back OWSCs shown in figure 2.13 versus the non dimensional parameter $kd$ for $d' = 50m$. (a) Magnitude of the excitation torque ($|F'|$); (b) Radiation damping ($\nu'$); (c) $q_{mod}$; (d) Added inertia ($\mu'$).

on an average the two OWSCs in such a configuration capture less power than two isolated WECs.

2.4.7 Two Wave Farms

In reality, an ideal wave energy site may encourage the deployment of two consecutive wave farms for energy harvesting because of high wave energy density. It is important to understand the dynamics of the system in such cases especially with one of the wave farm lying in the energy shadow of the other. A simplified case of two inline wave farm configurations, each comprising of 13 flaps is considered in normal wave incidence (see figure 2.16). The analysis is performed in constant water depth to understand the dominant interaction effects between the systems, although in reality, variations in depth are expected to modify the behavior slightly. The term $q_{farm}$ is utilised to understand the effect of the
Fig. 2.15 Variation of $q$ versus the incident wave period for the case of two back to back OWSCs with various values of the separation distance $d'$. 

interaction on each of the wave farms and is defined as

$$q_{farm} = \frac{P_{farm}}{P_{farmisolated}},$$

where $P_{farm}$ is the total power captured by a particular wave farm while $P_{farmisolated}$ is that by the same farm in an isolated environment. $q_{farm} > 1$ would mean that the presence of the other farm has a net beneficial influence on power absorption characteristics of the particular wave farm considered, while $q_{farm} < 1$ indicates otherwise. Figure 2.17 plots the variation of $q_{farm}$ versus the incident wave period of the two wave farms for various distances of separation. The oscillatory behavior of the $q_{farm}$ factor is similar to that of the $q$ factor observed in the two back-to-back OWSCs case (see figure 2.15), with a higher number of such oscillations occurring for larger distances of separation. For the range of distances considered, the $q_{farm}$ factor of wave farm 1 is always less than 1 which indicates
that such configurations will tend to have a detrimental influence on the farm located nearer to the shore. However, a steady upward shift in the $q_{farm}$ factor of wave farm 1 is observed as the distance is increased which can be explained due to the energy recovery in the rear side of wave farm 2 (see [4]). The rate of energy recovery is in fact quite slow and even for a distance of 2000 m, the $q_{farm}$ factor is still below 1. On the other hand, wave farm 2 has both detrimental and favourable interference effects. However, the magnitude of the oscillations in its $q_{farm}$ factor is much higher than that in wave farm 1. It is interesting that the bandwidths of the oscillations are almost the same for the distances considered.

Fig. 2.16 Layout of the two inline wave farm configurations separated by a distance $d'_{farm}$.

### 2.5 Conclusion

A mathematical model based on the linear potential flow theory has been used to analyze the hydrodynamic interaction between multiple flap-type WECs in a wave farm. The semi-analytical model can efficiently solve a reasonably sized OWSC wave farm which otherwise is difficult to evaluate with a complete numerical approach. It is shown that the dynamics of each individual OWSC in the wave farms considered in the analysis strongly depends on its location in the farm and the wave frequency. As the distance between the flaps increases, the mutual hydrodynamic interaction between them reduces and the behavior of the converters tends towards that of an isolated device. However, from an economic perspective, one would want to maximise the number of devices at a particular wave farm.
Fig. 2.17 Variation of $q_{farm}$ versus incident wave period for (a) wave farm 1 and (b) wave farm 2.

location to extract more power. This is important for nearshore devices like OWSCs as the space would be strictly limited unlike for offshore converters.

Wave absorption by an array of 13 OWSCs is studied for some of its possible layouts. For an inline configuration with normal incidence, a near resonant phenomenon is observed which becomes stronger as the number of flaps is increased. However, for oblique wave incidence there is a shift in the frequency of occurrence of this phenomenon with a slight increase in the resonant bandwidth associated with it. In a particular configuration of the large array (S3), a large enhancement in the performance of the front-most flap is observed. Such a behavior is attributed to a sharp increase in the excitation torque due to the focussing of waves by the other devices in the array. In general, the converters which are located in front of the array experience a noticeable positive interaction effect leading to a gain in their power capture. Such a favourable behavior in the performance of the foremost devices of the array is also reported in the recent study of [7]. An irregular wave analysis for the most probable sea at the Isle of Lewis reveals that S1 - the least staggered configuration, is a suitable layout for an OWSC wave farm. In the case of two back to back OWSCs located close to each other, the effect on the performance of the back flap is found to be detrimental across its entire operating range, while the front OWSC experiences regions of both positive and negative influences. And when two such flaps are considered as one system, the destructive interference effects are found to be more important than the constructive influences. Therefore such a system of two OWSCs is not recommended in reality. Also it
2.5 Conclusion

is shown that a system of two consecutive wave farms has in general a negative interaction effect on the net performance of the wave farm located downstream.

In a practical wave farm design however, the layout of an array configuration could be constrained by bathymetry variations which would affect the optimization process. Although no particular layout could be suggested which would lead to a gain in net wave farm energy output across the entire operating range of the device, since the constructive interference effects are usually accompanied by destructive influences as well, the study can help understand what sort of variability in the performance of individual OWSCs one can expect.
3 Optimization of wave energy converter arrays using machine learning

3.1 Introduction

Arrays of WECs have been extensively studied in the literature starting with the pioneering work of [9]. Notably, Budal [9] used the point-absorber approximation which assumes the dimensions of the WECs are much smaller than the wavelength of the incident wave field and therefore the diffracted wave field can be considered to be negligible. Since then, advances in numerical and analytical techniques (see e.g. [34, 42, 43, 75]) in the analysis of wave-structure interactions have enabled the investigation of the behavior of arrays of WECs of arbitrary shapes, taking into account the effects of both the diffracted and the...
3.1 Introduction

Fig. 3.1 Geometry of the physical system (a) side view of one of the WECs (b) top view of a wavefarm comprising of $M$ WECs.

Numerical boundary element codes (e.g. NEMOH and WAMIT developed at Ecole Centrale de Nantes and MIT, respectively) are widely used to understand the effects of the hydrodynamic interactions and to quantify the performance of the WECs and the array as a whole. However, such methodologies involve increased computational cost and resources. The general objective is to understand the effect of the interactions on the performance of the WECs and to determine layouts which would maximize the power captured from the whole system. In order to quantify the effects of the interactions on the performance of the array, Budal [9] defined the $q$ factor which is the ratio of the net power captured (ideally the maximum possible) by the array to the power absorbed by the same number of WECs in isolation. Budal [9] observed that it is possible to have $q$ factors much larger than 1 for particular wave frequencies. But the peaks in the $q$ factor are accompanied with wide troughs in its variation, and since a real ocean is polychromatic, Budal suggested that a properly designed array configuration should minimize the effect of the destructive influences. However, the identification of optimal layouts for a particular wave-climate is still a big challenge. The complexity of the optimization problem is manifold. The number of WECs in an unknown array can be arbitrary and so is the number of variables. And every single evaluation of the numerical/semi-analytical models has a computational cost which increases with the size of the array. In addition, there can be various constraints to such a problem (e.g. bathymetry variations for nearshore WECs). Child and Venugopal [12] presented a parabolic intersection method and a genetic algorithm to arrive at the optimal layouts of arrays. While the parabolic intersection approach uses simple calculations for a quick estimate of the array layouts, the genetic algorithm requires many evaluations of a semi-analytical (or numerical) method which is computationally expensive. Such a direct
3.1 Introduction

Fig. 3.2 The methodology considers only the interaction of the WECs with their neighbouring ones. As such the two edge WECs located at the two extremes of the wavefarm can be modelled with a simplified 2 WEC cluster configuration while those lying in the interior with a 3 WEC cluster arrangement.

Application of sequential optimization techniques also implies that if the number of WECs is changed, a new set of evaluations needs to be computed and analysed. In this work we propose a fast and scalable approach to address the challenge of determining the best layout for any number of WECs and arbitrary bathymetry constraints. We first use an active learning strategy to train a statistical emulator of the individual WECs inside the array. We then predict the performance of the whole wavefarm by evaluating only the quasi-instantaneous emulator. The optimization of the layout under the various constraints is then performed on the predicted performances with a genetic algorithm designed specifically for this task.

The analysis in this work is performed on a well known WEC- the Oscillating Wave Surge Converter. A number of studies are now available which have looked at various aspects of the device (see e.g. [26, 27, 56, 69, 71, 80, 81]). Most recently, a mathematical model was developed in [70] to analyse the behavior of the OWSCs in a wave farm and the simulations in this work will use this model. A Bretschneider spectrum is used (see [29]) to model the wave climate of the most probable sea state at the Isle of Lewis in Scotland, a planned site for wave farms, with a significant wave height of 1m and a significant wave period of 8.24s.

A wavefarm comprising of $M$ WECs (see figure 3.1) is considered, and so the $2M$ variables that needs to be optimized are $x'_1, y'_1, \ldots, x'_M, y'_M$, where $x'$ and $y'$ are the horizontal coordinates parallel and perpendicular to the incident wave direction, respectively. The overall performance of the array is decomposed as the sum of the powers captured by each individual WEC. In a realistic scenario, the seas are highly irregular, and the interaction effects on a particular WEC due to WECs located away from it are largely diminished.
3.1 Introduction

Fig. 3.3 The general layout of the 3-WEC configuration. The position of the central WEC is fixed while the positions of WEC-1 and 3 vary such that the left side edge of WEC-1 lies inside the striped square box and the right edge of WEC-3 inside the dotted box.

It is reasonable to assume that individual WECs in an array are predominantly influenced by those located very close to them. Our approach targets the approximated performance where we simplify the model of the individual WECs in order to take into account only a limited number of interactions. A WEC located inside the array is strongly influenced by the two WECs which are nearest to it, i.e. one on each side (see figure 3.2). To model the behavior of such a WEC, we consider a 3-WEC cluster and focus on the behavior of the central WEC. On the other hand, the edge WECs are modelled using a 2-WEC cluster. The effect of the interactions on the performance of the individual WECs can be quantified using the $q_{\text{mod}}$ factor (see [4]) which is defined as

$$q_{\text{mod},i} = \frac{P_i - P_{\text{isolated}}}{P_{\text{isolated}}}$$

where $P_i$ is the power captured by the $i$-th WEC in the array, while $P_{\text{isolated}}$ is the power captured by the same WEC in isolation. The $q$ factor (ratio of the total power captured from the array to that by the same number of WECs in isolation) and $q_{\text{mod}}$ factor can be related as

$$q = 1 + \frac{1}{M} \sum_{i=1}^{M} q_{\text{mod},i}.$$ 

We denote by $\tilde{q}_{\text{mod}}$ and $\tilde{q}$, the approximate form of the $q_{\text{mod}}$ and $q$ factor, considering only
3.1 Introduction

Fig. 3.4 The WEC-2 (in red) models the behavior of the edge WEC of the array. The position of WEC-1 (in blue) is varied such that its left edge is located within the striped square box.

the effects due to the local interactions. The additive model for $\tilde{q}$ can be expressed as

$$\tilde{q}(x'_1, y'_1, \ldots, x'_M, y'_M) = 1 + \frac{1}{M} \left( \tilde{q}_2^{\text{mod}}(x'_2 - x'_1, y'_2 - y'_1) + \tilde{q}_3^{\text{mod}}(x'_M - x'_{M-1}, y'_M - y'_{M-1}) + \sum_{i=2}^{M-1} \tilde{q}_3^{\text{mod}}(x'_i - x'_{i-1}, y'_i - y'_{i-1}, x'_{i+1} - x'_i, y'_{i+1} - y'_i) \right)$$

where $\tilde{q}_2^{\text{mod}}$ and $\tilde{q}_3^{\text{mod}}$ are respectively the models for the 2-WEC clusters (see figure 3.4) and the 3-WEC clusters (see figure 3.3). We predict the interaction factor of the individual WECs using two statistical emulators, one for $\tilde{q}_3^{\text{mod}}$ and one for $\tilde{q}_2^{\text{mod}}$. Our prediction function is based on Gaussian process regression trained with an active learning strategy, the Gaussian Process Upper Confidence Bound with Pure Exploration algorithm (GP-UCB-PE) [15]. Since the final goal of the work is to determine optimal layouts of an array, we are more interested in layouts which result in good performance of the individual WECs. The active learning strategy determines sequentially which configurations of the 3-WEC and 2-WEC clusters should be evaluated aiming to improve the predictions of the optimal clusters using the least number of evaluations possible. The algorithm performs a trade-off between exploitation and exploration looking for the maximum while exploring clusters.
3.2 Function Estimation through Gaussian Process Approach

with high uncertainty.

In our optimization methodology, we incorporate some constraints which are relevant to the problem. The OWSCs are nearshore WECs with depth specific designs, and as such bathymetry will play a significant role in deciding their locations. We consider an upper and a lower bound on the bathymetry contours, within which the placement of the center of the OWSCs is restricted. Although the mathematical model (for simulations) is based on a constant water depth assumption, the bathymetry constraints in the optimization problem take account of the spatial limitations imposed by the depth variations at real locations, in the placement of the WECs. In a practical situation, proper utilisation of space is also an important consideration and to account for that an upper bound in the distance separating the centers of the first and the last WEC in the \( y' \) direction (see figure 3.1) is fixed at a particular value.

The methodology of the statistical emulator and the machine learning algorithm will be illustrated in the following sections. Later in §3.4, case studies are performed on the optimization of a wavefarm comprising of 40 OWSCs.

### 3.2 Function Estimation through Gaussian Process Approach

We describe here how we predict the unknown values of a function \( f : \mathcal{X} \rightarrow \mathbb{R} \) given a data set of evaluations of \( f \). This approach will be used for predicting \( \tilde{q}_3^{mod} \) the interaction factor of a WEC with two neighbours (where \( \mathcal{X} \subset \mathbb{R}^4 \), the relative coordinates of the neighbours inside two 100m boxes as shown in figure 3.3) and \( \tilde{q}_2^{mod} \) the interaction factor of a WEC with a single neighbour (where \( \mathcal{X} \subset \mathbb{R}^2 \) as shown in figure 3.4).

#### 3.2.1 Probabilistic Setup

We assume that the unknown function \( f \) is a random sample from a Gaussian process (GP), which can be viewed as a general form of the multi-normal distribution. GP is a natural choice for our problem based on the observation that there is high correlation between layouts which are strongly similar. Let \( \{(x_1,y_1),\ldots,(x_\lambda,y_\lambda)\} \) be a training set of \( \lambda \) noisy
3.2 Function Estimation through Gaussian Process Approach

observations, where \( y_t = f(x_t) + \varepsilon_t \) and \( \mathcal{X} \) is an arbitrary vector space in \( \mathbb{R}^d \). We consider that the variables \( \varepsilon_t \) are distributed according to independent Gaussians \( \mathcal{N}(0, \sigma^2) \), they represent the numerical noise of the semi-analytical model. The GP is defined in terms of its mean \( m \) and covariance function \( k \) [54] as

\[
f \sim \mathcal{GP}(m, k)
\]

(3.1)

with \( m: \mathcal{X} \rightarrow \mathbb{R} \) and \( k: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R} \). Note in (3.1) the symbol \( \sim \) means ‘distributed according to’. For any set of input vectors \( x_1, x_2, \ldots, x_\lambda \) in \( \mathcal{X} \), the value of the function is given by a multivariate Gaussian random variable

\[
(f(x_1), f(x_2), \ldots, f(x_\lambda)) \sim \mathcal{N}(\mu, K)
\]

where the mean vector \( \mu \) and the covariance matrix \( K \) are expressed as

\[
\mu[x_i] = m(x_i)
\]

and

\[
K[x_i, x_j] = k(x_i, x_j).
\]

The widely used squared exponential kernel function for the covariance function can be written as

\[
k(x_1, x_2) = \exp\left(- (x_1 - x_2)^T D (x_1 - x_2) \right)
\]

where \( D \) is a diagonal matrix and the inverse square root of the diagonal terms defines the characteristic length scale. In our analysis, we use a modified version of squared exponential kernel function so as to incorporate the symmetrical nature of the problem with respect to the \( x' \) axis. As a matter of fact the behavior of the particular WEC which is modelled (e.g. WEC-2 in figure 3.3) doesn’t change if the \( y' \) coordinates of the WECs on its two sides are interchanged. The hyperparameters of the setting, which are the diagonal terms
of the matrix $D$ and the noise variance $\sigma^2$, are chosen so as to maximize the marginal likelihood of our training data set (see [54]). The marginal likelihood $\mathcal{L}(Y)$ is the integral of the likelihood times the prior

$$
\mathcal{L}(Y) = \int p(Y|F)p(F)dF
$$

where $Y[i] = y_i$, $F[i] = f(x_i)$ and $Y|F$ is the conditional $Y$ given $F$. With the prior belief that our function is sampled from a centered GP we obtain

$$
\log \mathcal{L}(Y) = -\frac{1}{2}Y^T(K + \sigma^2 I)^{-1}Y - \frac{1}{2} \log |K + \sigma^2 I| - \frac{\lambda}{2} \log 2\pi
$$

where $I$ is the identity matrix. Once the values of the hyperparameters are optimized with respect to $\mathcal{L}(Y)$, we can compute a prediction and an uncertainty for every input vector $x \in \mathcal{X}$ in the following way. Conditioned on the observations $Y$, we have that the distribution of $f$ is a GP such that $f(x)|Y \sim \mathcal{N}(\hat{\mu}(x), \hat{\sigma}^2(x))$, where

$$
\hat{\mu}(x) = k(X, x)^T[K + \sigma^2 I]^{-1}Y \quad (3.2)
$$

and

$$
\hat{\sigma}^2(x) = k(x, x) - k(X, x)^T[K + \sigma^2 I]^{-1}k(X, x) \quad (3.3)
$$

are the prediction and uncertainty respectively. Note in (3.2) and (3.3), $X$ indicates the training point and $k(X, x)$ is the vector of size $\lambda$ of the covariance terms between $X$ and $x$.

### 3.2.2 Search of Design Points

The larger the data set of observations is, the better the predictions are. But each training point requires to run a computationally expensive semi-analytical method. Since we are more interested in configurations leading to good performance, we use a sequential strategy to select the most informative training points, the Gaussian Process Upper Confidence Bound with Pure Exploration algorithm (GP-UCB-PE) [15]. This algorithm proceeds in
3.2 Function Estimation through Gaussian Process Approach

Fig. 3.5 A simple illustration of the GP-UCB-PE strategy showing the first two query points. The horizontal axis represents the $x$ coordinate while the vertical axis indicates the function value. The red continuous line is the prediction $\hat{\mu}$ while the black continuous lines bound the high confidence region. The lower confidence bound on the optimum is shown by the blue dotted line which is the maximum of the lower confidence bound. The intersection of the blue dotted line and the upper confidence bound defines the limits of our region of interest $\mathcal{R}$ which is shown in blue in the figure. The first point $x^0$ is chosen to be the maximum of the upper confidence bound. The dotted line in black shows the updated variance confidence after having selected $x^0$. The next query point $x^1$ is chosen such that it has the maximum updated uncertainty inside $\mathcal{R}$.

an iterative way so as to reduce the uncertainty of the prediction of the best configuration. It has recently been used in the estimation of maximum wave run-up of tsunami waves behind conical islands [78]. At iteration $\tau$ after having computed $\hat{\mu}_\tau$ and $\hat{\sigma}_\tau^2$, we simultaneously explore the configurations with high uncertainty and focus on the ones with high prediction. We choose a batch of $B$ unknown configurations using a tradeoff between $\hat{\mu}_\tau$ and $\hat{\sigma}_\tau^2$, we then query the semi-analytical model on those $B$ configurations to improve our prediction, and continue to iteration $\tau + 1$. The value of $B$ is arbitrarily equals to 20 in our setting, it is typically the number of runs of the semi-analytical model we can compute in parallel on a cluster of $B$ computers. The methodology of the active learning algorithm is illustrated in
3.2 Function Estimation through Gaussian Process Approach

the sequel.

The GP-UCB-PE Algorithm

We first build a high confidence interval within which the function $f$ is expected to be located with high probability as follows

$$
\hat{f}_\tau^+(x) = \hat{\mu}(x) + \beta_\tau \hat{\sigma}(x),
\hat{f}_\tau^-(x) = \hat{\mu}(x) - \beta_\tau \hat{\sigma}(x)
$$

where $\hat{f}_\tau^+$ and $\hat{f}_\tau^-$ are the upper and lower confidence bound on $f$ and the $\beta_\tau$ parameter regulates the width of the confidence region (see [15]). We then define the region $\mathcal{R}_\tau$ we are interested in, which contains the best configuration with high probability. Let $y_\tau^\circ$ be the maximum of the lower confidence bound $\hat{f}_\tau^-$

$$
y_\tau^\circ = \max_{x \in \mathcal{X}} \hat{f}_\tau^-(x)
$$

The value for $y_\tau^\circ$ describes the lower confidence bound for the optimum at each iteration and the region $\mathcal{R}_\tau$ is defined as

$$
\mathcal{R}_\tau = \left\{ x \in \mathcal{X} \mid \hat{f}_\tau^+(x) \geq y_\tau^\circ \right\}
$$

as shown in the figure 3.5. The first of the $B$ query points is chosen such that

$$
x_\tau^0 = \arg\max_{x \in \mathcal{X}} \hat{f}_\tau^+(x).
$$

Using the fact that the uncertainty does not depend on $Y$ but only on $X$, once we know $x_\tau^0$ we can update it to $\hat{\sigma}_{\tau,1}^2$ before having the associated observation. The remaining $B - 1$ points are chosen by Pure Exploration inside $\mathcal{R}_\tau$. We greedily select the configuration with highest updated uncertainty $\hat{\sigma}_{\tau,b}^2$

$$
x_\tau^b = \arg\max_{x \in \mathcal{X}} \hat{\sigma}_{\tau,b}^2(x),
$$
with \( b \) varying from 1 to \( B - 1 \). Under mild probabilistic assumptions the queries selected with this active learning strategy are shown to converge to the global optimum of the unknown function at a rate \( (\tau B)^{-\frac{1}{2}} \) (with omission of logarithmic factors).

**Experimental Design**

Since the GP-UCB-PE algorithm requires to fix the hyperparameters of the GP, we first optimize their value on an initial data set constructed following a classical experimental design approach. A simple choice would be a regular grid in which every coordinate is divided into a particular number of equally spaced intervals. But such an approach is highly inefficient because of the high computational costs associated with it, e.g. if there are \( l \) input variables, each of which is divided into \( p \) equidistant, non-overlapping intervals, then the total number of evaluations required to cover the input space is \( p^l \). In our analysis, we employ the latin hypercube sampling (LHS) in which the range of each input variable is divided into a particular number (say \( p \)) of equally probable, non-overlapping intervals. Then values are chosen randomly from each of the intervals for every variable and random combinations are made to arrive at the sample points. We then use the maximization of the minimum distance as an additional constraint in order to achieve maximal coverage of the input space. We chose 800 sample configurations by LHS to initialize the 3-WEC cluster predictor and 200 for the 2-WEC cluster.

**Stopping Criterion**

We stop the GP-UCB-PE algorithm when we know that is has discovered the best configuration with high probability, that is when the region \( \mathcal{R}_\tau \) shrinks to a single point. For the 3-WEC cluster predictor the active learning strategy ended after six iterations with \( B = 20 \), which means 120 evaluations in addition to the initial 800 from the LHS. In the case of the 2-WEC cluster predictor the algorithm ended after a single iteration with \( B = 20 \) which is explained by the highly predictable value of the \( \tilde{q}^{\text{mod}} \) factor with a single neighbouring WEC. In order to quantify the extent of the region \( \mathcal{R}_\tau \), a factor \( \xi_\tau \) is defined as the ratio of points inside the region of interest \( \mathcal{R}_\tau \) to the total number of uniform design points. Table 3.2 and 3.1 show the values of \( \max(\bar{\mu}_\tau) \) and \( \xi_\tau \) at each iteration \( \tau \) for the 3-WEC
3.3 Optimization of the Estimated Additive Model

and 2-WEC cluster, respectively. The number of uniform points was set to $10^6$ for the 3-WEC case and $10^5$ for the 2-WEC case. With each iteration, the value of $\varsigma_\tau$ decreases, signifying the reduction in the spatial extent of $\mathcal{R}_\tau$, while the maximum of the predictions of the performance factor $\tilde{q}^{mod}$ converges.

Table 3.1 3-WEC cluster iterations

<table>
<thead>
<tr>
<th>Iteration no.</th>
<th>$\varsigma_\tau$</th>
<th>$\max(\mu_\tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.057526</td>
<td>0.158511</td>
</tr>
<tr>
<td>2</td>
<td>0.008354</td>
<td>0.218310</td>
</tr>
<tr>
<td>3</td>
<td>0.001943</td>
<td>0.225993</td>
</tr>
<tr>
<td>4</td>
<td>0.000540</td>
<td>0.234827</td>
</tr>
<tr>
<td>5</td>
<td>0.000021</td>
<td>0.236200</td>
</tr>
<tr>
<td>6</td>
<td>0.000003</td>
<td>0.236036</td>
</tr>
<tr>
<td>7</td>
<td>0.000001</td>
<td>0.238663</td>
</tr>
</tbody>
</table>

Table 3.2 2-WEC cluster iterations

<table>
<thead>
<tr>
<th>Iteration no.</th>
<th>$\varsigma_\tau$</th>
<th>$\max(\mu_\tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00320</td>
<td>0.116084</td>
</tr>
<tr>
<td>2</td>
<td>0.00001</td>
<td>0.121031</td>
</tr>
</tbody>
</table>

3.3 Optimization of the Estimated Additive Model

The determination of the optimal layouts is a challenging task, due to the number of variables ($2M$), and the various constraints. Thanks to the quasi-instantaneous emulator, we are able to approximate the power captured by any layout in a fast and scalable manner. Once we trained the two statistical emulators of the performance of a WEC inside or at an edge of an array, it is easy to sum the outputs for the $M$ WECs to get a prediction $\tilde{q}$ of the
Optimization of the Estimated Additive Model

\[ \tilde{q}(x'_1, y'_1, \ldots x'_M, y'_M) = 1 + \frac{1}{M} \left( \tilde{q}^{\text{mod}}_2 (x'_2 - x'_1, y'_2 - y'_1) \right. \]
\[ + \left. \tilde{q}^{\text{mod}}_2 (x'_M - x'_{M-1}, y'_M - y'_{M-1}) \right. \]
\[ \left. + \sum_{i=2}^{M-1} \tilde{q}^{\text{mod}}_3 (x'_i - x'_{i-1}, y'_i - y'_{i-1}, x'_{i+1} - x'_i, y'_{i+1} - y'_i) \right) \]

where \( \tilde{q}^{\text{mod}}_2 \) and \( \tilde{q}^{\text{mod}}_3 \) are the predictions given by the statistical emulators. In order to optimize the array given some bathymetry constraints, we explore the space of valid layouts using a genetic algorithm [50]. The optimization routines of a genetic algorithm (GA) are inspired from the evolutionary ideas of natural selection and genetics. At each step of the iteration (called generation), a proportion of the existing population of solutions (called individuals) is selected based on their individual scores (called fitness value), to breed the next generation. In this particular problem, the fitness function is \( \tilde{q} \). The optimization was performed using the genetic algorithm solver of the MATLAB Global Optimization Toolbox. The population size of each generation of the GA is chosen to be \( 2^{11} \), since the computational time per unit evaluation ceased to improve beyond it. Two important operations in a genetic algorithm are crossover and mutation. In crossover, two parents having good performance are selected from the previous population and combined to create a child for the present population. While in the mutation, the later is obtained by making random changes to a selected parent. The initial population of the WEC layouts is selected such that the bathymetry constraints are all satisfied. In our approach, we utilize custom crossover and mutation operators such that all the constraints are kept satisfied. We illustrate our methodology for initialization, crossover and mutation in the following paragraphs.

**Selection of Initial Points** : The initial population of the WEC layouts is chosen randomly, while satisfying the required constraints at the same time. We first choose the spacings between the neighbouring WECs in the \( y' \) direction according to a uniform distribution satisfying the space constraint (total extent of the wave farm in the \( y' \) direction). Once the \( y' \) coordinates of the center of the WECs are fixed, the \( x' \) coordinates of the WECs are selected sequentially in the following way (illustrated in figure 3.6). Let us suppose the \( x' \) coordinate of the \( m \)-th WEC is already determined, we sample the \( x' \) coordinate of the
3.3 Optimization of the Estimated Additive Model

Fig. 3.6 Illustration of methodology for the selection of initial points and the mutation operation. The $y'$ coordinates of the WECs are selected randomly such that the distance between the edges of the neighbouring WECs is less than 100m. Once the $y'$ coordinates of the center of the WECs are fixed, the $x'$ coordinates are selected sequentially. The figure shows the selection of the coordinates of the $(m + 1)$-th WEC with the position of the $m$-th WEC already determined. The $x'$ coordinate must lie within the upper and lower limits of the bathymetry while also satisfying the square box constraint. The interval which satisfies both the mentioned constraints is shown in red in the figure, with the red circles indicating the limits, and the position of the WEC is selected randomly in this interval.

$m + 1$-th WEC uniformly in the region satisfying both the bathymetry constraints and the square box constraints relative to the $m$-th WEC. Here and in the following the bathymetry constraints are given by lower and upper limits of the feasible region. If it is not possible to satisfy all the constraints, we start again the operation from scratch.

**Crossover**: In crossover operation two good parent solutions from the present generation are combined to produce a child for the future generation. The $y'$ coordinates of the center of the WECs are taken to be the average of the ones of the two parents. The $x'$ coordinates are then determined in a manner, such that the center of the WECs satisfies the bathymetry constraints for all situations. The $x'$ coordinates of the WECs of the parents are first normalised with respect to the upper and lower bounds of the bathymetry. The mean of the normalised $x'$ coordinate of the WECs of the two parents is then computed
3.3 Optimization of the Estimated Additive Model

Fig. 3.7 Illustration of the crossover operation - (a) position of the $m$-th WEC of parent P1, (b) position of the $m$-th WEC of parent P2 and (c) position of the $m$-th WEC of the child. $(x_{(m,UB)}, y_{(m,UB)})$ and $(x_{(m,LB)}, y_{(m,LB)})$ are the coordinates of the upper bound and lower bound at the location of the $m$-th WEC. The $y'$ coordinate of the $m$-th WEC of the child is simply the average of the same of the parents. The $x'$ coordinate of the $m$-th WEC of the parents is first normalized with respect to the upper and lower bounds of the bathymetry, such that the normalized values of $x'_{(m,LB)}$, $x'_{(m,UB)}$ are 0 and 1, respectively. The average of the normalized $x'$ coordinate of the two parents is then computed and projected at the $y'$ coordinate $(y'_{C,m})$ of the child and un-normalized, to obtain the location of the $m$-th WEC.

and projected on to the already determined $y'$ coordinate of the child, with the bathymetry constraints being the bounds again, and this gives the position of the WECs of the child (see figure 3.7).
3.3 Optimization of the Estimated Additive Model

Fig. 3.8 In the mutation operation, \( n \) consecutive WECs (\( n=4 \)) are randomly selected and replaced by a new distribution of the same number of WECs. In this figure, four consecutive WECs, shown in blue in (a), are replaced by the new arrangement, shown in red in (b). The dashed red vertical lines, which are the edges of the WECs before and after the \( n \) consecutive WECs indicate the \( y' \) coordinate bounds within which the new configuration of the WECs is to be placed.

**Mutation** The mutation operation is done in the following way. Random alterations are made on \( n \) consecutive WECs of a particular parent, with the number \( n \) chosen from a Beta distribution. The Beta distribution is described over a finite interval and is parametrized by two positive shape parameters, which for the present problem are chosen such that the mean value of \( n \) for a 40 WEC array is 8. Given \( n \), we choose randomly \( n \) consecutive WECs from the array and replace them in a similar manner to the selection of the initial points, where we take care of the box constraints for the eventual neighbours of the \( n \) WECs (see figure 3.8).
The results and analysis are presented in three parts in this section. In the first part, the efficiency of the statistical emulator of the 3-WEC cluster and 2-WEC cluster are discussed. Then the prediction function is used to describe the variation of $\hat{q}$ for a wavefarm comprising of 40 WECs in inline and staggered configurations with uniform spacings and symmetrical arrangements. Lastly, optimizations of layouts of WECs in a wavefarm, subject to various constraints, are performed using GA.

### 3.4.1 3-WEC and 2-WEC clusters

We use the well-known 'leave-one-out’ strategy to validate the statistical model for predictions. One of the $\lambda$ training examples is left out and the statistical emulator is trained
3.4 Results and Analysis

Fig. 3.10 Prediction of the performance ($\tilde{q}^{mod}$) of a WEC for the 2-WEC configuration using 200 training examples.

using the remaining points. The left-out evaluation is then utilized for validation. The same process is repeated for all the $\lambda$ possible cases and the operation enables to quantify the accuracy of the predictions. Figures 3.9 and 3.10 show the prediction ($\tilde{q}^{mod}$) from the statistical emulator versus those obtained from the simulations of the mathematical model ($\tilde{q}^{mod}$) for the 3-WEC and 2-WEC clusters, respectively. The diagonal continuous line corresponds to perfect predictions. The circles plotted in the figure are the predictions while the vertical dotted lines are the 95% confidence intervals. For the 3-WEC cluster, there is good agreement given the small number of training examples and the complexity of the prediction problem. Almost all the actual results lie within the high-confidence interval of the predictions. The accuracy of the predictions is extremely good for values of $\tilde{q}^{mod}$ around zero. Note that in those figures the statistical model is trained using only the 800 and 200 points from the LHS for the 3-WEC and 2-WEC clusters respectively. We can remark that the prediction of the best 3-WEC is slightly under-estimated, which justifies
the need of the GP-UCB-PE algorithm to obtain better predictions for good clusters. In the case of the 2-WEC cluster, almost perfect predictions are observed with a smaller number of simulation points. This is largely due to the smaller dimension of the 2-WEC cluster problem (dimension two) than that of the 3-WEC cluster (dimension four).

3.4.2 Wavefarm of 40 WECs: Predictions

In this subsection, we analyse the variation of $\hat{q}$ for two simple configurations of 40 WEC arrays in the inline and staggered configurations. In both cases, we consider uniform spacing and symmetrical arrangement of the WECs for simplicity, so that the output can be easily quantified in terms of the input variables which are the distances of separation between the neighbouring WECs in the $x'$ and $y'$ directions (see figure 3.11).

**Inline**

This case corresponds to the situation in which the pitching axes of all the WECs are coaligned i.e. there is no separation in the $x'$ direction. The only variable is the lateral distance of separation in the $y'$ direction - $y'_d$. In figure 3.12, the prediction $\hat{q}$ is plotted as function of the lateral spacing between the neighbouring WECs. There is clear maximum of $\hat{q}$ which occurs at a separation distance of 76.5m between the centers of the neighbouring WECs for the spectrum considered in this study. The inline system of a finite array of OWSCs is well known to result in a near-resonant phenomenon [55]. In fact resonance occurs for an infinite array of OWSCs or an OWSC in a channel at/near cut-off frequencies of the symmetric transverse sloshing modes in a channel. An extensive literature now exists on the inline system and the readers are referred to them (see [56],[58],[55], [59]). In [70],
3.4 Results and Analysis

![Graph showing variation of \( \hat{q} \) versus the distance of separation between the centers of the WECs for an inline configuration of 40 WECs with uniform spacings.](image)

Fig. 3.12 Variation of \( \hat{q} \) versus the distance of separation between the centers of the WECs for an inline configuration of 40 WECs with uniform spacings.

A single simulation of a 40 OWSC inline wave farm was performed, with a separation of 20m (\( y'_d = 46 \)m) between the edges of the neighbouring OWSCs, and the layout resulted in a \( q \) factor of 0.9. Using the prediction algorithm presented in this paper, a \( \hat{q} \) factor of 0.89 is observed with the same separation of the inline case.

**Staggered**

A staggered array has two rows of OWSCs. Here again we consider uniform spacing between the neighbouring devices. However, there are now two parameters instead of one - \( x'_d \) and \( y'_d \). The variation of \( \hat{q} \) with respect to the two spacing parameters is plotted in figure 3.13. For small distances of separation, the cumulative effect of the interactions on the performance of the array is mostly destructive. The highest value of \( \hat{q} \) is not much different from that of the inline case. In fact in figure 3.13, the \( x'_d = 0 \)m line corresponds to the inline configuration. However, by staggering the array configurations, there is now a wider possible range of layouts by which the maximum of \( \hat{q} \) can be achieved.
3.4 Results and Analysis

3.4.3 Optimal Layouts

The optimization of the layouts of a wavefarm comprising of 40 WECs is considered in this work. In order to understand the general arrangement of the WECs and to demonstrate the robustness of the presented methodology, we consider five different arbitrary bathymetry constraints which take account of the limitations in the placement of the WECs due to depth variations. The center of the WECs must be located within the upper and lower bounds of the bathymetry constraints. The upper bound for the spatial extent of the wavefarm in the \( y' \) direction (perpendicular to the direction of incident waves) is set at 2500m.

**Genetic Algorithm**

Figure 3.14 shows five layouts of wavefarms of 40 WECs with different bathymetry constraints, obtained after \( 100(2M) = 8000 \) generations, that is over 16 million predictions. The optimum value of \( \tilde{q} \) obtained for all the cases is similar. The layouts are well distributed along the \( y' \) direction, so that the full spatial extent is properly used. Even in the
3.4 Results and Analysis

![Graphs showing bathymetry and optimized wavefarm configurations](image)

The optimal configuration of a wavefarm comprising of 40 OWSCs subject to various constraints, obtained with the presented methodology using genetic algorithm. The OWSCs are shown in red. The blue and the pink coloured arbitrary curves indicate the upper and lower bathymetry constraints, respectively, and the centers of the WECs must lie between them. In addition, the total extent of the wavefarm in the y’ direction must be less than 2500m. The arrow from the top indicates the direction of incident waves.

In the case of the perfect inline and staggered layouts (see figure 3.12 and 3.13), we observe that close spacings lead to strong destructive influence on the overall performance of the array. This suggests that clustering of devices should be avoided while planning wavefarms and can be considered as a guideline for designers. Also note that $\hat{q}$ for the optimized layouts
of the five bathymetry configurations is marginally less than the maximum $\hat{q}$ for the inline and staggered cases (maximum of $\hat{q} \sim 1.03$, see figure 3.12 and 3.13). This is due to the constraints imposed by the bathymetry contours and the total spatial extent of the array in the $y'$ direction, which restricts the selection of layouts having higher values of $\hat{q}$. In general, the WECs are oriented in arrangements which can be considered as a combination of inline and slightly staggered layouts, while following the bathymetry contours.

Table 3.3 Comparison of $\hat{q}$ (predictions) and $q$ (simulations) for the optimal configurations identified by the presented methodology. The $q$ factors are obtained from the simulations of the mathematical model which considers the interactions between all the WECs, while $\hat{q}$ are the predictions using the simplified interaction approach.

<table>
<thead>
<tr>
<th>Bathymetry no.</th>
<th>$\hat{q}$</th>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.021</td>
<td>1.005</td>
</tr>
<tr>
<td>2</td>
<td>1.016</td>
<td>1.006</td>
</tr>
<tr>
<td>3</td>
<td>1.012</td>
<td>1.006</td>
</tr>
<tr>
<td>4</td>
<td>1.014</td>
<td>1.003</td>
</tr>
<tr>
<td>5</td>
<td>1.02</td>
<td>1.004</td>
</tr>
</tbody>
</table>

To check the accuracy of the results, simulations of the identified optimal arrangement of WECs are performed using the mathematical model of [70], where the interactions between all the WECs are accounted for. Table 3.3 shows the comparison of $\hat{q}$ (predictions) and $q$ (simulations) for the optimal configurations identified by the GA. The good agreement between $\hat{q}$ and $q$ demonstrates the robustness of the methodology presented in this paper, despite the use of a simplified interaction approach.

**Monte Carlo Optimization**

In order to check the effectiveness of the proposed optimization methodology using GA, the same optimization is also performed using a Monte Carlo type method. In this approach, $\hat{q}$ is evaluated for a particular number of randomly selected arrangement of the WECs. The methodology to obtain the random distribution of the WECs is the same as that used for the selection of initial points and mutation operation of the GA. The layout corresponding to the maximum value of $\hat{q}$ is considered to be the optimal arrangements of the WECs. Over 16 million evaluations (same number as that required with the GA approach) of the
3.4 Results and Analysis

Fig. 3.15 The optimal configurations obtained using a Monte Carlo type approach, with the number of evaluations of the prediction function being the same as that required using GA. The maximum values of $\hat{q}$ are less than that obtained using GA for all the five cases.

prediction function were used for each case of the bathymetry. Figure 3.15 plots the layout of the optimal configurations obtained using the Monte Carlo approach. The maximum of $\hat{q}$, corresponding to the optimal configurations was always less than those obtained using the GA. Notably, the arrangement of the WECs is much more arbitrary than that of the latter (see figure 3.14).
3.5 Conclusion

A methodology for the prediction and optimization of layouts of WECs in a wave farm is presented using an innovative active learning approach. The predictions are performed using a Gaussian process regression method. Such a strategy is relevant when the performance of similar layouts of WECs are strongly correlated. The problem is simplified into an additive model with local interaction using 3-WEC and 2-WEC clusters. The evaluations of the statistical emulators are quasi-instantaneous and so once the hyper-parameters of the GP are computed, one can immediately predict the variation for an unknown arrangement of the WECs. A recently developed active learning strategy, the GP-UCB-PE algorithm, was then used to train the predictor in regions of interest of the problem. The algorithm focuses on the maxima and regions of high uncertainty of the function. This in-turn enables the algorithm to make better predictions in the process of determining the optimal layout of the WECs. The predictions of the statistical emulator are reasonably good despite the complexity of the problem and small number of training data used.

Variation of the performance of an inline and a staggered array of 40 WECs and uniform spacing is analysed using the prediction function. The $\hat{q}$ factor for the inline configuration has a maximum for a separation distance of 76.5m between the centers of the neighbouring WECs. Optimization of the layout of a wavefarm comprising of 40 WECs with arbitrary bathymetry constraints and space limitations is then performed using a genetic algorithm. It is shown that it is possible to achieve a $\hat{q}$ factor greater than one even with various restrictions. It is also suggested that clustering of devices is to be avoided while designing layouts of wave energy converter arrays.

Another significant advantage of the presented methodology is its scalability to large
3.5 Conclusion

problems. The prediction formulation once developed needs no modification, and optimization using GA can be performed for a wavefarm with an arbitrary number of WECs and constraints. A similar approach of decomposing a problem into smaller problems and applying machine learning/optimization techniques may be pursued in solving other problems in the field of marine renewable energy research.
The modular concept of the Oscillating Wave Surge Converter

4.1 Introduction

The OWSC is already recognised as a robust and efficient wave energy conversion device. By nature of its operating principles, the OWSC concepts are nearshore based as they try to exploit the amplification in the horizontal surge motion of the water particles in shallow waters [81]. One of the best known OWSC is the Oyster device developed by Aquamarine Power. The design of this wave energy converter (WEC) has evolved significantly since its inception and a lot of research is still focussed on modifications which can address some of its shortcomings. One of the disadvantages of having a single flap of such large width

Collaborators: Kenneth Doherty, Frederic Dias
is the large wave loads acting on the common foundation at the bottom especially in extreme wave conditions which have been observed on the Oyster800 prototype installed at the European Marine Energy Centre test site, Orkney, Scotland. A possible mechanism to mitigate such destructive effects is to divide the flap into smaller components. A new concept that has emerged based on the above philosophy is a modular form of the OWSC [see 82], which is analyzed in this thesis (see figure 4.1). In fact, experimental results presented in [82] show a reduction in the parasitic foundation loads such as the yaw and roll twisting moments. In addition to this, the breakdown of the structure can potentially help in its fabrication and installation. However, it is not yet understood how such a design alteration would impact the hydrodynamics and performance of the OWSC system.

The studies on Venice gates were probably the first investigations on the behavior of ocean based modular systems [see e.g. 48, 64, 65]. The purpose of these barriers is to control the flooding of the Venice lagoon and they are at present under construction. The research work on the gates was mostly focussed on understanding the subharmonic resonance of the system of gates which resulted in large out-of-phase oscillations of the coupled gates. A linear theory was developed in [48] to explain the resonant phenomenon which occurred at half the frequency of the incident wave and was verified with experimental findings as well. Later the method was extended in [37] to examine inclined gates using a hybrid element method. In [36], the natural modes of a long barrier comprising of a sequence of discrete gates were determined, while in [2] the gate system was analyzed in a semi-channel open to the sea. Recently, the potential of exploiting the subharmonic resonant mechanisms in harnessing energy was explored in [63]. However, the resonant phenomenon depends strongly on the parameters e.g. the inertia of the gates, incident wave frequency, water depth. This sensitivity limits the application of the system to a real ocean environment. The key differences between the modular gate system and the modular flap system discussed in this study include their purpose, their dimensions (width of each modular gate similar to a rigid OWSC) and their application in different layouts (gates in a channel, OWSC in the open ocean). The hydrodynamics of a single wide OWSC has been studied extensively since the initial works of [26, 27, 32, 80], and is well understood now. An abundant theoretical literature now exists on it, starting from understanding its behavior in a channel [56], in the open
4.1 Introduction

Fig. 4.1 A 3D graphical illustration of a rigid flap and a modular flap-type WEC.

...
first we develop a mathematical model for the analysis of the system shown in figure 4.2.

In §4.3, computations are performed for some possible power take-off strategies, along with a discussion of the general hydrodynamics and the multiple resonant characteristics of the modular system. In §4.4, an optimization of the power take-off damping coefficients is performed using genetic algorithm. And lastly, in §4.5, the same modular system with a gap underneath is considered and analyzed by using the mathematical model of [74].

### 4.2 Mathematical Model

The modular flap-type WEC is considered to be situated in an ocean of constant water depth $h'$, and comprises of a total of $M$ modules. The incoming waves of amplitude $A'_I$ are considered to be obliquely incident making an angle $\psi$ with the negative $x'$-axis. Each of the modules independently performs oscillatory motion about a horizontal hinge which is located at a distance $c'$ above the sea bed. The fluid is considered to be inviscid and incompressible, and the flow irrotational. Therefore there exists a velocity potential $\Phi'$ which satisfies the Laplace equation in the fluid domain. The scalar potential also satisfies

---

**Fig. 4.2 Geometry of the physical system a) top view and b) cross-section of the j-th module.**
4.2 Mathematical Model

the linearised kinematic-dynamic free surface boundary condition

$$\Phi_{,tt'} + g\Phi_{,z'} = 0, \quad z' = 0,$$ (4.1)

where $g$ is the acceleration due to gravity, and the no-flux boundary condition on the seabed

$$\Phi_{,z'} = 0, \quad z' = -h'.$$ (4.2)

The individual modules of the WEC are modelled as cylinders and the kinematic boundary condition on them yields

$$\Phi_{,r'} j = -\theta_j \rho (z' + h' - c') H (z' + h' - c') \cos \xi_j, \quad r'_j = a'_j, \quad 0 < \xi'_j < 2\pi,$$ (4.3)

where $H$ is the Heaviside step function. The non-dimensional system of variables is chosen as

$$(x, y, z, r_j) = (x', y', z', r'_j)/h', \quad t = \sqrt{g/h'} t', \quad \Phi = \frac{\Phi'}{\sqrt{gh'}A'_j}, \quad \varepsilon \theta = \Theta'_j,$$ (4.4)

where $\varepsilon = A'_j/h'$ is the small parameter of the problem. Assuming the motion to be simple harmonic, we obtain

$$\theta_j = Re\{\Theta_j \exp^{-i\omega t}\}, \quad \Phi = Re\{\phi(r_j, \xi_j, z) \exp^{-i\omega t}\},$$ (4.5)

where $\omega = \omega' \sqrt{h'/g}$ and $\Theta_j$ are respectively, the angular frequency and amplitude of oscillation of the $j$-th module, while $\phi(r_j, \xi_j, z)$ is the complex spatial velocity potential in the co-ordinate system of the $j$-th module. In order to analyze the modular WEC, we first determine the scattering matrices of an isolated module and then use them to obtain the solution for an array of such bodies. The methodology is similar to that in [74], with a system of heaving truncated cylinders.
4.2 Mathematical Model

Fig. 4.3 Top view of two modules and the coordinate systems. The terminologies shown here are utilised to change from the $i$-th coordinate system to the $j$-th one.

### 4.2.1 Isolated Module

For an isolated module, the coordinate system is located at its center. The spatial velocity potential $\phi$ is decomposed into the scattering potential $\phi^{(S)}$ and radiation potential per unit velocity $\phi^{(R)}$ as follows

$$
\phi = \phi^{(S)} + V \phi^{(R)} = (\phi^{(I)} + \phi^{(D)}) + V \phi^{(R)},
$$

where, $\phi^{(I)}$ is the incident wave potential, $V = i \omega \Theta_0$ is the complex angular velocity of the isolated module, and $\phi^{(D)}$ is the diffracted wave potential. In our description, the subscript index $j = 0$ will be used to indicate the behavior of the module in isolation (i.e. no other module is present). The general form of the spatial potential for the radiation ($R$) and the scattering ($S$) problem can be written as

$$
\begin{bmatrix}
\phi^{(R)} \\
\phi^{(S)}
\end{bmatrix} = \sum_{l=-\infty}^{\infty} e^{il\xi_0} \sum_{n=0}^{\infty} Z_n(z) \begin{bmatrix}
H_1^{(1)}(\kappa_nr_0) \\
H_1^{(1)}(\kappa_na_0)
\end{bmatrix} \begin{bmatrix}
\alpha_{0ln}^{(R)} \\
\alpha_{0ln}^{(D)}
\end{bmatrix} + \frac{J_l(\kappa_nr_0)}{J_l(\kappa_na_0)} \begin{bmatrix}
0 \\
\beta_{0ln}^{(I)}
\end{bmatrix},
$$

(4.7)
4.2 Mathematical Model

where \( H^{(1)}_l \) is the Hankel function of first kind and order \( l \), \( J_l \) is the Bessel function of the first kind and order \( l \), \( \alpha^{(R,D)}_{0ln} \) are the unknown scattering coefficients,

\[
\beta^{(l)}_{0ln} = -\frac{A_l}{\omega Z_0(0)}(-i)^{l+1}J_l(ka_0)e^{i\psi_0}\delta_{ln}
\]  

are the coefficients of the ambient incident wave potential, \( Z_n(z) \) are the normalised vertical eigenmodes (see (2.16)) and \( \kappa_n \) are the solutions to the dispersion relations (see (2.17)). For the scattering problem, the spatial potential needs to satisfy

\[
\phi^{(S)}_{r_0} = 0, \quad r_0 = a_0.
\]  

Using the orthogonality of \( e^{il\xi_0} \) and \( Z_n(z) \) over the range \( 0 \leq \xi_0 \leq 2\pi \) and \( -h \leq z \leq 0 \) respectively, (4.9) yields

\[
\alpha^{(D)}_{0ln}\left(\kappa_n\frac{H^{(1)\prime}_l(\kappa_na_0)}{H^{(1)}_l(\kappa_na_0)}\right) + \beta^{(l)}_{0ln}\left(\kappa_n\frac{J'_l(\kappa_na_0)}{J_l(\kappa_na_0)}\right) = 0.
\]  

This equation (4.10) can then be written in a matrix form as

\[
\begin{bmatrix} M^A_l & M^B_l \end{bmatrix} \begin{bmatrix} \alpha^{(D)}_{0l} \\ \beta^{(l)}_{0l} \end{bmatrix} = 0,
\]  

which gives the solution to the unknown coefficients

\[
\alpha^{(D)}_{0l} = -(M^A_l)^{-1}M^B_l\beta^{(l)}_{0l}
\]

\[
= B_l\beta^{(l)}_{0l}
\]  

For the radiation problem, the boundary condition on the module for the radiation potential per unit velocity is

\[
\phi^{(R)}_{r} = (z + h - c)H(z + h - c)\cos\xi_0, \quad r = a,
\]  

\[
81
\]
4.2 Mathematical Model

which on application of the orthogonality of $e^{il\xi_0}$ and $Z_n(z)$ gives

$$\alpha_{0l}^{(R)} \left( \frac{H^{(1)'}(\kappa_n a_0)}{H^{(1)}(\kappa_n a_0)} \right) = \frac{1}{2} f_n \delta_{1|l|}, \quad (4.14)$$

where

$$f_n = \sqrt{2} [\kappa_n (h - c) \sinh(\kappa_n h) + \cosh(\kappa_n c) - \cosh(\kappa_n h)] \kappa_n^2 (h + \omega^{-2} \sinh^2(\kappa_n h))^{1/2}. \quad (4.15)$$

Expressing (4.14) in a matrix form

$$M_i^A \alpha_{0l}^{(R)} = \alpha_{0l}^*, \quad (4.16)$$

gives the solution

$$\alpha_{0l}^{(R)} = (M_i^A)^{-1} \alpha_{0l}^* = D_i \quad (4.17)$$

The matrices $B_i$ and $D_i$ (see equations (4.12) and (4.17)) will now be used to solve the modular system of the OWSC.

4.2.2 Array of modules

The WEC analyzed in this study comprises of a finite number of modules placed adjacent to each other. Since it is a multi-body system, there are be as many radiation problems as the number of modules. The spatial velocity potential in this case can be decomposed into

$$\phi = \phi^{(S)} + \phi^{(R)}$$

$$= (\phi^{(I)} + \phi^{(D)}) + \sum_{m=1}^{M} V_m \phi^{(m)} \quad (4.18)$$
4.2 Mathematical Model

where

\[
\phi^I = -iA_I \frac{\cosh k(z + h)}{\cosh kh} e^{-ikx_j' \cos \psi + iky_j' \sin \psi} \sum_{l=-\infty}^{\infty} e^{il(\xi_j + \psi)} J_l(kr_j)(-i)^l
\]  

(4.19)

is the incident wave potential in the co-ordinate system of the \( j \)-th module. In (4.19), \( x_j' \) and \( y_j' \) are the horizontal coordinates of the center of the \( j \)-th module with respect to the global coordinate system. In (4.18), \( \phi^D \) is the diffracted wave potential, \( \phi^{(m)} \) is the radiation potential per unit velocity induced by the motion of the \( m \)-th module when all the other modules are held fixed and \( V_m = i\omega \Theta_m \) is the complex angular velocity of the moving flap.

To solve the multi-body interaction problem, we use the approach introduced by [34]. The waves scattered by the \( i \)-th body can be transformed as an incident wave on the \( j \)-th body using the addition theorem for the Bessel function

\[
H_p^{(1)}(\kappa_n r_i) e^{ip(\xi_i - \varepsilon_{ij})} = \sum_{l=-\infty}^{\infty} H_p^{(1)}(\kappa_n L_{ij}) J_l(kr_j) e^{il(\pi - \xi_j + \varepsilon_{ij})},
\]  

(4.20)

and therefore the scattering coefficients \( (\alpha_{i\rightarrow j}^{(m,D)}) \) can be expressed as incident coefficients \( (\beta_{i\rightarrow j}^{(m,D)}) \) on the \( j \)-th body as

\[
\beta_{j\rightarrow i}^{(m,D)} = \sum_{p=-\infty}^{\infty} T_{i\rightarrow j}^{p \rightarrow} \alpha_{i\rightarrow j}^{(m,D)},
\]  

(4.21)

where

\[
T_{i\rightarrow j}^{p \rightarrow} = \frac{H_p^{(1)}(\kappa_n L_{ij})}{H_p^{(1)}(\kappa_n \alpha)} J_l(kr_j) e^{i(p-l)\varepsilon_{ij}}.
\]  

(4.22)

Note, the terms in (4.20) are described in figure 4.3, with \( \varepsilon_{ij} \) being the angle made by the line joining the centers of the \( i \)-th and \( j \)-th module with the positive \( x_j' \) coordinate axis. The general form of the spatial potential for the \( m \)th radiation and the scattering (S) problem
can be written in the coordinate system of the $j$th module as

$$
\left\{ \begin{array}{c}
\phi_j^{(m)} \\
\phi_j^{(S)} \\
\end{array} \right\} = \sum_{l=-\infty}^{\infty} e^{il\xi_j} \sum_{n=0}^{\infty} Z_n(z) \left[ H_l^{(1)}(k_n r_j) \alpha_{jn}^{(m)} \right] + J_l(k_n r_j) \left[ J_l^{(D)}(\kappa_n a) \right] + \left[ \beta_j^{(m)}(\kappa) + \beta_j^{(f)}(\kappa) \right],
$$

(4.23)

where $\beta_j^{(m,D)}$ are the incident coefficients on the $j$th module due to the scattering by the other bodies and $\beta_j^{(I)}$ are the coefficients of the ambient incident wave potential in the coordinate system of the $j$-th module expressed as

$$
\beta_j^{(I)} = \frac{A_I}{\omega Z_0} e^{-ik_j^c \cos \psi - iky_j^c \sin \psi} (\text{e}^{-i})^{l+1} J_l(k_j a) e^{i\psi} \delta_{ln},
$$

(4.24)

Note, the superscripts $(m,D)$ are used to denote the coefficients for either the $(m)$-th radiation or $(D)$ diffraction problem. Let $\alpha_{jn}^{(m,D)}$ be the vector of the scattering coefficients $\alpha_j^{(m,D)}$, where $j = 1, 2, \ldots M$, $\beta^{(I)}$ be the vector of coefficients of the ambient incident wave $(\beta_j^{(I)})$, and $\beta^{(D)}$ be the incident coefficients on the $j$th body due to the scattering by the other modules. We denote by $B^{array}$ and $D^{array}_m$ the array form of the matrix $B_l$ and $D_l$ respectively. Therefore, for the scattering problem, the incident and the scattering coefficients can be written as

$$
\alpha^{(D)} = B^{array} (\beta^{(I)} + \beta^{(D)}),
$$

(4.25)

Making use of the transformation $\beta^{(D)} = \mathcal{F} \alpha^{(D)}$, the solution to the unknown coefficients $\alpha^{(D)}$ is obtained as

$$
\alpha^{(D)} = (I - B^{array} \mathcal{F})^{-1} B^{array} \beta^{(I)},
$$

(4.26)

where $\mathcal{F}$ is the coordinate transfer matrix and $I$ is the identity matrix. For the radiation problems, the relations are

$$
\alpha^{(m)} = B^{array} \beta^{(m)} + D^{array}_m.
$$

(4.27)
Using $\beta^{(m)} = \mathcal{F} \alpha^{(m)}$ gives

$$\alpha^{(m)} = (I - B^{array} \mathcal{F})^{-1} D_m^{array}. \quad (4.28)$$

To obtain the solutions, the infinite series in the vertical eigenmodes is truncated at a finite number $N$, while for the harmonics $2L + 1$ terms are considered (i.e. $l$ from $-L$ to $L$). For the range of wave periods considered, it was found that convergence with a relative error of $O(10^{-3})$ was obtained with the first 6 vertical eigenmodes and 17 harmonics. The solutions to the unknown coefficients $\alpha^{(m,D)} (m = 1, 2, ..., M$ indicate the radiation modes) are then utilized to obtain the hydrodynamic coefficients of the problem, which are later used to solve for the body equation of the modules.

### 4.2.3 Hydrodynamic Parameters

The solution for the velocity potential is then used to solve the equation of motion of each individual flap in the frequency domain. Suppose for the $j$-th module, $I_j = I'_j/(\rho h^5)$ is the second moment of inertia and $C_j = C'_j/(\rho gh^4)$ is the coefficient of the flap restoring buoyancy torque. Note, $\rho$ is the density of water. The magnitudes $I_j$ and $C_j$ are chosen such that their cumulative magnitudes are the same as that of a rigid flap, with the value per unit-width of every module being constant. Then non-dimensional equation of motion of the $j$-th module can be expressed as shown in [55]

$$[-\omega^2 (I_j + \mu_{jj}) + C_j - i\omega (v_{jj} + v_{j}^{mo})]\Theta_j - \sum_{i=1 \atop i \neq j}^{M} [\omega^2 \mu_{ij} + i\omega v_{ij}]\Theta_i = F^S_j, \quad (4.29)$$

where

$$F^S_j = i\omega a_j \pi f_0 \left[ \mathcal{H}_j^{(D)} - \frac{iA_j}{\omega Z_0(0)} \left[ J_{-1}(ka_j)(-i)^{-1}e^{-i\psi} + J_{1}(ka_j)(-i)^{1}e^{i\psi} \right] \right], \quad (4.30)$$
4.2 Mathematical Model

is the excitation torque, while

$$\mu_{ij} = \pi a_j \Re \left\{ \sum_{n=0}^{\infty} f_n \mathcal{M}^{(i)}_{jn} \right\}$$  \hspace{1cm} (4.31)

is the added inertia and

$$\nu_{ij} = \omega \pi a_j \Im \left\{ \sum_{n=0}^{\infty} f_n \mathcal{M}^{(i)}_{jn} \right\}$$  \hspace{1cm} (4.32)

is the radiation damping, where

$$\mathcal{M}^{(i,D)}_{jn} = \left[ \alpha^{(i,D)}_{j(-1)n} + \sum_{q=1}^{M} \sum_{p=-\infty}^{\infty} \alpha^{(i,D)}_{qpn} \frac{H_{1}^{(1)}(\kappa_n L_{qj})}{H_{1}^{(1)}(\kappa_n a_j)} J_{-1}(\kappa_n a_j) e^{i(p+1)e_{qj}} \right]$$  \hspace{1cm} (4.33)

In (4.29), $v_{j}^{pto} = \frac{v_{j}^{pto}}{\rho h^{5} \sqrt{g/h_{j}^{'}}} \sqrt{g/h_{j}^{'}}$ is the power take-off (PTO) damping coefficient of the $j$-th module. In the literature, the PTO damping of an oscillating device in an array is usually set to the optimal damping of the same device in isolation [see e.g. 4, 12, 55, 69, 70]. In monochromatic waves, the optimal PTO damping employed to extract maximum power at a particular wave period in the case of an isolated module is given by

$$v_{j}^{pto1} = \sqrt{\frac{(C_j - (I_j + \mu_{j}^{open})\omega^2)}{\omega^2} + (v_{j}^{open})^2},$$  \hspace{1cm} (4.34)

where $\mu_{j}^{open}$ and $v_{j}^{open}$ are respectively the added moment of inertia and radiation damping of the $j$-th module isolated in the open ocean. We will refer to this PTO strategy as PTO1. However, for closely packed systems, like the one considered here, such an approximation can be non-optimal as the mutual interaction terms contribute significantly. Another possible option is to consider the PTO damping per unit width of the modules to be same as that of a rigid flap (extracting maximum power) of width equivalent to the combined width of all the modules. We will denote the magnitude of the applied PTO damping on the $j$-th...
4.2 Mathematical Model

Fig. 4.4 Variation of excitation torque versus incident wave period for a six-module flap system with $A' = 0.4m$ and normal wave incidence. Module 1 is located at the edge while module 3 is one of the central modules.

Fig. 4.5 Variation of (a) added inertia ($\mu'_{ij}$) and (b) radiation damping ($\nu'_{ij}$) versus wave period of the third module of a six module flap-type WEC. The terms ($\mu'_{ij}$) and ($\nu'_{ij}$) indicate the effect of the mutual interaction on the $j$-th module when the $i$-th module is oscillating and all the others held fixed.

module in this case as $v^\text{PTO2}_j$ and call it the PTO2 strategy. Since one of the main objectives of this study is to compare the performance of the modular system with that of a rigid flap, most of the computations will be performed using $v^\text{PTO2}_j$.

According to the theory of damped oscillating systems (see [24]), the average extracted
4.3 Results

The power by the $i$-th module at a particular wave period is

$$P_{captured}^i = \frac{1}{2} \omega^2 v_i^{pto} |\Theta_i|^2,$$

(4.35)

and the cumulative power extracted by all the modules together then becomes

$$P_{captured}^{total} = \frac{1}{2} \omega^2 \sum_{i=1}^{M} v_i^{pto} |\Theta_i|^2.$$  

(4.36)

In order to quantify the performance of each module, we use the capture factor $C_F^i$ term defined as the ratio of the power captured by a particular module $i$, to that incident across its diameter:

$$C_F^i = \frac{P_{captured}^i}{2a_iP_{incident}},$$

(4.37)

where $P_{incident}$ is the incident wave power per unit wave front. Also, to understand the dynamics of the system as a whole, a term $C_F^{total}$ is introduced

$$C_F^{total} = \frac{P_{captured}^{total}}{2 \sum_{i=1}^{M} a_iP_{incident}}.$$  

(4.38)

The latter is the ratio of the net power captured by all the modules to the wave power incident across a width equivalent to the sum of the diameters of all the modules.

4.3 Results

The computations are performed here for a single rigid flap of width $w' = 24m$ divided into six modules of equal dimensions. So the radius of each of the cylindrical components is $a_j' = a' = 2m$, $j = 1, 2, \ldots M$. For practical reasons, a separation of $0.1m$ between the edges of the consecutive modules is introduced. However the net moment of inertia and buoyancy restoring torque of all the modules combined are kept the same as that of the rigid flap of equivalent total width. The modular OWSC is considered to be located at a water depth $h' = 13m$, with the normally incident waves having an amplitude $A_j' = 0.4m$. 

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4.3 Results

Fig. 4.6 Variation of the amplitude of rotation versus wave period of the respective modules for a six module flap-type WEC using $v^{pro2}$.

4.3.1 Excitation Torque

The excitation torque of the modules is plotted versus wave period in figure 4.4. As one moves towards the central modules (module 3 and 4) of the system, the excitation torque increases. In fact, the excitation torque is maximum for module 3, while lowest for module 1. The behavior is similar to that of a thin rigid flap: where the difference in pressure across its two sides, which produces the excitation torque, is maximum in the central region, while at the edge of the thin-rigid flap, the difference in the velocity potential tends to zero resulting in low excitation torque at such locations. The variation of the excitation torque is also qualitatively similar to that of a rigid OWSC of equivalent width [see 57].

4.3.2 Radiation Parameters

Figure 4.5 shows the variation of the added inertia and of the radiation damping of the third module. The contributions of the mutual interaction terms on a particular flap are quite significant compared to that due to its own motion. Such a behavior is expected given the closely packed nature of the system. Therefore the mutual interaction terms play a
4.3 Results

Fig. 4.7 Behavior of the capture factor $C_F$ of the individual modules versus wave period.

Fig. 4.8 Variation of the different applied PTO strategies versus incident wave period.

pivotal role in determining the performance of the devices, as shown below.
4.3 Results

Fig. 4.9 Behavior of the amplitude of rotation of the modules of a six module flap-type WEC for the various applied PTO strategies.

4.3.3 Amplitude of Rotation

Figure 4.6 plots the variation of the amplitude of rotation of the three modules versus the incident wave period with $\nu_{\text{PTO2}}$ as the PTO damping. Again the nature of the variation is similar to that observed for a rigid flap, increasing monotonically as one moves towards higher wave periods. In the operating range of the device (5-15s), there is an increase in the amplitude of rotation of the modules as one moves from the edge towards the center.

4.3.4 Capture Factor

Figure 4.7 plots the variation of the capture factor $C_F$ of the individual modules versus the incident wave period using the PTO2 strategy. Again in the operating range of the device, the trend is similar to that of the excitation torque with the capture factor of the modules increasing from the edge towards the center. The behavior suggests that in the case of modular flaps of large widths, it may be wise to use the modules at the edge as dummy/stationary structures as they capture the least amount of power.
4.3 Results

4.3.5 PTO strategy

One of the challenges of modelling this particular converter is to determine an optimal PTO strategy. The device is a strongly interacting multi-body oscillating system, yet as a whole it may be treated as a single entity. The individual modules have their own natural frequency of resonance, while the dynamics of the device as a whole, is similar to a rigid OWSC (with $\nu_{pto}^2$ damping). In the computations so far, we have applied the PTO2 strategy. Now, we will consider the other extreme of the possible PTO strategy (i.e. PTO1) and will also assess the behavior with some other values of the PTO damping which lie in between the two extremes, as shown in figure 4.8. It can be seen that in the operating range of the device, PTO1 has the lowest magnitude of damping, while PTO2 has the maximum. Figure 4.9 plots the amplitude of rotation of the three modules for the various PTO strategies versus the incident wave period. There is a general increase in the amplitude of rotation of all the modules as the magnitude of the PTO damping is reduced from $\nu_{pto}^2$. In the case of PTO1, large peaks can be observed in the amplitude of rotation of the modules located at the edges in particular. We now look more closely into the variation of the amplitude of rotation of the modules using PTO1 (see figure 4.10) and we note the occurrence of another peak at a higher period.

To look for a possible explanation, let us consider the unforced, undamped case of the system (i.e. $v_{ij}=0$, $v_{pto}^j=0$, $F_{Sj}=0$ in equation 4.29 for $j=1,2,..M$). The objective is to identify the natural modes of the system which are relevant to the problem investigated. Since normal wave incidence is considered in this study, the modular system exhibit a symmetrical behavior about the centerline $y = (y^c_3 + y^c_4)/2$ which is halfway between the nearest edges of the third and the fourth module, and so only the even natural modes of the modular system are analyzed. This implies that $\Theta_1 = \Theta_6$, $\Theta_2 = \Theta_5$ and $\Theta_3 = \Theta_4$, and the number of independent equations of motion the system reduces (from 6) to 3.

It is well known that a single isolated oscillating system has a unique natural frequency given by $\omega = \sqrt{C/(I+\mu)}$, where $C$ is the coefficient of the buoyancy restoring torque, $I$ is the moment of inertia and $\mu$ is the coefficient of added inertia of the rotating body. However, for non-trivial solutions to the system of equations for a 6-module flap system,
4.3 Results

Fig. 4.10 Variation of the amplitude of rotation of the modules of a six module flap-type WEC using $\nu^{PTO1}$ as the applied PTO damping coefficient (PTO1 strategy).

\[
\begin{vmatrix}
-\omega^2(I + \mu_{11} + \mu_{16}) + C & -\omega^2(\mu_{12} + \mu_{15}) & -\omega^2(\mu_{13} + \mu_{14}) \\
-\omega^2(\mu_{12} + \mu_{62}) & -\omega^2(I + \mu_{22} + \mu_{52}) + C & -\omega^2(\mu_{32} + \mu_{42}) \\
-\omega^2(\mu_{13} + \mu_{63}) & -\omega^2(\mu_{23} + \mu_{53}) & -\omega^2(I + \mu_{33} + \mu_{43}) + C
\end{vmatrix} = 0
\] (4.39)

Let us denote the matrix in (4.39) as $E$. The variation of Det[$E$] is plotted in figure (4.11)(a) with respect to the incident wave period. The plot shows three distinct positive values of the wave period for which relation (4.39) is satisfied. At these periods, the system is expected to resonate and in fact large spikes in the amplitude of rotation (see figure 4.11) are observed. A similar resonant behavior at multiple frequencies has also been reported in the case of three inline OWSCs [see 59]. Note that the amplitude of rotation plotted in figure 4.11(b) is for the case with no applied PTO damping. Interestingly, there is an increase in the bandwidth of the resonances as one moves towards higher wave periods. Nevertheless, in reality viscous (and PTO) damping will reduce the extent of the peaks.
Fig. 4.11 (a) Plot of Det[E] (see 4.39) and (b) amplitude of rotation of the modules with radiation damping but no applied PTO damping ($v_{pto j}^j = 0; j = 1, 2, ..., M$) versus incident wave period for a six module flap-type WEC.

depending on their magnitude. In the operating range of the device, the magnitude of $v_{pto1}$ is the lowest (see figure 4.8) and resonant peaks are manifested in the variation of the amplitude of rotation (see figure 4.9). On the other extreme, the resonant peaks are invisible when the PTO2 strategy is employed where the magnitude of the PTO damping is the largest. Figure 4.12 plots the total capture factor $C_{total}^F$ of the system versus the incident
4.3 Results

Fig. 4.12 Variation of total capture factor $C_F^{total}$ versus wave period of a six module flap-type WEC.

wave period. The efficiency levels for all the PTO strategies are similar except for PTO1, where a huge reduction in the net power captured is observed at intermediate wave periods, and it can be attributed to the low magnitudes of the PTO damping coefficients (see figure 4.8).

4.3.6 Larger number of modules

Some computations are performed with a larger number of modules to understand the effect of the width of the system. The geometry of the modules and the separation between them are kept the same i.e. $d' = 2m$ and $s' = 0.1m$ along with PTO2 ($\nu^{PTO2}$) strategy. Figure 4.13(a) plots the excitation torque of the respective modules of a system comprising of twenty modules. Module 1 or rather the module located at the edge, has the lowest magnitude of torque across the entire range of wave periods. Although some variations are observed in the excitation torque of the other modules at low periods, beyond a period of 7s there is a clear increasing trend in torque as one moves from the edge towards the center. A similar behavior is also observed in the variation of the capture factor $C_F$ of the individual modules at higher periods. However, there appears to be a saturation in both the parameters $|F'|$ and $C_F$ as one moves towards the central module for longer waves.

Figure 4.14 plots the total capture factor $C_F^{total}$ for systems with a larger number of modules. There is a general decrease in the efficiency with an increase of the number
4.3 Results

Fig. 4.13 Variation of (a) excitation torque $|F'|$ and (b) capture factor $C_F$ versus wave period of a twenty module flap-type WEC. The radius of each of the modules is $a' = 2\text{m}$ with a distance of separation $s' = 0.1\text{m}$ between their edges.

of modules. For longer waves, there is a decrease in $C_{total}^F$ as the number of modules is increased. This is because at higher periods, there is a significant reduction in the power captured by the modules at the edges, which deteriorates the performance of the system as a whole. Therefore, for practical applications, the modules at the edges may be considered as dummy structures.
4.4 Optimal PTO damping

Fig. 4.14 Comparison of the variation of the capture factor $C_{total}^F$ versus incident wave period with different number of modules.

4.4 Optimal PTO damping

One of the major difficulties in modelling a multiple oscillating body system is to obtain the optimal PTO damping of the modules which maximizes the power captured by the system as a whole. In order to obtain the optimal PTO damping coefficients, we use a genetic algorithm (GA) [see e.g. 50], which is general enough to be used for a large number of modules. It is a heuristic approach based on the theory of evolution of species. The function one wants to optimize is known as the fitness function which in this particular problem is the total power captured by all the modules together. An individual is a point (vector) where the fitness function is evaluated and in our case it comprises of the PTO damping coefficients. And the magnitude of the fitness function of a particular individual is known as its score. An array of such individuals is known as population. Individuals in the current population having higher fitness values, known as parents, are selected and then used to produce children for the next generation. Two major operations are performed to arrive at the new generation - crossover and mutation. The crossover operation is analogous to reproduction and two parents are combined to produce new children, while the mutation
Fig. 4.15 Comparison of the total capture factor $C_{total}^F$ obtained using PTO coefficients determined using a genetic algorithm and that for the other cases of a six module WEC. The black line denotes the case where the PTO damping of all the modules are considered to be same while the red dashed line is for the case when the PTO damping coefficients of the modules are allowed to vary, enabling the system to capture larger power. The green line shows the capture factor of a rigid OWSC and is obtained using the mathematical model of [57].

operation involves the alteration in the set of parameters which define a possible solution and are essential in maintaining heterogeneity.

We will consider two cases, one in which the PTO damping coefficient is taken to be the same for all the modules, and the other where they can vary for each module. However, an upper and lower bound is set on the PTO damping coefficients to discard unrealistic values. Figure 4.15 plots the optimized total capture factor for such a system with the upper and lower limit on $\nu^{PTO}$ set at 0.002 and 0.05 respectively. Note that $\nu^{PTO}$ is a non-dimensional quantity. The best performance is achieved in the case where the PTO damping coefficients vary for all the modules (red dashed line), with multiple peaks in its variation. While in
the case of fixed damping (black solid line), where the damping is same for every module, a single peak is observed at a higher period. Note that this peak exactly coincides with the third resonant period of the oscillating system (see figure 4.11(b)). A comparison with an equivalent rigid OWSC shows that even at low wave periods, away from the resonant frequencies, the modular system performs equally well. While at higher periods, multiple resonances occur for a system of modules (see again figure 4.11(b)) and this phenomenon enables a properly tuned modular system to capture more power.

4.5 Modules with gap

In this section, we perform the analysis of the modular flaps with the bottom foundation beneath the oscillating flaps absent. The whole modular system is then assumed to be supported centrally or at the edges through structural components which are of negligible dimensions compared to the radius of the cylinders. For convenience of mathematical modelling, the modules can now be treated as truncated cylinders. The region underneath each truncated cylinder is identified as interior region and that outside as exterior region specific to that cylinder. Arrays of truncated cylinders have been modelled in quite a few studies using the interaction theory of [34]. The diffraction problem was solved in [86] for a group of vertical truncated cylinders. The same approach was then adopted in [85] to solve for the radiation problem of a group of cylinders oscillating as a single unit. In recent years, a similar methodology was adopted in [74] to analyze the interaction in an array of truncated cylinders oscillating independently, however, without the application of any external forcing. The same method was used in [12] to obtain optimal configurations of heaving wave energy converters approximated as truncated cylinders.

In our analysis, we follow the approach of [74]. The decomposition of the problem into scattering and radiation problems is the same as in §4.2 and will not be described here again to avoid redundancy. The general form of the spatial velocity potential in the exterior region of the \( j \)-th cylinder is the same as that of the previous problem (see (4.23)) while, in
the interior region it can be expressed as

\[
\left\{ \begin{array}{l}
\tilde{\phi}_j^{(m)} \\
\tilde{\phi}_j^{(s)}
\end{array} \right\} = \sum_{l=-\infty}^{\infty} e^{i l \xi_j} \left[ \frac{1}{2} \left( \frac{r_j}{a} \right)^{|l|} \right] \left\{ \begin{array}{l}
\gamma_j^{(m)} \\
\gamma_j^{(D)}
\end{array} \right\} + \sum_{s=1}^{\infty} \frac{I_l \left( \frac{r_j s \pi}{c} \right)}{I_1 \left( \frac{a s \pi}{c} \right)} \cos \left( \frac{s \pi}{c} (z + h) \right) \\
\left\{ \begin{array}{l}
\gamma_{jls}^{(m)} \\
\gamma_{jls}^{(D)}
\end{array} \right\} - \left\{ \begin{array}{l}
\frac{1}{2c} \left( (z + h)^2 r_j - \frac{r_j^3}{4} \right) \cos \xi_j \delta_{jm} \\
0
\end{array} \right\} \right.
\] (4.40)

where \( I_l \) is the modified Bessel function of first kind and order \( l \). These solutions have been extensively used in modelling truncated cylinders. The last term inside curly brackets in (4.40) is the inhomogeneous component of the velocity potential (when the module performs pitching motion in the absence of incident waves), while the other part (before the minus sign) in (4.40) is the homogeneous potential which satisfies the boundary condition on the bottom surface of the truncated cylinder when it is at rest [see e.g. 74, 84].

The matching conditions for the continuity of potential and velocity underneath the
4.5 Modules with gap

Fig. 4.17 Geometry of the \( \mathcal{J} \)-th module of system with a gap underneath a) cross-section and b) top view.

The truncated module respectively are

\[
\phi_j^{(\text{m}, \mathcal{S})} = \bar{\phi}_j^{(\text{m}, \mathcal{S})}, \quad r_j = a, \quad -h \leq z \leq -h + c, \tag{4.41}
\]

\[
\frac{\partial \phi_j^{(\text{m}, \mathcal{S})}}{\partial r_j} = \frac{\partial \bar{\phi}_j^{(\text{m}, \mathcal{S})}}{\partial r_j}, \quad r_j = a, \quad -h \leq z \leq -h + c, \tag{4.42}
\]

while the kinematic boundary condition on the sides of the modules require

\[
\frac{\partial \phi_j^{(\mathcal{S})}}{\partial r_j} = 0, \quad r_j = a, \quad -h + c \leq z \leq 0 \tag{4.43}
\]

for the scattering problem and

\[
\frac{\partial \phi_j^{(\text{m})}}{\partial r_j} = (z + h - c) \cos \xi_j, \quad r_j = a, \quad -h + c \leq z \leq 0, \tag{4.44}
\]
4.5 Modules with gap

\[
\frac{\partial \tilde{\phi}_j^{(m)}}{\partial z} = -r_j \cos \xi_j, \quad z = -h + c, r_j \leq a,
\]

(4.45)

for the radiation problem. Following the methodology of [74], the solution to the unknown coefficients of the scattered and radiated potentials due to an isolated truncated module is first obtained (see C), which are then used to obtain the solutions for an array of modules (see D). With these solutions the hydrodynamic parameters of the individual modules are evaluated. These hydrodynamic coefficients can then be employed to solve for the body equation of the modules (see (4.29)). The excitation torque is expressed as

\[
F^S_j = i \omega a_j \pi \left\{ \sum_{n=0}^{\infty} f_n \mathcal{M}_{jn}^{(D)} - \frac{i A_j}{\omega Z_0(0)} f_0 \left[ J_{-1}(ka_j)(-i)^{-1} e^{-i\psi} + J_1(ka_j)(-i)^{1} e^{i\psi} \right] + \mathcal{N}_{jn}^{(D)} \right\}
\]

(4.46)

while the radiation parameters are given by

\[
\mu_{ij} = \pi a_j \text{Re} \left\{ \sum_{n=0}^{\infty} f_n \mathcal{M}_{jn}^{(i)} + \mathcal{N}_{jn}^{(i)} - \left( \frac{a^3 c}{8} - \frac{a^5}{48c} \right) \delta_{ij} \right\}
\]

(4.47)

and

\[
\nu_{ij} = \omega \pi a_j \text{Im} \left\{ \sum_{n=0}^{\infty} f_n \mathcal{M}_{jn}^{(i)} + \mathcal{N}_{jn}^{(i)} - \left( \frac{a^3 c}{8} - \frac{a^5}{48c} \right) \delta_{ij} \right\}
\]

(4.48)

where

\[
\mathcal{N}_{jn}^{(i,D)} = (\gamma_{j(-1)}^{(i,D)} + \gamma_{j(10)}^{(i,D)}) \frac{a^2}{8} + \sum_{s=1}^{\infty} (\gamma_{j(-1)s}^{(i,D)} + \gamma_{j(1s)}^{(i,D)}) \frac{ac}{s \pi} \frac{I_2}{I_1} \frac{a \pi}{c} (-1)^s.
\]

(4.49)

These expressions (4.46),(4.47) and (4.48) are then utilized to solve for the body equation of motion of the modules, and thereafter the performance indicators are evaluated.

Figure 4.18 plots the variation of the excitation torque of the individual modules versus incident wave period for the system with gap underneath and comparison is also made with the modular system with rigid foundation. At low wave periods, the excitation torques
4.5 Modules with gap

Fig. 4.18 Variation of excitation torque $|F'|$ versus wave period module flap-type WEC with gap underneath it. The radius of each of the modules is $a' = 2m$ with a distance of separation $s' = 0.1m$ between their edges and the gap underneath is $c' = 4m$.

Fig. 4.19 Comparison of the net capture factor $C_{F_{total}}^E$ of a modular flap with rigid foundation and that with bottom gap for two different applied power take-off damping mechanisms (a) $\nu_{pto1}$ and (b) $\nu_{pto2}$. are similar for both systems, however, for longer waves, there is a clear reduction in the excitation torque on all the modules for the case. In figure 4.19, a comparison of the total capture factor $C_{total}^E$ is made for two different applied power take-off mechanisms - PTO1
and PTO2. In general, in both cases, the modular system with rigid foundation captures more power except in short waves, where the behaviors of both systems are similar. In the case of PTO2 (see figure 4.19b), there is almost a steady decrease in the net capture factor with increasing period. This has a lot of practical implications, because in the PTO2 mechanism, the net PTO damping is the same as that of the rigid flap and consequently the qualitative behaviors of the performance of the two systems (modular and rigid OWSC with PTO2 strategy) are similar as well (see figure 4.15). From this, it can be reasonably inferred that there would be a reduction in the efficiency of a rigid OWSC as well, with the absence of the bottom foundation. Such a behavior can also be expected on observation of the variation of the excitation torque (figure 4.18). Most of the theoretical work on rigid OWSC includes a bottom foundation which makes it easier to model the system. And it is well known that the dynamics of a rigid OWSC is primarily governed by the diffraction phenomenon. Therefore a reduction in the excitation torque is associated with a decrease in the performance as well. The presence of a gap beneath the structure provides a free passage to the waves. Although the bottom foundation is static and doesn’t contribute directly to wave power extraction, its presence provides a blockage to the waves which consequently produces a larger difference in potential across the two sides of the device resulting in a greater excitation torque than in the case without blockage.

4.6 Conclusion

The hydrodynamics of the modular concept of the OWSC is analyzed in this work. The idea is to decompose a rigid flap into a series of modules. The possible advantage of the breakdown is that it enables the distribution of wave loads acting on the common foundation, reduction in the twisting wave forces and flexibility in the operation of the system. It is shown that the modular-flap WEC is equally good in capturing power as well and can be a very effective and promising concept. The behavior of the system is found to be strongly dependant on the applied power take-off mechanism. When the magnitude of the applied PTO-damping is low, strong resonating effects are observed which are attributed to the closely packed nature of the system resulting in strong mutual interactions. On the
4.6 Conclusion

On the other hand, the variation of total capture factor \( C_{total}^{F} \) of a six module flap system of total width 24m is found to be similar to that of a rigid flap when the applied PTO damping per unit width is the same for both the cases (see figure 4.15). In such a case, the modules at the center have higher amplitudes of oscillations and capture more power than those at the edges. So for very large systems, one can envisage having the modules at the edges as dummy or stationary structures, however, they would alter the hydrodynamics of the system if undamped. In order to maximize the net power captured by the modular system, an optimization of the power take-off damping coefficients is performed using a genetic algorithm. At lower periods, away from the resonant frequencies, similar levels of efficiency as that of a rigid flap are obtained using the optimized coefficients, while at higher periods, the occurrence of multiple resonances enables the modular system to capture more power than that of a rigid flap. An analysis is also performed for a modular system with a gap underneath. It is shown that such a system captures less power than a system with rigid foundation. The mathematical model used in this paper is based on linear potential flow theory and gives a valuable understanding and insight into the fundamental dynamics of the modular OWSC. However, in the real ocean, viscous, nonlinear and other effects would modify the behavior of the system, especially at high wave periods and at the locations of resonance where large amplitudes of oscillations of the modules occur.
An Oscillating Wave Surge Converter near a straight coast

5.1 Introduction

With the OWSC emerging as one of the frontrunners in the pursuit of harnessing energy from the ocean [81], there is a growing interest in understanding its behavior in the context of various physical environments. Recently, Renzi and Dias developed the first theoretical models of the OWSC in a channel [56], of an infinite array of OWSCs [58] and of a single OWSC in the open ocean [57]. However, the effect of a straight coast on the performance of the OWSC has not been investigated yet. Such an analysis is relevant for the device under investigation, since proposed OWSC wave farms are to be located in shallow wa-

Collaborators: Emiliano Renzi, Frederic Dias
5.1 Introduction
ters along the rocky coasts of western Ireland and Scotland [see for example 3]. So far, few studies have tried to investigate the behavior of wave energy converters (WECs) near a coast [20, 40, 41]. In particular, [20] studied the variation of maximum efficiency of a point absorber in front of a vertical coastline. He observed the occurrence of peaks in the efficiency of the device when the latter is placed at specific distances from the coast. However, [20] did not provide an explanation of such behavior. At the state of the art, the following research questions are still unanswered: How is the hydrodynamics of a flap-type WEC in front of a straight coast altered in relation to that in the open ocean? And if there are any significant differences in the hydrodynamic behavior, then how does that impact the performance of the device? [21] had shown that for a thin vertical rolling plate in water of finite depth and in front of a rigid vertical wall, there is extreme resonance in the hydrodynamic coefficients for a frequency corresponding to the first sloshing mode between two infinitely long vertical barriers. However, it is uncertain as to how such resonance would affect the performance of similar systems used for energy capture. [77] have shown that for a system of two independent vertical rolling power absorbers in infinite water depth, nearly 100% power capture is in fact possible. However, complete absorption of energy could not be achieved due to the finite length of the plates, which results in leakage of energy beneath them. But both the studies of [21] and [77] are two-dimensional and are therefore insufficient to describe the behavior of the systems in a three-dimensional wave field.

In this paper, following the procedure of [56], a three dimensional mathematical model is developed based on the assumption of the fluid being inviscid and incompressible and the flow irrotational. The analysis is then undertaken in the framework of the linear theory using Green’s integral theorem and Green’s function for a semi-infinite domain. The linearity of the problem facilitates its decomposition into two separate - radiation and scattering - components. The variations of the excitation torque and of the radiation coefficients are discussed . The effect of the hydrodynamic behavior on the device performance is then investigated. It is shown that the presence of the vertical barrier on the leeward side of the OWSC has in general a kind of 'oxymoron’ effect on the device performance.
5.2 Mathematical Model

5.2.1 Governing Equations

Consider an OWSC of width $w'$ located in an ocean of constant water depth $h'$ and at a distance $d'_c$ away from the coastline, represented as a fully reflecting vertical barrier as shown in figure 5.1. Primes denote physical dimensional variables. The origin of the coordinate system is located on the mean free surface at the intersection of the straight coast and the vertical plane passing through the center of the flap and perpendicular to the coast. The $x'$ axis is directed outwards from the vertical barrier, the $y'$ axis is directed along the coast and $z'$ points upwards. The flap is hinged to a bottom foundation and oscillates about the horizontal axis located at a distance $h' - c'$ from the mean free surface. Monochromatic waves of amplitude $A'_I$ and period $T'$ incident from the right, making an angle $\psi$ with the $x'$-axis cause the flap to oscillate, thereby extracting energy from a generator linked to it. The amplitude of the waves is considered to be small such that $A'_I/w' \ll 1$. As a consequence, the governing equations of motion can be linearised by taking only the first-order terms of the perturbation series expansion in $A'_I/w'$ [see 56]. The velocity potential $\Phi'$ satisfies the Laplace equation

$$\nabla'^2 \Phi' = 0,$$

(5.1)
in the fluid domain. The linearised kinematic-dynamic boundary condition on the free surface gives

\[ \Phi_{\eta^\prime} + g \Phi_{z^\prime} = 0, \quad z^\prime = 0, \]  

(5.2)

where \( g \) is the acceleration due to gravity. The no-flux condition at the sea bed gives

\[ \Phi_{x^\prime} = 0, \quad z^\prime = -h^\prime, \]  

(5.3)

while the absence of normal flow through the vertical barrier results in

\[ \Phi_{x^\prime} = 0, \quad x^\prime = 0. \]  

(5.4)

Finally the kinematic condition on the lateral surfaces of the flap yields

\[ \Phi_{x^\prime} = -\theta_{x^\prime}(z^\prime + h^\prime - \epsilon^\prime)H(z^\prime + h^\prime - \epsilon^\prime), \quad x^\prime = d^\prime_c \pm \epsilon^\prime, \epsilon^\prime \to 0, \quad |y^\prime| < \frac{w^\prime}{2}, \]  

(5.5)

where thin plate approximation has been used [see 39, 56]. In (5.5), \( \theta(t^\prime) \) is the unknown amplitude of oscillation of the flap, positive if anticlockwise (see again figure 5.1), while \( H \) is the Heaviside step function.

### 5.2.2 Solution

Let us introduce the non-dimensional system of variables as follows

\[ (x, y, z, d_c, r) = (x^\prime, y^\prime, z^\prime, d_c^\prime, r^\prime)/w^\prime, \quad t = \sqrt{\frac{g}{w^\prime}}t^\prime, \quad \Phi = \frac{\Phi^\prime}{\sqrt{gw^\prime A^\prime}}, \quad \theta = (w^\prime/A^\prime)\theta^\prime, \]  

(5.6)

where \( r^\prime = \sqrt{x'^2 + y'^2} \). Assuming the oscillations of the flap to be simple harmonic in nature, the time dependence of the variables can be separated out as

\[ \theta = \text{Re}\{\Theta e^{-i\omega t}\}, \quad \Phi = \text{Re}\{(\phi^R + \phi^S)e^{-i\omega t}\}, \]  

(5.7)

where \( \omega = \omega^\prime \sqrt{w^\prime/g} \) and \( \Theta \) are respectively, the angular frequency and amplitude of oscillation of the flap, while \( \phi^R(x, y, z) \) and \( \phi^S(x, y, z) \) are the complex spatial radiation and
5.2 Mathematical Model

scattering potential, respectively. The scattering potential can in turn be resolved into

\[ \phi^S = \phi^I + \phi^F + \phi^D \]  \hspace{1cm} (5.8)

where

\[ \phi^F = -\frac{iA_I \cosh k(z+h)}{\omega \cosh kh} e^{ikx \cos \psi + iky \sin \psi}. \]  \hspace{1cm} (5.9)

In (5.8), \( \phi^I \) is the incident wave potential (see (2.8)), \( \phi^F \) the potential of the reflected wave from the coast and \( \phi^D \) the diffracted wave potential, which is unknown. On substitution of the factorisation (5.7) and (5.8) in the governing equations (5.1)–(5.5), we obtain a boundary-value problem in terms of the spatial radiation and scattering potentials. The latter satisfy the Laplace equation

\[ \nabla^2 \phi^{(R,D)} = 0, \]  \hspace{1cm} (5.10)

where the notation \( \phi^{(R,D)} \) denotes either potential, the linearised free-surface boundary condition

\[ -\omega^2 \phi^{(R,D)} + \phi^{(R,D)}_z = 0, \quad z = 0, \]  \hspace{1cm} (5.11)

the no-flux boundary conditions at the sea bed and at the straight coast

\[ \phi^{(R,D)}_z = 0, \quad z = -h, \]  \hspace{1cm} (5.12)

\[ \phi^{(R,D)}_x = 0, \quad x = 0, \]  \hspace{1cm} (5.13)

respectively, and the kinematic conditions

\[
\begin{align*}
\left\{ \begin{array}{l}
\phi^{R}_{x} \\
\phi^{D}_{x}
\end{array} \right\} &= \left\{ \begin{array}{l}
V(z + h - c)H(z + h - c) \\
-\phi^I_x - \phi^F_x
\end{array} \right\}, \quad x = d_c \pm \varepsilon, \varepsilon \to 0, |y| < \frac{1}{2},
\end{align*}
\]  \hspace{1cm} (5.14)

on the lateral surfaces of the flap. Finally, both \( \phi^R \) and \( \phi^D \) are required to be outgoing disturbances of the wave field [49]. In (5.14) \( V = i\omega \Theta \) and the thin plate approximation has been used [see 39, 56]. The vertical dependence can now be isolated out of the three
5.2 Mathematical Model

dimensional governing system (5.10)–(5.14) by using the separation [see 49]

\[ \psi^{(R,D)}(x,y,z) = \sum_{n=0}^{\infty} \psi_n^{(R,D)}(x,y)Z_n(z), \]  

(5.15)

where \( Z_n(z) \) are the normalized vertical eigenmodes (see (2.15)) with \( \kappa_0 = k \) and \( \kappa_n = ik_n \) satisfying the dispersion relation (2.17). Using the decomposition (5.15) and the orthogonality relation (2.16) yields a two-dimensional governing system for \( \psi_n^{(R,D)} \), where the Laplace equation (5.10) becomes the Helmholtz equation

\[ (\nabla^2 + \kappa_n^2) \psi_n^{(R,D)} = 0, \]  

(5.16)

the no-flux condition at the straight coast (5.13) transforms into

\[ \psi_{n,x}^{(R,D)} = 0, \quad x = 0, \]  

(5.17)

and the kinematic condition on the flap (5.14) becomes

\[
\begin{cases}
  \psi_{n,x}^R = V f_n, \\
  \psi_{n,x}^D = A f_n
\end{cases}
\]

\( x = d_c \pm \varepsilon, \varepsilon \to 0, \quad |y| < \frac{1}{2}, \)  

(5.18)

In the latter

\[ f_n = \frac{\sqrt{2} [\kappa_n (h - c) \sinh(\kappa_n h) + \cosh(\kappa_n c) - \cosh(\kappa_n h)]}{\kappa_n^2 (h + \omega^{-2} \sinh^2(\kappa_n h))^{1/2}} \]  

(5.19)

and

\[ d_n = -\frac{\sqrt{2} ik \cos \psi \sin(k \cos \psi d_c)}{\omega \cosh kh} (h + \omega^{-2} \sinh^2(kh))^{1/2} \delta_{0,n} \]  

(5.20)

are constants depending on the geometry of the system. Finally the \( \psi_n^{(R,D)} \) must be outgoing disturbances for \( r \to \infty \). Following the method of [56], the boundary value problem (5.16)–(5.18) is solved using Green’s integral equation formulation and appropriate Green’s function for the semi-infinite fluid domain. The procedure, described in Appendix E, allows one to obtain a solution for the spatial velocity potentials \( \psi_n^{(R,D)} \) (5.15) as a fast converging
semi-analytical form. As a result, the radiation potential is expressed as

\[
\phi^R(x,y,z) = -\frac{i}{8}V \sum_{n=0}^{+\infty} \kappa_n Z_n(z) \sum_{p=0}^{P} \alpha_{pn} \int_{-1}^{1} (1-u^2)^{1/2}U_p(u) \times \left\{ H_1^{(1)} \left( \kappa_n \sqrt{(x-d_c)^2 + (y-u^2/2)} \right) \right. \\
\left. \sqrt{(x-d_c)^2 + (y-u^2/2)} \right\} (x-d_c) - \left\{ H_1^{(1)} \left( \kappa_n \sqrt{(x+d_c)^2 + (y-u^2/2)} \right) \right. \\
\left. \sqrt{(x+d_c)^2 + (y-u^2/2)} \right\} (x+d_c) \right\} du, \\
\]

(5.21)

where \( H_1^{(1)} \) is the Hankel function of the first kind and first order, \( U_p \) is the Chebyshev polynomial of the second kind and order \( p \), \( p = 0, 1, ..., P \in \mathbb{N} \) and the \( \alpha_{pn} \) are the complex solutions of a system of equations, which is solved numerically using a collocation scheme (see Appendix E for details). Similarly, the diffraction potential is given by

\[
\phi^D(x,y,z) = -\frac{i}{8}A_I k Z_0(z) \sum_{p=0}^{P} \beta_{p0} \int_{-1}^{1} (1-u^2)^{1/2}U_p(u) \times \left\{ H_1^{(1)} \left( k \sqrt{(x-d_c)^2 + (y-u^2/2)} \right) \right. \\
\left. \sqrt{(x-d_c)^2 + (y-u^2/2)} \right\} (x-d_c) - \left\{ H_1^{(1)} \left( k_0 \sqrt{(x+d_c)^2 + (y-u^2/2)} \right) \right. \\
\left. \sqrt{(x+d_c)^2 + (y-u^2/2)} \right\} (x+d_c) \right\} du, \\
\]

(5.22)

where the \( \beta_{p0} \) are the complex solutions of a system of equations, again solved numerically. Note that in \( \phi^D \) (5.22) only the 0th order vertical mode is present, the flap being a walled structure in the scattering problem (i.e \( \varphi_n^D = 0 \) for \( n > 0 \)). Using the above expressions (5.21) and (5.22), the equation of motion of the flap can be now solved.

### 5.2.3 Hydrodynamic Parameters

The non-dimensional equation of motion of the flap in the frequency domain is that of a damped harmonic oscillator [see 49, 57] :

\[
[-\omega^2 I + C - i\omega \nu_{pto}] \Theta = F, \\
\]

(5.23)
5.2 Mathematical Model

In the latter, \( I' = I'/\left(\rho w^5\right) \) is the second moment of inertia of the flap, \( C = C'/\left(\rho gw^4\right) \) the coefficient of the net restoring flap buoyancy torque, \( v_{pto} = v'_{pto}/\left(\rho w^5 \sqrt{g/w'}\right) \) the power take-off (PTO) damping coefficient and

\[
\Delta \phi = \phi(d_c - \epsilon, y, z) - \phi(d_c + \epsilon, y, z) = [\phi^D(d_c - \epsilon, y, z) + \phi^R(d_c - \epsilon, y, z)] - [\phi^D(d_c + \epsilon, y, z) + \phi^R(d_c + \epsilon, y, z)],
\]

(5.25)

with \( \epsilon \to 0 \), is the potential difference across the two sides of the flap. Using (5.15), (E.5) and (E.11) and decomposing the complex hydrodynamic torque due to the radiation potential into real and imaginary components, (5.23) finally becomes

\[
[-\omega^2(I + \mu) + C - i\omega(v + v_{pto})] \Theta = F,
\]

(5.26)

where

\[
\mu = \frac{\pi}{4} Re \left\{ \sum_{n=0}^{\infty} f_n \alpha_{0n} \right\}
\]

(5.27)

is the added inertia due to the torque,

\[
v = \frac{\pi \omega}{4} Im \left\{ \sum_{n=0}^{\infty} f_n \alpha_{0n} \right\}
\]

(5.28)

is the radiation damping and

\[
F = -\frac{\pi \omega}{4} iA_1 \beta_{00} f_0
\]

(5.29)

is the excitation torque. According to the theory of damped oscillating systems [see 24], the average extracted power by the generator over a wave period is

\[
P = \frac{1}{2} \frac{\omega^2 v_{pto} |F|^2}{[C - (I + \mu) \omega^2]^2 + (v + v_{pto})^2 \omega^2}.
\]

(5.30)
5.3 Discussion

To obtain the optimal PTO damping for maximum power capture away from body resonance [see 56–58], we need \( \frac{\partial P}{\partial \nu_{pto}} = 0 \), which gives

\[
\nu_{pto} = \sqrt{\frac{|C - (1 + \mu)\omega^2|^2}{\omega^2} + v^2},
\]

and the optimum generated power is

\[
P_{opt} = \frac{|F|^2}{4(\nu_{pto} + v)}. \tag{5.32}
\]

In order to quantify the efficiency of the device, the capture factor is used which is defined as the ratio of the power extracted by the device per unit flap width to the incident wave power per unit crest width, i.e.

\[
C_F = \frac{P_{opt}}{\frac{1}{2} C_g A_f^2}, \tag{5.33}
\]

where

\[
C_g = \frac{\omega}{2k} \left( 1 + \frac{2kh}{\sinh 2kh} \right) \tag{5.34}
\]

is the group velocity of the incident waves. In the next section, the behavior of the hydrodynamic parameters of the system is discussed. Parametric analysis is then undertaken which shows the sensitivity of the system with respect to the distance \( d_c' \) between the flap and the coast.

5.3 Discussion

In this section, the behavior of the hydrodynamic parameters derived in §5.2.3 is discussed. Numerical computations are performed with a flap of width \( w' = 26 \text{m} \), in water of depth \( h' = 13 \text{m} \) and with the distance \( c' = 4 \text{m} \). The values of the physical variables closely resemble those of the newer version of the flap-type wave energy conversion device Oyster800 developed by Aquamarine Power Ltd. (www.aquamarinepower.com). Computations performed in this study are shown only for normally incident waves (i.e. \( \psi = 0^\circ \)) since, in shallow waters the incoming wave fronts are predominantly parallel to the shoreline, which
is also known to let the OWSC generate maximum power [see 81]. In fact normal wave incidence is common in practice with this device [see 56–58].

5.3.1 Excitation Torque

The variation of the excitation torque versus the non-dimensional parameter $k'd'_c = kd_c$ is shown in figure 5.2 for $d'_c = 50m$. Zeros in the excitation torque occur for $kd_c = m\pi$, $m = 1, 2, \ldots$, while maxima are slightly below the intermediate locations between two neighbouring zeros, i.e. $kd_c \simeq (m + 1/2)\pi$, $m = 1, 2, \ldots$ Such behavior can be explained with the following argument. Let us consider a semi-infinite fluid domain in the absence of the flap. In steady state, the interaction of the incident and reflected waves from the straight coast leads to the formation of a two-dimensional standing wave field, as illustrated in figure 5.3. The locations $kd_c = m\pi$ correspond to the antinodes of the standing wave field, where the movements of the water particles are in the vertical direction [see 17]. Now consider wave diffraction by a flap held fixed in its upright position, located at one of the antinodes of the standing wave field represented in figure 5.3. In this configuration, the no-flux boundary condition at the flap lateral surfaces is satisfied automatically and, being of negligible thickness, the flap apparently does not introduce any discontinuity in the wave field. In
other words, the flap pretends as if it is invisible to the wave field. As a consequence, there is no diffracted wave, which manifests as zero excitation torque on the flap for $kd_c = m\pi$, as obtained in the simulations of figure 5.2. However, for locations of the flap different from those mentioned above, a three-dimensional (3D) diffracted wave field is produced, as a deviation from the original 2D configuration.

Such deviation is naturally stronger near the locations $kd_c = (m + 1/2)\pi, m = 0, 1, 2, ...$, corresponding to the nodes of the standing waves where the trajectories are horizontal, and results in the resonant peaks of figure 5.2. Those are slightly below $kd_c = (m + 1/2)\pi$, due to the radial dispersion of the 3D scattered wave field, which represents a source of damping for the system and lowers its resonant periods with respect to the 2D scenario.

5.3.2 Radiation parameters

The behaviors of the added inertia torque and the radiation damping are shown in figure 5.4 for the same configuration as above. Spikes in added inertia and radiation damping coefficients are observed, each below $kd_c = (m + 1/2)\pi, m = 1, 2, ...$. This behavior, although different, has some similarity with that of the antisymmetric modes of motion of a thin vertical plate lying on the centerline of a straight channel of width $2d_c'$ and aligned with the channel walls [see 38]. For such a system, [38] found that the radiation coefficients for sway motion exhibit sharp spiky but non-singular behavior at frequencies a little below
5.3 Discussion

Fig. 5.4 Behavior of (a) added inertia ($\mu = \mu'/\rho w'^5$, see 5.27) (b) radiation damping ($\nu = \nu'/(\rho w'^4\sqrt{gw'})$, see 5.28) for $w' = 26m$, $d_{c'} = 50m$, $h' = 13m$, $c' = 4m$, $\psi = 0^\circ$ and $A'_{r} = 0.3m$. The vertical lines indicate the location of the peaks in radiation damping: $kd_{c} = 4.28, 6.84$ and $9.8$. 
5.3 Discussion

\( kd_c = (m + 1/2)\pi \). Each of the latter corresponds to a simple pole in the lower half of the complex frequency plane. [38] also showed that the complex force coefficient \( F_c = \nu + i\mu \) traces a circle as \( \omega \) moves on the real axis close to the pole and that the total extent of the spikes in radiation coefficients correspond to the diameter of the circle. The system analyzed here can be thought of to be the vertical flap of [38] with the straight coast as one of the channel walls while the other wall is absent. Compared with the sharp spikes of [38], the variation in added inertia and radiation damping of figure 5.4 exhibits blunt peaks with wider bandwidth, suggesting an imperfect form of resonance. Note that the peaks in radiation damping correspond to locations \( kd_c \) where the magnitude of the slope of the added inertia (\( \mu' \)) is maximum. This is also similar to the behavior of the radiation coefficients in [38]. However there are some notable differences between the two systems.

The variation of the complex force coefficient as \( \omega \) moves on the real axis is plotted in figure 5.5 for the system of figure 5.4. First, note that the complex force coefficient shown in figure 5.5, follows approximate ellipses rather than circles [as in 38], due to the unequal extent of the spikes in added inertia and radiation damping coefficients. Second, the traces of the ellipses do not go through the origin. This is because the magnitude of the radiation coefficients \( \nu \) and \( \mu \) away from the resonant locations is not insignificant compared to the peaks, as opposed to the channel problem. Lastly, unlike in [38], the trace of the complex force coefficient never completes any particular orbit in the \((\nu, \mu)\) plane (see again figure 5.5). This is due to a likely shift of the poles further away from the real axis, because of the larger dissipation of energy by radiation in the straight coast problem than in the channel. Since each pole is further away from the real axis, the neighbouring poles now influence the variation of \( F_c \). As a consequence, the complex force moves on a near elliptical orbit as \( \omega \) moves on the real axis close to a particular pole. However, as \( \omega \) moves further away from that particular pole, the influence of the neighbouring pole becomes increasingly dominant and therefore \( F_c \) cannot complete a particular orbit and moves to another orbit.

5.3.3 Energy Capture

Figure 5.6(a) shows the behavior of the amplitude of rotation \( |\Theta'| \) versus the parameter \( k'd'_c \). Unlike in the open ocean, where the amplitude of rotation of the flap increases mono-
5.3 Discussion

Fig. 5.5 Path of the complex force coefficient $F_c = \nu + i\mu$ as $\omega$ moves on the real axis, showing the variation of added inertia coefficient $\mu$ versus radiation damping coefficient $\nu$ for $d'_c = 50$ m. The parameters of the system are those of figure 5.4.

...
5.3 Discussion

Fig. 5.6 Behavior of (a) amplitude of rotation and (b) capture factor with $k'd_c'$ in physical variables for $d'_c = 50m$. The values of the other physical variables are as mentioned in §5.3.

5.3.4 Parametric Analysis

The variation of power capture shown in figure 5.6(b) suggests that even a slight movement away from the peak periods would drastically reduce the performance of the device and could even lead to no power capture at all. Therefore, from a designer’s perspective, the challenge is to avert such detrimental effects on the device performance by identifying a layout in which the zeros of power capture can be avoided, for a given frequency range of the incident waves. As already shown in figure 5.6(b), the zeros of $C_F$ occur each time the parameter $kd_c'/\pi$ passes through a positive integer value. This would suggest that to avoid the zeros, it must be $\max(k)d_c'/\pi < 1$, which would mean $d'_c < \min\lambda'/2$, i.e. the distance $d'_c$ should be within half a wavelength of the smallest wave considered. This would then circumvent the possibility of the flap being located at an anti-node of the standing wave field (see again figure 5.3).

As an example, the behavior of the excitation torque versus $k'd_c'$ and time period $T'$ is plotted in figure 5.7 and figure 5.8 shows the variation of the capture factor $C_F$ with time period $T'$. In both figures, $d'_c = 12m$ and $k'd_c' < \pi$. In this configuration, not only the zeros of the excitation torque are avoided (see figure 5.7) but also a significant enhancement in
the average capture factor over that in an open ocean is observed (see figure 5.8). This suggests that the presence of the coast can have a strong beneficial influence on the device performance if the distance $d'_c$ is appropriately chosen. The enhancement is primarily due to the magnification in excitation torque which interestingly occurs at $k'd'_c$ values much less than $\pi/2$.

### 5.3.5 OWSC, point absorbers and future research directions

As mentioned earlier, [20] studied the performance of a point absorber near a straight coast. In the case of normal incidence, a similar irregular behavior in the performance of the point absorbers was observed, with periodic occurrence of maxima and minima in the variation of maximum efficiency with $kd_c$. However, for the point absorber the capture factor is zero when $k'd'_c \approx (m + 1/2)\pi$, while it is maximum at $k'd'_c = m\pi$. This behavior is in contrast and exactly opposite to that of the OWSC (see again figure 5.6b) and can be explained in the following manner. The locations $kd_c = m\pi$ correspond to the antinodes of the standing wave field where the movement of the water particles is completely vertical. On the other hand, $kd_c = (n + 1/2)\pi$ corresponds to the nodes where the movement is horizontal. The heaving point absorber utilizes the vertical movement of the water particles to capture power. Therefore it comes to a complete standstill at the nodes because of no excitation force, but captures maximum power when located at the antinodes. This simplified explanation is possible because of the small dimension of the point absorbers which produce an insignificant diffracted wave field. The OWSC exploits the amplified surge motion of the water particles in shallow water and therefore behaves conversely to the point absorber. This would then suggest that using a proper combination of OWSCs and point absorbers near a straight coast could be beneficial for enhancing the efficiency of a combined wave farm. Such investigation is envisaged as an intriguing future research direction. Another interesting topic for future research would be to investigate the behavior of the device on a sloped bottom using a numerical approach.
5.4 Conclusion

Using Green’s function for a semi-infinite domain, a three dimensional semi-analytical method is developed to analyze the behavior of the OWSC in front of a straight vertical coast. For locations corresponding to \( kd_c = m\pi \), the incident, diffracted and reflected waves produce a two-dimensional standing wave field which brings the flap to a standstill. Hence the OWSC does not capture any power corresponding to these locations. The further the flap is from the coast, the higher is the number of standing wave modes for given range of monochromatic incident wave periods. Conversely, peaks in the radiation coefficient occur for \( kd_c \) a little below the resonant frequencies \( (m + 1/2)\pi \) of a swaying flap aligned with the walls of a straight channel [see 21]. Surprisingly, when located very close to a coast, the device reproduces the qualitative behavior it would have in the open ocean, with only a single peak in the capture factor curve. For a given range of incident wave fields, it is found that the formation of the standing waves is avoided when \( d'_c \) is less than half the wavelength of the incident wave field. Therefore one can wisely utilise the presence of a coast for capturing more power by effectively tuning the distance between the coast and the flap. Taking inspiration from the above phenomenon, one might in a realistic scenario reproduce the influence of the straight coast by introducing a long vertical breakwater on
Fig. 5.8 Comparison of the capture factors for $d_c' = 12m$ and for the open ocean with the geometry of the physical system as mentioned in §5.3. Values for the open ocean are calculated with the model of [57] on the leeward side of the OWSC, to enhance its performance.
6.1 Introduction

A plethora of wave energy conversion systems are now known in the literature and the prototypes of a few are already tested. The race to produce commercially viable ocean wave power is still an open challenge though. Some of the concepts have emerged to be promising, the OWSC is one of them. The OWSC has a wide bandwidth of consistent performance levels which has generated a lot of interest in such systems. A lot of research efforts
6.1 Introduction

are now focussed in identifying mechanisms to enhance the performance and address the shortcomings of the existing design (see e.g. [66]). The recent study by [67] showed that a straight coast has a ‘oxymoronic’ kind of influence on the OWSC performance. However, when the flap is located close to the straight coast, it can consistently capture much higher levels of energy. The present study tries to investigate whether a breakwater present on the lee-side of the OWSC can realize such predominant favourable effects so that it can further enhance the performance of the device. The two-dimensional (2D) analysis of a similar system comprising of two flaps was investigated in [77], and they found that a combination of two flaps can result in higher efficiency. In fact in the 2D analysis, the later showed that maximum efficiency upto 1 is possible which corroborated with the maximum efficiency obtained for a 2D analysis of a single OWSC located near a straight coast [67]. But in the 3D description of the straight coast problem, much higher levels of efficiency is shown to be possible [see 67, 71]. The goal of this study is to investigate if it is artificially possible to produce similar hydrodynamic characteristics as that of the straight coast with the introduction of a breakwater. The study is also important because in general the presence of coastal structures can influence the performance of the OWSC. Breakwaters are installed in coastal regions for various reasons. Construction of the breakwater by itself is a costly investment. With the implementation of this hybrid form of system, one incurs an additional expenditure but then there is the possibility of recovering which may ensure the economic viability of such projects. Some research studies in the literature have proposed hybrid systems which can serve both - coastal protection and energy generation. For example, [73] presented a WEC integrated with a caisson breakwater where the dynamic wave pressure exerted on underwater opening on the front side of the breakwater produces a flow of water into a ramp with gradual constriction, and then through a turbine located at the rear, and finally into the sea. Although it may be stated that a very long breakwater would be like a straight coast and therefore result in similar hydrodynamics of the OWSC as that near a straight coast, it is important to quantify the physical dimension of the breakwater. The analysis is also important for the economic perspective. So if one tries to compromise on the width of the breakwater, then how does that impact the performance of the converter? Is it then wise to use a breakwater under such circumstances?
6.2 Mathematical Model

The OWSC is located in an ocean of constant water depth $h'$ with the breakwater located at a distance $d'$ behind the flap. Waves of amplitude $A'$ are incident from the right making an angle of $\psi$ with the x-axis as shown in figure. The origin is located at the center of the mean free surface on the flap with $x'$ pointing away from the breakwater and the flap and $z'$ directed upwards. The center of the flap and breakwater are oriented along the same x co-ordinate. The velocity potential satisfies the Laplace equation in the fluid domain. The
6.2 Mathematical Model

Linearised kinematic-dynamic boundary condition on the free surface gives

$$\Phi'_{x'}, g\Phi'_{z'} = 0, \quad z' = 0, \quad (6.1)$$

where $g$ is the acceleration due to gravity. The no-flux condition at the sea bed gives

$$\Phi'_{z'} = 0, \quad z' = -h'. \quad (6.2)$$

And lastly the kinematic condition on the lateral surfaces of the flap is expressed as

$$\Phi'_{x'} = -\theta' (z' + h' - c') H(z' + h' - c'), \quad x' = 0 \pm \varepsilon', \varepsilon' \to 0, |y'| < \frac{w'}{2} \quad (6.3)$$

and that on the breakwater to be

$$\Phi'_{x'} = 0, \quad x' = -d' \pm \varepsilon', \varepsilon' \to 0, \quad |y'| < \frac{b'}{2} \quad (6.4)$$

using the thin plate approximation. Consider the following non-dimensional system of physical variables

$$(x, y, z, d, b) = \left(x', y', z', d', b'\right)/w', t = \sqrt{\frac{g}{w'} t'}, \Phi = \frac{\Phi'}{\sqrt{gw'A'}} \theta' = \varepsilon \theta, \quad (6.5)$$

where $\varepsilon = A'/w'$ is the small parameter for the problem. Separating out the time dependence out of the unknown system of variables with the assumption of simple harmonic oscillation gives

$$\begin{cases} \theta \\ \Phi \end{cases} = \text{Re} \left( \begin{cases} \Theta \\ \phi^I(x, y, z) + \phi^D(x, y, z) + V \phi^R(x, y, z) \end{cases} \right) e^{-i\omega t}, \quad (6.6)$$

where $V = i\omega \Theta$, $\omega = \omega' \sqrt{w'/g}$ and $\Theta$ are respectively, the angular frequency and amplitude of oscillation of the flap, while $\phi^I(x, y, z)$, $\phi^R(x, y, z)$ and $\phi^S(x, y, z)$ are the complex spatial incident, radiation and scattering potential, respectively. The incident wave potential is a known parameter for the problem (see (2.8)) where $\psi$ is the angle of incidence with
6.2 Mathematical Model

respect to the negative x axis as shown in figure 6.1. The unknown potentials can be further simplified by separating out the vertical dependence [see 49] in the form of normalised vertical eigenmodes $Z_n(z)$ (see (2.15)) with $\kappa_0 = k$, $\kappa_n = ik_n$ are the solutions to the dispersion relations (see (2.17)). Application of the factorisation (6.6) and (5.15) and using the orthogonality relation (2.16) yields a two dimensional boundary-value problem in terms of the spatial radiation and diffraction potentials, where the Laplace equation becomes the Helmholtz equation

$$ (\nabla^2 + \kappa_n^2) \begin{cases} \phi^R_n \\ \phi^D_n \end{cases} = 0, \quad (6.7) $$

and the kinematic condition on the flap (6.3) becomes

$$ \begin{cases} \phi^R_{n,x} \\ \phi^D_{n,x} \end{cases} = \begin{cases} f_n \\ A_1d_{1n}e^{ikysin\psi} \end{cases} \quad x = x_m \pm \epsilon, \epsilon \to 0, \quad -\frac{w}{2} < y < \frac{w}{2} \quad (6.8) $$

where

$$ d_{1n} = -\frac{k\cos\psi(h + \omega^{-2}\sinh^2kh)^{1/2}}{\sqrt{2}\omega \cosh kh} \delta_{0,n} \quad (6.9) $$

and $f_n$ (see (5.19)) are constants. Similarly the kinematic condition on the breakwater (6.4) yields

$$ \begin{cases} \phi^R_{n,x} \\ \phi^D_{n,x} \end{cases} = \begin{cases} 0 \\ A_1d_{2n}e^{ikysin\psi} \end{cases} \quad x = -d \pm \epsilon, \epsilon \to 0, \quad -\frac{b}{2} < |y| < \frac{b}{2} \quad (6.10) $$

with

$$ d_{2n} = -\frac{k\cos\psi(h + \omega^{-2}\sinh^2kh)^{1/2}}{\sqrt{2}\omega \cosh kh} (\cos(kd \cos\psi) + i \sin(kd \cos\psi)) \delta_{0,n}. \quad (6.11) $$

And lastly, both $\phi^R$ and $\phi^D$ are required to be outgoing disturbances of the wave field [49]. The above system of equations (6.7), (6.8) and (6.10) is then solved using the procedure outlined in [56] with the application of Green’s integral theorem (see Appendix for the solutions details). Finally solutions for the spatial radiation and diffraction potentials are
obtained as
\[
\phi^R(x, y, z) = -\frac{i}{8} \sum_{n=0}^{\infty} \kappa_n Z_n(z) w \sum_{p=0}^{P} \int_{-1}^{1} (1 - u^2)^{1/2} U_p(u) \left\{ a_{pn1} \frac{H_1^{(1)}(\kappa \sqrt{x^2 + (y - \frac{uw}{2})^2})}{\sqrt{x^2 + (y - \frac{uw}{2})^2}} + a_{pn2} \frac{H_1^{(1)}(\kappa \sqrt{(x + d)^2 + (y - \frac{ub}{2})^2})}{\sqrt{(x + d)^2 + (y - \frac{ub}{2})^2}} \right\} du,
\]

(6.12)

and
\[
\phi^D(x, y, z) = -\frac{i}{8} A_k Z_0(z) w \sum_{p=0}^{P} \int_{-1}^{1} (1 - u^2)^{1/2} U_p(u) \left\{ b_{p01} \frac{H_1^{(1)}(k \sqrt{x^2 + (y - \frac{uw}{2})^2})}{\sqrt{x^2 + (y - \frac{uw}{2})^2}} + b_{p02} \frac{H_1^{(1)}(k \sqrt{(x + d)^2 + (y - \frac{ub}{2})^2})}{\sqrt{(x + d)^2 + (y - \frac{ub}{2})^2}} \right\} du,
\]

(6.13)

respectively, where \( H_1^{(1)} \) is the Hankel function of the first kind and first order, \( U_p \) is the Chebyshev polynomial of the second kind and order \( p \), \( p = 0, 1, \ldots, P \in \mathbb{N} \), \( a_{pn1}, a_{pn2}, b_{p01} \) and \( b_{p02} \) are the complex solutions of a system of equations, which are solved numerically using a collocation scheme (see Appendix for details). The expressions for the radiation and diffraction potential given by (6.12) and (6.13) respectively have a similar form to solutions obtained for the same potentials in the other recently solved problems e.g. a flap in an open ocean [57], channel [56] and near a straight coast [71]. The above solutions for the potentials are then utilised to solve the body equation of motion of the flap.

### 6.2.1 Hydrodynamic Parameters

While the breakwater is fixed, the bottom hinged flap undergoes pitching motion about its mean vertical position due the pressure difference across its two sides. The non-
6.2 Mathematical Model

Fig. 6.2 Variation of the excitation torque versus the non-dimensional parameter $k'd'$ for $d' = 50m$. Comparison is made with the behavior of the OWSC near a straight coast for the same distance of separation (black solid line).

dimensional equation of motion of the flap is expressed as

$$[-\omega^2(I + \mu) + C - i\omega(v + v_{pto})]\Theta = F_1,$$  \hspace{1cm} (6.14)

where

$$\mu = \frac{\pi w}{4} Re \left\{ \sum_{n=0}^{\infty} f_n a_{0n1} \right\}, \hspace{0.5cm} v = \frac{\pi \omega w}{4} Im \left\{ \sum_{n=0}^{\infty} f_n a_{0n1} \right\}, \hspace{0.5cm} F_1 = -\frac{\pi \omega w}{4} iA_j b_{001} f_0$$  \hspace{1cm} (6.15)

are the added inertia, radiation damping and excitation torque due to the pitching motion of the flap respectively, while $I = l'/(\rho w^5)$ is the second moment of inertia of the flap, $C = C'/(\rho gw'^3)$ the coefficient of the net restoring flap buoyancy torque and $v_{pto} = v_{pto}'/(\rho w'^3 \sqrt{g/w'})$ the power take-off (PTO) damping coefficient. The performance of the device is quantified in terms of the capture factor $C_F$ (see equation 5.33), using the expressions for optimal power take-off damping $v_{pto}$ (see equation 5.32) and $P_{opt}$ (see equation 5.32).
6.3 Results

Fig. 6.3 Variation of the hydrodynamic parameters of the OWSC (a) added inertia $\mu'$ and (b) radiation damping $\nu'$ versus the non-dimensional parameter $k'd'$ for $d' = 50m$.

6.3 Results

The computations are performed with parameter values given in table 6.1. Figure 6.2

<table>
<thead>
<tr>
<th>$w'$</th>
<th>$c'$</th>
<th>$h'$</th>
<th>$A_f'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>26m</td>
<td>4m</td>
<td>13m</td>
<td>0.3m</td>
</tr>
</tbody>
</table>

plots the variation of excitation torque versus the non-dimensional parameter $k'd'$. For the case of the breakwater of width $2w'$, the variation of the excitation torque are marked by alternate maxima and minima as in the case of the straight coast problem. In the case of the straight coast the zeros in the excitation torque occur exactly at $k'd' = m\pi, m = 1, 2, ...$

The phenomenon was explained due to the formation of a 2D standing wave field which brings the flap to standstill at its vertical position.

However, with a breakwater, a 3D wave field will always be produced because of scattering due to the finite length effect. This would in turn modify the wave field around the flap, leading to a potential difference across its two sides and thereby producing a net torque. Nevertheless, as one increases the width of the breakwater, the effect of the edge generated scattered wave field would reduce and one would obtain similar behavior of the excitation torque as that in the straight coast. The minima in the excitation torque for breakwater problem occurs at frequencies which are little higher than the integral values of $k'd'/\pi$. 

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6.3 Results

Figure 6.4 shows the variation of the capture factor $C_F$ with the incident wave period $T'$ for various widths. The variation of the capture factor $C_F$ with the incident wave period is shown in figure 6.4 and compared with that of a single flap near a straight coast and one in an open ocean. For small breakwaters, the performance of the OWSC is only slightly perturbed from that in the open ocean. Increasing the width of the breakwater makes the performance behavior resemble more of that found in the straight coast problem.

So, in general we find that the hydrodynamic behavior of the OWSC with a small breakwater is similar to that in the open ocean, while larger breakwaters induce resonat-
ing effects that are observed in the straight coast problem. In fact this is what intuition would suggest. Now let us turn to the primary objective of this investigation. How well a breakwater of reasonable finite width is capable of reproducing performance features similar to that of the straight coast when the flap is located close to the breakwater. Figure 6.5 plots the variation of capture factor \( C_F \) versus the wave period \( T' \) for \( d' = 12 \text{m} \). For equal widths of the flap and the breakwater, a strong enhancement in the performance is seen at low periods in comparison to that in the open ocean. However at high periods, the levels of capture factor is less than in open ocean but qualitatively it maintains the consistency in performance observed in the later at high periods. As the breakwater width is enlarged, the bandwidth of high performance levels increases and in fact tends to that observed in the straight coast problem. The peak of the maximum capture factor associated with the straight coast problem is almost achieved with a breakwater of width twice that of the OWSC. Observing the trend of the performance behavior, it can be inferred that as the breakwater width is increased even further, the device response would resemble more closely the straight coast variation.

Oblique Waves: The dynamics of the OWSC is primarily governed by the diffraction phenomenon unlike a point absorber where radiation effects are dominant. Therefore it is expected that its performance would decrease in oblique waves and to be non-functioning for grazing wave incidence \((\psi = 90^\circ)\) as the excitation torque becomes zero. Previous studies have shown that the performance of the OWSC decreases almost as function of \( \cos^2 \psi \) in oblique waves [80]. However, the OWSC is a shallow water WEC where the waves are directional. And therefore it is expected to be located in regions with predominantly normal wave incidence or rather can be oriented along the most predominant direction. However, since the problem addressed here describes a modification to the original WEC configuration, a few computations are performed for near normal incidence to assess the possibility of any drastic changes in the behavior. Figure 6.6 shows the variation of the capture factor \( C_F \) for three different angles of incidence. Qualitatively the variation of the performance features remain the same with however, the magnitudes in capture factor dropping down. Comparing the rate of decrease in the peaks of the capture factor curve, a OWSC with a breakwater is marginally more sensitive to directional wave incidence than
6.4 Random Seas

The behavior of the system in random seas is investigated here. The analysis is performed for six of the total ten sea states that were identified by Aquamarine Power Ltd. to represent the annual wave climate at the European Marine Energy Centre wave site [13], and a methodology similar to that presented in [68] is adopted. Computation for the four sea states that are not performed here correspond to high sea states where the spectral components from large periods which cause excitation of body resonance of the flap are not negligible and may lead to significant error in the results [68]. Also, the exploitable wave energy as a percentage of the total wave energy incident would be significantly reduced in higher sea states than in low seas because of restrictions in the power take-off system, which act as a kind of performance inhibitor (see [81]). It is worth mentioning that the oc-
6.4 Random Seas

Fig. 6.6 Variation of the capture factor $C_F$ of the OWSC versus the incident wave period with $d' = 12m$ in case of oblique wave incidence (a) $\psi = 10^\circ$, (b) $\psi = 20^\circ$ and (c) $\psi = 30^\circ$ for various widths of the breakwater.

Fig. 6.7 plots the capture factor for six sea states considered. A breakwater of the same width as that of flap captures more power than that of a single flap in the open ocean with the exception of sea number 6. But definitely if the mentioned sea states were to

The Bretschneider spectrum for fully developed wind waves [29] has been used here and is given by the form

$$S(f) = 0.257H_{1/3}^2 T_{1/3}^{-4}f^{-5} \exp[-1.03(T_{1/3}f)^{-4}],$$

(6.16)

where is $S$ is the spectral density, $H_{1/3}$ is the significant wave height and $T_{1/3}$ is the significant wave period. The six different sea states considered in this analysis are described in Table 6.2.

Figure 6.7 plots the capture factor for six sea states considered. A breakwater of the same width as that of flap captures more power than that of a single flap in the open ocean with the exception of sea number 6. But definitely if the mentioned sea states were to

occurrence of the higher sea states (which are not analyzed here) in the annual wave climate representation are much smaller than those investigated here. And therefore the analysis would give a fair picture although slightly incomplete.
6.4 Random Seas

Fig. 6.7 Comparison of the capture factor $C_F$ of the OWSC in random seas, located at a distance $(d')$ of 12m from the breakwater for various widths of the breakwater and six different seas.

represent the complete wave climate for a particular location, the annual energy production would then be considerably higher for equal widths of the two structures. For breakwater of width $b = 1.5w$, higher capture factor is observed across all the seas analyzed, with $C_F > 2 \times (C_F)_{\text{OpenOcean}}$ for the first three seas. Capture factors of at least twice that in the open ocean across all the seas are seen for the breakwater of width twice that of the flap. In fact for sea number 3 the capture factor is almost three times that in the open ocean.

Table 6.2 Representation of six sea states

<table>
<thead>
<tr>
<th>Sea Number</th>
<th>$H_{1/3}$ (m)</th>
<th>$T_{1/3}$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6</td>
<td>6.1</td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
<td>6.1</td>
</tr>
<tr>
<td>3</td>
<td>1.0</td>
<td>7.9</td>
</tr>
<tr>
<td>4</td>
<td>1.9</td>
<td>9.0</td>
</tr>
<tr>
<td>5</td>
<td>1.1</td>
<td>9.8</td>
</tr>
<tr>
<td>6</td>
<td>2.0</td>
<td>10.5</td>
</tr>
</tbody>
</table>

The random sea analysis gives an insight into the behavior of such systems in realistic
6.5 Conclusion

sea conditions. The representation of the wave climate is strongly site specific. But definitely, the breakwater can be used as a performance enhancer for the OWSC at locations where the occurrence of the low seas are significantly predominant.

In comparison to the hybrid system presented in [73], the three-dimensional dynamics of the OWSC-breakwater system enables it to have much higher levels of efficiency than the former. Also, in the present system, the breakwater is utilised to enhance the performance of a well known WEC, and as such design modifications are not required.

6.5 Conclusion

The possibility of using a breakwater to enhance the performance of an OWSC is explored in this paper. A three-dimensional mathematical model is presented in the framework of linear potential theory to analyze the hydrodynamics and performance of the OWSC in such circumstances. Results show that when the OWSC is located very close to the breakwater, it is possible to significantly enhance the performance, with the absorption bandwidth depending on the width of the breakwater. The general behavior tends toward that observed in the case of a straight coast when the breakwater width is enlarged. As far as the hydrodynamic variations are concerned, spikes in the radiation parameters $\mu'$ (added inertia) and $\nu'$ (radiation damping) are observed near $k'd' = (m + 1/2)\pi, m = 1, 2, \ldots$, which are similar to the resonance observed in the straight coast problem. The excitation torque on the OWSC tend to zero for integral values of $k'd'/\pi$ as the breakwater width is enlarged which is explained due to a standing wave phenomenon in the straight coast problem. The random sea analysis reveals that this kind of system would be most effective for locations with higher occurrence of low sea states. From the perspective of economics, a fairly good enhancement in the performance of the OWSC can be obtained with a breakwater of width one and a half times the width of the OWSC. In a realistic situation, a breakwater can span over a long distance and an array of OWSC can be implemented in such a situation. It would be interesting to understand the effect of the mutual interactions between the OWSCs in presence of the breakwater. Such a topic could be a future research direction to this work.
7

Interactions between an Oscillating Wave Surge Converter and a Heaving Wave Energy Converter

7.1 Introduction

It is well recognised now that to make wave energy commercially viable, a large number of wave energy converters (WECs) must be deployed at designated sites. Wave farm devices like Oyster (developed by Aquamarine Power) are already planned at the Isle of Lewis in Scotland. The hydrodynamics of a large number of OWSC has been studied recently by [71], while [7] examined the interaction between a large number of Heaving Wave Energy

Collaborators: Emiliano Renzi, Frederic Dias
Converters (HWEC) using a numerical approach. However, the studies mentioned essentially comprise of devices of the same kind. In reality, circumstances might entail locating devices of more than one kind at nearby locations. It is also important to understand the effect of the mutual interactions in such cases. The dynamics of such systems has not been studied before, and in this paper we attempt to develop a mathematical model to address the problem. The complexity in modelling such systems is the contrasting geometry of the devices coupled with the different modes of motion they perform.

Wave interaction in an array of WECs has been studied by many starting with the seminal work of [9], followed by many others. In general it has been shown that the interactions in an array lead to regimes of constructive and destructive interference effects on the performance of the individual WECs and also on the array as a whole. The nature of such interactions depend strongly on the configuration of the array, the geometry of the systems and also on the incident wave frequency. [4] had investigated the effect of the interactions between two WECs of the same kind for large distances of separation. In this study, we are going to focus on a similar sparse distribution of two WECs but of different kind. However, the motivation is not just to see the impact of the separating distances, but also to understand the essence of the interactions given their disparate attributes. A three dimensional mathematical model is developed based on linear potential flow theory, which assumes the fluid to be inviscid and incompressible and the flow irrotational. The HWEC is modelled as a truncated vertical cylinder, while thin rigid-plate approximation of [39] has been utilized for the OWSC.

The isolated hydrodynamic characteristics of a truncated cylinder has been studied extensively in the literature. Garrett [28] was the first to analyze the diffraction problem using the series expansion of eigenfunctions of the velocity potential, while Yeung [84] extended the approach to obtain the solution to the radiation problem. There are also numerous studies which have investigated the interaction in an array of truncated cylinders. Most of them have employed the interaction theory of [34] which utilises the addition theorems of the Hankel and Bessel functions. The theory of [34] is in fact quite general and can be applied to arbitrary geometries provided their isolated diffraction solution is known.

Recent studies have analyzed the hydrodynamics of both - isolated and arrays of the
7.2 Mathematical Model

flap-type OWSC thoroughly. The method of the solution employs Green’s integral equation formulation and Green’s function yielding hypersingular integrals which are solved numerically via a collocation scheme, using the Chebyshev polynomials of the second kind. The mathematical approach followed in the latter is distinctly different from that adopted for truncated cylinders. In our model, we unify the above two methods and express the final solution as a combination of the isolated cases of the respective bodies. The net spatial potential is then required to satisfy the boundary conditions on the OWSC and HWEC simultaneously. While for the boundary condition on the OWSC, a numerical collocation scheme is used, the method of matched eigenfunctions is employed between the core and exterior region of the truncated cylinder. In the following sections, we first derive the theory for the interaction between the two systems. Then we perform a few case studies demonstrating the effect of the distance and location on the performance of the individual devices.

7.2 Mathematical Model

The OWSC and the HWEC are considered to be situated in an ocean of constant water depth $h'$. The global co-ordinate system coincides with the local co-ordinate system of the OWSC which is located at the center of the flap. The coordinates of the center of the HWEC with respect to the origin of the global coordinate system is denoted by $(x'_c, y'_c)$.
7.2 Mathematical Model

For convenience, we refer to $x'_c$ as the axial distance and $y'_c$ as the lateral distance between the center of the two WEC’s. The incoming waves are considered to be obliquely incident making an angle $\psi$ with the negative $x'$-axis. The fluid is considered to be inviscid and incompressible, and the flow irrotational. Therefore there exists a velocity potential $\Phi'$ which satisfies the Laplace equation in the fluid domain. The scalar potential also satisfies the linearised kinematic-dynamic free surface boundary condition

$$\Phi'_{x'} + g \Phi'_{z'} = 0, \quad z' = 0, \quad (7.1)$$

where $g$ is the acceleration due to gravity and the no-flux boundary condition on the sea bed

$$\Phi'_{z'} = 0, \quad z' = -h'. \quad (7.2)$$

The two devices perform different modes of motion yielding separate forms of the kinematic boundary condition on their respective surfaces. The OWSC is modelled using the thin-rigid plate approximation and the kinematic boundary condition on it yields

$$\Phi'_{x'} = -\theta_{x'}(z' + h' - c')H(z' + h' - c'), \quad x' = \pm \epsilon', \quad |y| < w/2, \quad (7.3)$$

and that on the bottom and lateral surface of the HWEC gives

$$\Phi'_{z'} = \zeta', \quad z' = -h' + b', \quad r' \leq a', \quad 0 < \vartheta < 2\pi \quad (7.4)$$

and

$$\Phi'_{r'} = 0, \quad -h' + b' < z' < 0, \quad r' = a', \quad 0 < \vartheta < 2\pi. \quad (7.5)$$

respectively. Note in (7.3), $H$ is the Heaviside step function and $\epsilon' \to 0$. The nondimensional system of variables is chosen as

$$(x, y, z, r) = (x', y', z', r')/w', \quad t = \sqrt{g \omega/\nu}, \quad \Phi = \Phi'_{\sqrt{g\omega A_I'}}, \quad \zeta = \zeta'/A_I', \quad \epsilon\theta = \theta' \quad (7.6)$$
7.2 Mathematical Model

where \( \varepsilon = (A_f/w') \) is the small parameter of the problem. Assuming the motion to be simple harmonic in nature, we obtain

\[
(\theta, \zeta) = \text{Re}\{(\Theta, \zeta_0)e^{-i\omega t}\}, \quad \Phi = \text{Re}\{\phi(x,y,z)e^{-i\omega t}\},
\]

(7.7)

where \( \omega = \sqrt{w'/g} \), \( \Theta \) and \( \zeta_0 \) are respectively, the angular frequency and amplitude of oscillation of the flap and HWEC, while \( \phi(x,y,z) \) is the complex spatial velocity potential. The spatial potential is further decomposed into

\[
\phi = \phi^S + \phi^R = (\phi^f + \phi^D) + (V_1\phi^{(R1)} + V_2\phi^{(R2)})
\]

(7.8)

where

\[
\phi^f = -\frac{iA_I\cosh k(z+h)}{\omega \cosh kh} e^{-ik(x \cos \psi + y \sin \psi)}
\]

\[
= -\frac{iA_I\cosh k(z+h)}{\omega \cosh kh} e^{-ik(x \cos \psi - y \sin \psi)} \sum_{l=-\infty}^{\infty} e^{il(\theta + \psi)} J_l(kr)(-i)^l.
\]

(7.9)

In (7.8), \( \phi^D \) is the diffracted wave potential, \( \phi^{(R1)} \) is the radiation potential induced by the motion of the flap when the HWEC is held fixed and vice-versa for \( \phi^{(R2)} \). Also in (7.8), \( V_1 = -i\omega \zeta_0 \) and \( V_2 = i\omega \Theta \) are respectively the complex angular velocity of the moving flap and the heave velocity of the HWEC. Note in (7.9), \( J_l \) is the Bessel function of the first kind. The most general form of the diffracted/radiated potential due to an isolated OWSC is given by

\[
\phi^{(R,D)}(x,y,z) = -\frac{i}{8} \sum_{n=0}^{\infty} \kappa_n x Z_n(z) \sum_{p=0}^{\infty} \beta_{pn}^{(R,D)} \int_{-1}^{1} (1-u^2)^{1/2} U_p(u) H_1^{(1)}\left(\kappa \sqrt{x^2 + (y - \frac{uw}{2})^2}\right) \frac{\sqrt{x^2 + (y - \frac{uw}{2})^2}}{\sqrt{\kappa^2 + \frac{(y - \frac{uw}{2})^2}}}
\]

(7.10)
and that due to an isolated HWEC in the exterior region is expressed as

\[
\phi^{(R,D)}(r, \vartheta, z) = \sum_{n=0}^{\infty} Z_n(z) \sum_{l=-\infty}^{\infty} \alpha_{nl}^{(R,D)} \frac{H_1^{(1)}(\kappa_n r)}{H_1^{(1)}(\kappa_n a)} e^{il\vartheta}.
\]  

(7.11)

Note in (7.10), \( U_p(u) \) is the Chebyshev polynomial of the second kind, \( H_1^{(1)} \) is the Hankel function of first kind and first order and \( \kappa_0 = k, \kappa_n = ik_n \) are the solutions to the dispersion relations (see (2.17)). Also, in (7.10) and (7.11), the superscript with comma denotes either of the two potentials (radiated/diffracted) and \( Z_n(z) \) are the normalised vertical eigenmodes (see (2.16)). Both the above potentials -(7.10) and (7.11)- automatically satisfy the free surface kinematic dynamic boundary condition, the no-flux boundary condition at the sea bed and the radiation condition at large distances away from the two bodies. Now the solution to the complete problem would require a linear combination of (7.10) and (7.11) to satisfy the kinematic boundary conditions on the two bodies simultaneously. Considering the two bodies to be spaced at a sufficient distance apart, it is reasonable to assume that the effect of the evanescent modes produced by one of the bodies would be absent or be negligible in the vicinity of the other body. So the total disturbed spatial potential near the OWSC can expressed as

\[
\phi^{(R,D)}(x, y, z) = -\frac{i}{8} \sum_{n=0}^{\infty} \kappa_n x Z_n(z) \sum_{p=0}^{P} \beta_p^{(R,D)} \int_{-1}^{1} (1 - u^2)^{1/2} \times
\]

\[
\frac{H_1^{(1)}(\kappa_n \sqrt{x^2 + (y - u w)^2})}{\sqrt{x^2 + (y - u w)^2}} du + Z_0(z) \sum_{l=-\infty}^{\infty} \alpha_{bl}^{(R,D)}
\]

\[
\times H_1^{(1)}(k \sqrt{(x - x_c)^2 + (y - y_c)^2}) e^{il\arctan((y-y_c)/(x-x_c))},
\]

(7.12)
7.2 Mathematical Model

and this potential has to satisfy the kinematic boundary condition on the OWSC. Similarly the net disturbed potential in the exterior region around the HWEC is given by

$$\phi(R, \theta, z) = -\frac{i}{8} k(x_c + r \cos \theta)Z_0(z) \sum_{p=0}^{P} \beta_{p0}^{(R,D)} \int_{-1}^{1} (1 - u^2)^{1/2}$$

\[
H_1^{(1)} \left( \frac{k}{(x_c + r \cos \theta)^2 + (y_c + r \sin \theta - \frac{uw}{2})^2} \right) \times U_p(u) \frac{1}{\sqrt{(x_c + r \cos \theta)^2 + (y_c + r \sin \theta - \frac{uw}{2})^2}}
\]

\[
+ \sum_{n=0}^{\infty} Z_n(z) \sum_{l=-\infty}^{\infty} \frac{\alpha_{nl}^{(R,D)} H_1^{(1)}(\kappa_n r)}{H_1^{(1)}(\kappa_n a)} e^{il\theta},
\]

which needs to satisfy the respective boundary and matching condition on and below the HWEC. Note that both potentials are expressed in the local co-ordinates of the nearest body. To obtain the final solution, there are three different problems that need to be solved separately - the scattering problem when both the bodies are held fixed in incoming waves, the R1 radiation problem when the HWEC is moving while the OWSC is at rest and finally the R2 radiation problem when the OWSC oscillates with the HWEC at rest. The solution to the three problems in the core region beneath the cylinder can be expressed as

\[
\begin{cases}
\phi_S^C \\
\phi_{R1}^C \\
\phi_{R2}^C
\end{cases} = \sum_{l=-\infty}^{\infty} e^{il\theta} \left[ \begin{cases}
\gamma_{10}^S \\
\gamma_{10}^{R1} \\
\gamma_{10}^{R2}
\end{cases} \right] + \frac{1}{2} \left( \frac{r}{a} \right)^{|l|} + \sum_{s=1}^{\infty} \left[ \begin{cases}
\gamma_{ls}^S \\
\gamma_{ls}^{R1} \\
\gamma_{ls}^{R2}
\end{cases} \right] \times \left( \frac{r s \pi}{b} \right) \cos \left( \frac{s \pi}{b} (z + h) \right),
\]

\[
\begin{cases}
I_l \left( \frac{rs \pi}{b} \right) \frac{1}{2b} \left( (z + h)^2 - \frac{a^2}{2} \right) \\
I_l \left( \frac{as \pi}{b} \right)
\end{cases}
\]

(7.14)

The solution to the undetermined coefficients in the expression for the velocity potentials is obtained using the kinematic boundary conditions on the OWSC and the matching conditions at the interface between the core and the exterior region at the edge of the HWEC. Let us look at the conditions separately.
7.2 Mathematical Model

7.2.1 Boundary Condition on the OWSC

The boundary condition on the OWSC requires

\[
\begin{align*}
\left\{ \begin{array}{l}
\phi_{E,x}^D \\
\phi_{E,x}^{R1} \\
\phi_{E,x}^{R2}
\end{array} \right. = \left\{ \begin{array}{l}
-\phi^l_x \\
0 \\
(z + h - c)H(z + h - c)
\end{array} \right. \quad x = \pm \epsilon, \quad |y| < w/2, \\
\end{align*}
\]

(7.15)

with \( \epsilon \to 0 \). Using the orthogonality of \( Z_n(z) \) over the range \(-h \leq z \leq 0\), we obtain

\[
\begin{align*}
\sum_{p=0}^P \left\{ \begin{array}{l}
\beta_{p0}^D \\
\beta_{p0}^{R1} \\
\beta_{p0}^{R2}
\end{array} \right\} & \left[ (p+1)U_p(v) - i k \int_{-1}^1 \frac{(1-u^2)^{1/2}U_p(u)}{|v-u|}R_n\left(\frac{1}{2}kw|v-u|\right)du \right] \\
+ \sum_{l=-\infty}^{\infty} \left\{ \begin{array}{l}
\alpha_{0l}^D \\
\alpha_{0l}^{R1} \\
\alpha_{0l}^{R2}
\end{array} \right\} & \frac{e^{il\varphi}}{H_{l}^{(1)}(ka)} \left[ -kx_c \frac{H_{l}^{(1)'}}{(x_c)^2 + (\frac{v_w}{2} - y_c)^2} - \right. \\
& \left. \frac{i\beta_{l}^{(1)}}{k} \left( \frac{\left(\frac{v_w}{2} - y_c\right)}{(x_c)^2 + (\frac{v_w}{2} - y_c)^2} \right) \right] = \left\{ \begin{array}{l}
A_l \frac{k\cos \psi}{\omega Z_0(0)} e^{ik(v/2)\sin \psi} \\
0 \\
f_0
\end{array} \right\} \\
\end{align*}
\]

(7.16)

and for \( n > 0 \)

\[
\begin{align*}
\sum_{p=0}^P \left\{ \begin{array}{l}
\beta_{pm}^D \\
\beta_{pm}^{R1} \\
\beta_{pm}^{R2}
\end{array} \right\} \left[ (p+1)U_p(v) - i k \int_{-1}^1 \frac{(1-u^2)^{1/2}U_p(u)}{|v-u|}R_n\left(\frac{1}{2}kw|v-u|\right)du \right] & = \left\{ \begin{array}{l}
0 \\
0 \\
f_n
\end{array} \right\}. \\
\end{align*}
\]

(7.17)

In (7.16) and (7.17),

\[
f_n = \frac{\sqrt{2} |\kappa_n(h - c) \sinh \kappa_n h + \cosh \kappa_n c - \cosh \kappa_n h|}{\kappa^2 (h + \omega^{-1/2} \sinh^2 \kappa_n h)^{1/2}}. \\
\]

(7.18)
7.2 Mathematical Model

7.2.2 Continuity of pressure at \( r = a \)

The matching condition of pressure between the core and the exterior region at \( r = a \) requires

\[
\begin{pmatrix}
\phi_E^D + \phi_r^l \\
\phi_E^{R1} \\
\phi_E^{R2}
\end{pmatrix} = \begin{pmatrix}
\phi_C^S \\
\phi_C^{R1} \\
\phi_C^{R2}
\end{pmatrix}, \quad r = a, \quad -h \leq z \leq -h + b. \tag{7.19}
\]

Using the orthogonality of \( e^{-i\theta} \) in \( 0 \leq \theta \leq 2\pi \) and of \( \cos((z+h)s\pi/b) \) over the range \( -h \leq z \leq -h + b \), we obtain

\[
\sum_{p=0}^{\infty} \left\{ \frac{\beta_p^D}{\beta_p^{R1}} \right\} \left[ -\frac{i}{8} k \int_0^{2\pi} (x_c + a \cos \theta) e^{-i\theta} \int_{-1}^{1} (1 - u^2)^{1/2} U_p(u) \frac{H_1^1(k\Psi)}{\Psi} du \int d\theta \right] C_{p0} + \\
\sum_{n=0}^{\infty} \left\{ \frac{\alpha_n^D}{\alpha_n^{R1}} \right\} \left( \frac{\gamma_{ls}^S}{\gamma_{ls}^{R1}} \gamma_{ls}^{R2} \right) b \pi = \begin{pmatrix}
\frac{iA_l}{\omega} e^{-ikx_c \cos \Psi + ikx_c \sin \Psi} 2\pi j_l(ka) (-i)^l \frac{e^{i\Psi}}{Z_0(0)} C_{s0} \\
\frac{\pi}{b} \int_{-h}^{-h+b} (z+h)^2 - \frac{a^2}{2} \cos \left( \frac{s\pi}{b} (z+h) \right) dz \delta_{il}
\end{pmatrix}
\]

where

\[
\Psi = \sqrt{(x_c + a \cos \theta)^2 + (y_c + a \sin \theta - u/2)^2}. \tag{7.21}
\]

and

\[
C_{sn} = \int_{-h}^{-h+b} Z_n(z) \cos \left( \frac{s\pi}{b} (z+h) \right) dz. \tag{7.22}
\]

7.2.3 Continuity of velocity at \( r = a \)

To ensure continuity of velocity between the core and the exterior region, we require

\[
\begin{pmatrix}
\phi_{E,r}^D + \phi_{r,r}^l \\
\phi_{E,r}^{R1} \\
\phi_{E,r}^{R2}
\end{pmatrix} = \begin{pmatrix}
\phi_{C,r}^S \\
\phi_{C,r}^{R1} \\
\phi_{C,r}^{R2}
\end{pmatrix}, \quad r = a, \quad -h \leq z \leq -h + b. \tag{7.23}
\]
and the boundary condition on the vertical sides of the truncated cylinder

$$\begin{bmatrix} \phi_{E,r}^D + \phi_{r}^l \\ \phi_{E,r}^R \\ \phi_{E,r}^R \end{bmatrix} = \begin{cases} 0, & r = a, \ -h + b \leq z \leq 0. \end{cases}$$ (7.24)

Again employing the orthogonality of $e^{-ilt}$ in $0 \leq \theta \leq 2\pi$ and of $Z_n(z)$ over the range $-h \leq z \leq 0$, we obtain for $n = 0$

$$\sum_{p=0}^{P} \left\{ \frac{\beta^D_{p0}}{\beta^R_{p0}} \right\} \frac{-ik}{8} \int_{0}^{2\pi} e^{-ilt} \cos \theta \int_{-1}^{1} (1 - u^2)^{1/2} H_1^{(1)}(k\gamma) \frac{U_p(u)}{Y} du - (x_c + a \cos \theta) \times$$

$$\int_{-1}^{1} (1 - u^2)^{1/2} U_p(u) \left( (x_c + a \cos \theta) \cos \theta + (y_c + a \sin \theta - u/2) \sin \theta \right) kH_2^{(1)}(k\gamma) \frac{U_p(u)}{Y^2} du \right] d\theta$$

$$+ \left\{ \begin{array}{c} \beta^D_{p0} \\ \beta^R_{p0} \end{array} \right\} \frac{2\pi k}{\pi} \left( \begin{array}{c} H_1^{(1)}(ka) \\ H_1^{(1)}(ka) \end{array} \right) \frac{\pi |l|}{a} C_{60} - \frac{2\pi}{\pi} \sum_{s=1}^{\infty} \left\{ \begin{array}{c} \gamma_{l0}^S \\ \gamma_{l0}^R \end{array} \right\} \frac{I_l(\frac{a\pi}{b})}{I_l(\frac{s\pi}{b})} (s\pi \frac{a}{b}) C_{60}$$

$$= \left\{ \begin{array}{c} 2\pi i A_{l} e^{-ikz_c \cos \psi + ikz_c \sin \psi} J_l(ka)(-i)^l e^{ill} \\ -\frac{\pi a}{b} C_{60} \\ 0 \end{array} \right\}$$ (7.25)

and for $n > 0$

$$-2\pi \sum_{s=1}^{\infty} \left\{ \begin{array}{c} \gamma_{l0}^S \\ \gamma_{l0}^R \end{array} \right\} \frac{I_l(\frac{a\pi}{b})}{I_l(\frac{s\pi}{b})} (s\pi \frac{a}{b}) C_{6n} = \left\{ \begin{array}{c} -\frac{\pi a}{b} C_{6n} \\ 0 \end{array} \right\}$$ (7.26)
7.2 Mathematical Model

The infinite system of equations are truncated with $N_1$ evanescent modes in the vicinity of the HWEC, $N_2$ evanescent modes near the OWSC, $S$ cores modes and $(2L+1)$ Fourier harmonics. It is worth noting that terms with $\hat{\beta}_{pn}^{(R,D)}$, $n > 0$ are not present in the system of equations (7.20) and (7.25) and can be independently determined from (7.17). As for the other terms $\hat{\beta}_{p0}^{(R,D)}$, $\alpha_{nl}^{(R,D)}$ and $\gamma_{ls}^{(R,D)}$ - form a $(P+1)+(2L+1)(N_2+1)+(2L+1)(S+1)$ coupled system of linear equations which are solved simultaneously. This differential distribution in the number of evanescent modes near each body is extremely useful in attaining fast convergence in results. So, while for the OWSC, $N_1 = 5$ is sufficient for convergence, that required for the HWEC is $N_2 = 50$.

7.2.4 Hydrodynamic Parameters

The solution for the velocity potential is then used to solve for the equation of motion of each of the individual bodies in the frequency domain. Let $M = M'/(\rho w'^3)$, $C_1 = C_1'/(\rho gw'^2)$ and $v_{pi}^{pt} = v_{pi}^{pt'}/(\rho w'^3\sqrt{g/w'})$ be the non-dimensional mass, coefficient of restoring force and power take-off damping of the HWEC respectively. The equation of
7.2 Mathematical Model

Fig. 7.3 Comparison of the mutual interaction terms (a) $\mu_{12}$ and $\mu_{21}$, (b) $\nu_{12}$ and $\nu_{21}$ versus incident wave period for $(x'_c,y'_c)=(200,0)m$.

motion of the HWEC can then be written as

$$
[-\omega^2(M+\mu_{11})+C_1-i\omega(\nu_{11}+\nu_{11}^{pr})]\zeta_0 - \omega^2\mu_{21} + i\omega\nu_{21}]\Theta = F_1. \tag{7.27}
$$

where

$$
F_1 = i\omega \pi a^2 \left[ \gamma_{00}^D \frac{I_1}{2} + \sum_{s=1}^{\infty} \gamma_{0s}^D \frac{I_1}{b} \left( \frac{as\pi}{b} \right) (-1)^s \right] \tag{7.28}
$$

is the heave excitation force and

$$
\mu_{11} = Re\left\{ p^{R1} \right\}, \quad \nu_{11} = Im\left\{ p^{R1} \right\}, \tag{7.29}
$$

are the added mass and radiation damping of the HWEC. In the equations above,

$$
p^{R1} = \pi a^2 \left[ \gamma_{00}^{R1} \frac{I_1}{2} + \sum_{s=1}^{\infty} \gamma_{0s}^{R1} \frac{I_1}{b} \left( \frac{as\pi}{b} \right) (-1)^s + \left( \frac{b}{2} - \frac{a^2}{8b} \right) \right]. \tag{7.30}
$$

Finally,

$$
\mu_{21} = Re\left\{ p^{R2} \right\}, \quad \nu_{21} = Im\left\{ p^{R2} \right\}, \tag{7.31}
$$
7.2 Mathematical Model

Fig. 7.4 Variation of the hydrodynamic parameters (a) added mass $\mu_{11}$, (b) radiation damping $\nu_{11}$ of the HWEC and (c) added inertia $\mu_{22}$, (d) radiation damping $\nu_{22}$ of the OWSC for $(x'_c, y'_c)=(200,0)$m.

where

$$p^{R2} = \pi a^2 \left[ \frac{\gamma R^2}{2} + \frac{2}{a} \sum_{s=1}^{\infty} \frac{I_1 \left( \frac{as\pi}{b} \right)}{I_0 \left( \frac{as\pi}{b} \right)} (-1)^s \right]. \tag{7.32}$$

Suppose $I = I'/(\rho w'^5)$ is the second moment of inertia of the OWSC and $C_2 = C_2'/(\rho gw'^4)$ is the coefficient of the flap restoring buoyancy torque. Then its non-dimensional equation of motion can be expressed as shown in [55]

$$[-\omega^2 (I + \mu_{22}) + C_2 - i\omega (\nu_{22} + \nu_{22}^{pti})] \Theta - [\omega^2 \mu_{12} + i\omega \nu_{12}] \zeta_0 = F_2. \tag{7.33}$$

In the latter, $v_2^{pti} = \frac{v_2^{pti}}{(\rho w'^5 \sqrt{g/w'})}$ is the power take-off (PTO) damping coefficient,

$$F_2 = -\frac{\pi \omega w'}{4} \frac{i A f \rho_{00}}{f_0} \tag{7.34}$$
7.2 Mathematical Model

Fig. 7.5 Behavior of the (a) excitation force on the HWEC versus \( k' a' \) and (b) excitation torque on the OWSC versus incident wave period \( T' \) for \((x'_c, y'_c)=(200,0)\) m.

is the excitation torque,

\[
\mu_{22} = \frac{\pi w}{4} Re \left\{ \sum_{n=0}^{\infty} f_n^w R_0^{R_2} \right\}, \quad \nu_{22} = \frac{\pi \omega w}{4} Im \left\{ \sum_{n=0}^{\infty} f_n^w \beta_{0n}^{R_2} \right\} \quad (7.35)
\]

is the added inertia due to the torque and radiation damping respectively, while

\[
\mu_{12} = \frac{\pi w}{4} Re \left\{ \sum_{n=0}^{\infty} f_n^w R_0^{R_1} \right\}, \quad \nu_{12} = \frac{\pi \omega w}{4} Im \left\{ \sum_{n=0}^{\infty} f_n^w \beta_{0n}^{R_1} \right\} \quad (7.36)
\]

are the mutual interaction terms. The power captured by each of the devices are expressed as

\[
P_1 = \frac{1}{2} \omega^2 v_{p1}^{\text{pto}} |\xi_0|^2, \quad P_2 = \frac{1}{2} \omega^2 v_{p2}^{\text{pto}} |\Theta|^2, \quad (7.37)
\]

where, \( v_{p1}^{\text{pto}} \) and \( v_{p2}^{\text{pto}} \) are optimized to capture maximum power for every monochromatic incident wave in the respective isolated cases of the WECs. In order to understand the effect of the interactions, we use the \( q^{\text{mod}} \) factor (see (2.31)). \( q^{\text{mod}} > 0 \) implies that the interactions are beneficial to the performance of the WEC i.e. it captures more power than in its isolated case for a similar wave condition, while \( q^{\text{mod}} < 0 \) signifies destructive interference effects.


7.3 Results

The computations were performed with an incident wave of amplitude $A'_i = 1m$ in an ocean of constant water depth $h' = 13m$. The physical dimensions of the two WECs are given in table 7.1, with $w'$ as the width of the OWSC. The physical parameters of the OWSC that resembles the device Oyster800, are obtained from Aquamarine Power and are held fixed throughout the study. Unless and otherwise mentioned, the system parameters used for the computations are that shown in table 7.1.

<table>
<thead>
<tr>
<th>$w'$</th>
<th>$c'$</th>
<th>$a'$</th>
<th>$b'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>26m</td>
<td>4m</td>
<td>8m</td>
<td>9m</td>
</tr>
</tbody>
</table>

The capture factor ($C_F$) of a WEC is defined as the ratio of the power captured by that particular device to the incident wave power across its width. Figure 7.2 plots the variation of the capture factor of the two WECs in their respective isolated cases. It is essential to understand that the primary mechanism of energy extraction by the two WECs is different. While the OWSC is a non-resonant device driven by excitation torque, the HWEC is a resonant system. In figure 7.2, the $C_F$ of the HWEC is shown for two cases - one in which the PTO strategy is same as that of the OWSC and the other with the PTO damping taken to be equal to that at resonance for all frequencies. However, in order to follow a consistent approach, we use the PTO damping strategy adopted for the OWSC in both the cases.

The results are compared with that of the respective isolated systems to understand the dynamics of the interaction. Figure 7.3 plots the mutual interaction terms between the WECs corresponding to the two radiation problems for $x'_c = 0m$ and $y'_c = 200m$. Note, the axial distance $x'_c$ between the centers of the two WECs is the shortest in this case amongst all the case studies that are performed here. Good agreement is observed in the variations of the mutual interaction terms which highlights the robustness of the mathematical model. Figure 7.4 and 7.5 plot the behavior of the hydrodynamic parameters of the two WECs respectively. The variations of the hydrodynamic variables are seen to be oscillatory around that in their respective isolated cases. Infact, similar form of such hydrodynamic behavior has been observed in the case of truncated cylinders [74], truncated elliptical cylinders [11]
or even for multiple flap-type systems [70].

A slight reduction in the excitation torque on the OWSC is observed across the entire range of incident wave periods (see figure 7.5b). The excitation torque on the flap-type WEC system is produced by the difference in potential across its two sides, which is maximum in case of normal wave incidence. So the general decrease in the excitation torque is because the disturbed wave field due to the presence of the HWEC in front of the OWSC doesn’t enable the generation of the same levels of potential difference as in the isolated case.
7.3 Results

To understand the effect of the interaction on the respective systems, we now consider the case when the HWEC is located right upfront of the OWSC i.e. the lateral separation \( y'_c = 0 \) m. Figure 7.6 plots the variation of \( q^{\text{mod}} \) of the two WECs for five cases. Among the two systems, the performance of the HWEC is significantly altered in presence of the OWSC. This can be due to several reasons. Firstly, in the scattering problem, the OWSC resembles a breakwater and act as a strong reflector of waves. Such reflections produces both - constructive and destructive interference effects depending upon the incident wave frequency and the distance of separation, and results in the pronounced oscillatory behavior of the HWEC. As a consequence, the variability in the excitation force on the HWEC is higher than that of the excitation torque on the OWSC (see e.g. figure 7.5). Also the fact that the OWSC is a dipole-mode radiator while the HWEC radiates waves equally in all directions, can contribute to such disparity. Another important thing to note is that the OWSC extends throughout the water depth, unlike the HWEC and therefore is more

Fig. 7.7 Variation of \( q^{\text{mod}} \) versus incident wave period when \((x'_c,y'_c)=(500,0)\) m for various angles of oblique wave incidence (a) \( \psi = 30^\circ \), (b) \( \psi = 45^\circ \), (c) \( \psi = 60^\circ \) and (d) \( \psi = 90^\circ \).
7.3 Results

Fig. 7.8 Variation of $q^{\text{mod}}$ versus incident wave period when $(x'_c,y'_c)=(1000,0)$ m for various angles of oblique wave incidence (a) $\psi = 30^\circ$, (b) $\psi = 45^\circ$, (c) $\psi = 60^\circ$ and (d) $\psi = 90^\circ$.

dominating in the disturbance that it creates. This is despite the fact that the displacement of the HWEC is larger than that of the OWSC. It can be seen from the $q^{\text{mod}}$ variations that the effects of the interactions on the OWSC are mostly felt at lower wave periods, while at higher periods it behaves unperturbed to the presence of the HWEC.

7.3.1 Oblique wave incidence

Let us now consider the case of oblique wave incidence. Figure 7.7 plots the variation of the $q^{\text{mod}}$ of the two WECs versus the incident wave period for four angles of oblique wave incidence when $(x'_c,y'_c)=(500,0)$ m. It can be observed that the $q^{\text{mod}}$ variations of the OWSC are significant and comparable to that of the HWEC. In fact for $\psi = 60^\circ$, the maximum of the $q^{\text{mod}}$ variations of the OWSC is much larger than that of the HWEC. This is because the OWSC is strongly sensitive to the angle of oblique wave incidence. In
7.3 Results

Fig. 7.9 Variation of $C_F$ of the HWEC versus wave period with $(x_c', y_c')=(500,0)m$ for various radius of the cylinder - (a) $a'=4m$, (b) $a'=6m$, (c) $a'=10m$ and (d) $a'=12m$.

fact in its isolated case with $\psi=90^\circ$, the OWSC doesn’t capture any power at all. Note, the leading order behavior of the power captured by the OWSC reduces as a function of $\cos^2 \psi$, in case of oblique wave incidence. This sensitive behavior is also visible through the interaction effect that it creates on the HWEC (see figure 7.7d). The amplitude of the variations induced by the OWSC on the HWEC reduces as the angle of incidence is increased. For complete oblique wave incidence ($\psi=90^\circ$), the HWEC behaves similar to that in its isolated case. Similar behavior for oblique wave incidence is also observed for $(x_c', y_c')=(1000,0)m$, with however reduced magnitudes of the peaks due to weakened interactions (see figure 7.8).

7.3.2 Influence of radius of HWEC

Figure 7.9 shows the variation of the $C_F$ of the HWEC versus the incident wave period for different radii of the cylinder. Note, the draft of the HWEC is held fixed at 4m for
Fig. 7.10 Variation of $q_{\text{mod}}$ versus incident wave period with $(x'_c, y'_c)=(500,0)$ m for various radius of the cylinder - (a) $a'=4\text{m}$, (b) $a'=6\text{m}$, (c) $a'=10\text{m}$ and (d) $a'=12\text{m}$.

all the cases shown in figure 7.9. As the radius of the HWEC is increased, there is a reduction in the peak of $C_F$ of the HWEC accompanied by a widening of the bandwidth and a shift in the peak towards higher periods. Also, one can notice that at higher periods, the effect of the interactions are enhanced for larger radii of the cylinder. The variation of the $q_{\text{mod}}$ factor of the two WECs for the same cases are plotted in figure 7.10. There is an increase in the interactions between the two devices which is manifested by higher magnitude of the amplitude in $q_{\text{mod}}$ variations as the radial dimension of the HWEC is enhanced. Interestingly, for the OWSC it is the destructive interference effects at low periods which are more pronounced and is in fact largest for $a'=12\text{m}$. So on increasing the radius of the cylinder, not only does it exert more influence on the OWSC but also it experiences larger variations. Note, the dimensions of the OWSC are kept constant for all the cases that are considered here.
Fig. 7.11 Variation of $q^{\text{mod}}$ versus incident wave period with $(x'_c, y'_c)=(500,0)$m for various drafts of the cylinder - (a) $b'_1 = 11$m, (b) $b'_2 = 10$m, (c) $b'_3 = 8$m and (d) $b'_4 = 7$m. Note, the draft of the HWEC is expressed as $h' - b'$.

### 7.3.3 Influence of draft of HWEC

The variation of $q^{\text{mod}}$ of the WECs is plotted versus time period $T'$ in figure 7.11 for four different values of the draft of the HWEC. In fact the maxima of the $q^{\text{mod}}$ variations of the HWEC are similar in all the cases that are presented. Interestingly, with larger drafts, there is an increase in the maximum of the $q^{\text{mod}}$ factor of the OWSC. Even in this case, the effect of the interactions has more impact on the performance of the HWEC across the range of drafts that are considered. However, the influence on the HWEC is not as severe as is in the case of radius alteration.

### 7.4 Conclusion

The hydrodynamic interaction between an pitching flap-type wave energy converter and heaving truncated cylinder used for power extraction is investigated. A semi-analytical
7.4 Conclusion

A mathematical model is described which combines the isolated disturbed potentials of the two diverse geometrical systems performing different modes of motion. In the approach adopted in this paper, the interaction between the two bodies by their evanescent modes is neglected. One of the advantages of the model is that - in the near field of a particular body, the evanescent modes generated due to its own presence are retained while the interaction takes place only through the progressive wave mode. The method enables the use of appropriate number of vertical eigenmodes in the locality of a particular body which can vary from that of another body. The variations in the performance features of the HWEC due to the interactions are found to be stronger than in the case of the OWSC, although the HWEC has a much larger displacement. Also from the case studies performed, it is found that the interaction effects on the HWEC’s performance are more sensitive to its radius than its draft. It would be interesting to see how an offshore wavefarm of HWECs affects a nearshore wavefarm comprising of OWSCs. Such an analysis could be a topic of future research.
Future Direction

The OWSC is a shallow water device where the waves are mostly directional. However, it would be worth extending the mathematical model to analyze the behavior of the device in short crested seas and investigate the effect of directionality. The semi-analytical approach is based on linear theory which can give good estimates of the performance and general hydrodynamics of the device. However, the real sea and the behavior of the OWSC is nonlinear, and the assumptions of linearity fails in extreme wave conditions. The theory presented in this work may be extended to higher orders, to account for the effects of nonlinearity.

The development of the Oyster device is still at an early stage. Improvements in the design are investigated which can possibly reduce the wave loads, without compromising on the performance of the device. The new modular OWSC concept analyzed in Chapter 4 could be a possible alternative. However, more detailed investigation of such a system using computational fluid dynamic tools needs to be undertaken.

Another important topic that requires investigation is to analyze the effect of the wave energy farm on the wave climate. Assessment of environmental impacts is necessary as a part of the planning process so as to anticipate the consequences of proposed projects on wave energy. The wave models can first be used to compute the undisturbed wave field at the location of the WECs. The mathematical model presented in Chapter 2 can then be used to determine the wave perturbation created by an array of WECs. The modified wave energy flux is to be introduced along a wave generation line in the wave model, which encloses all the WECs. The flux can be calculated by depth integration of the pressure and
velocity terms at locations along the wave generation line.

There is scope for improvement in the array optimization problem presented in Chapter 3, with the general methodology remaining the same. The future of the work would be to perform the optimization using a three-dimensional wave spectrum and so the simulations of the mathematical model would include the effects of directionality. A more accurate description of the performance of the individual devices may be produced by taking into account the interactions of more WECs (although their effects are mostly negligible). That would involve considering a 5-WEC cluster to model WECs well inside the array (2 WECs on each side instead of 1 as for 3-WECs system), 4-WEC cluster to predict the performance of WECs just beside the ones at the edges and a 3-WEC cluster for prediction of the edge WECs. Such an analysis would obviously require much computational resources, time and increase in complexity of machine learning algorithm due to the increase in size of the input parameter space. The methodology can also be extended in determining the optimal layouts of other WECs e.g. the devices which capture energy by heaving motion. Such devices which are located offshore are not constrained by bathymetry variations, although different limitations can exist. One difference in modelling the statistical emulator for the offshore array is that the neighbouring WECs can be located at all possible directions with respect to a particular WEC, with larger distances of separation between them. For such a case, a different simplified interaction approach for predicting the performance of a particular device in the array needs to be considered.

In Chapter 6, it is shown that a breakwater of finite width can significantly enhance the performance of the OWSC. However, in a realistic situation, a breakwater can span over a long distance and an array of OWSC can be implemented in such a situation. It would be interesting to understand the effect of the mutual interactions between the OWSCs in presence of the breakwater.
References


References


References


References


References


Expansion of torque terms in the equation of motion

The three terms inside the first brackets within the integration in equation (1.35) - $z$, $\varepsilon \Phi_t$ and $\varepsilon^2 \frac{1}{2} |\nabla \Phi|^2$ are evaluated separately. The derivation is similar to that performed in [62]. The torque due to the fluid pressure $T_p$ is broken down into six components

$$T_p = T_p^{a1} + T_p^{a2} + T_p^{b1} + T_p^{b2} + T_p^{c1} + T_p^{c2},$$  \hspace{1cm} (A.1)

where $T_p^{a1}$, $T_p^{b1}$ and $T_p^{c1}$ represent the integration of the terms with $z$, $\varepsilon \Phi_t$ and $\varepsilon^2 \frac{1}{2} |\nabla \Phi|^2$ between $(-h+c)$ to 0, respectively, and $T_p^{a2}$, $T_p^{b2}$ and $T_p^{c2}$ represent the integration of the same terms between 0 and the free surface.
Torque due to the term $z$

Integration of the first term from $-h+c$ to 0 gives

$$T_{p1} = - \int_{-w/2}^{w/2} \int_{-h+c}^{0} z \left[ (z+h-c) - a \varepsilon \theta + (z+h-c) \varepsilon^2 \theta^2 - \frac{5}{6} a \varepsilon^3 \theta^3 \right] dzdy$$

$$+ \int_{-w/2}^{w/2} \int_{-h+c}^{0} z \left[ (z+h-c) + a \varepsilon \theta + (z+h-c) \varepsilon^2 \theta^2 + \frac{5}{6} a \varepsilon^3 \theta^3 \right] dzdy$$

$$= -w \left( \varepsilon a (h-c)^2 \theta + \varepsilon^3 \frac{5}{6} a (h-c)^2 \theta^3 \right) \quad (A.2)$$

while integration from 0 to the free surface simplifies as

$$T_{p2} = - \int_{-w/2}^{w/2} \int_{0}^{\varepsilon \zeta} z \left[ (z+h-c) - a \varepsilon \theta + (z+h-c) \varepsilon^2 \theta^2 - \frac{5}{6} a \varepsilon^3 \theta^3 \right] dzdy$$

$$+ \int_{-w/2}^{w/2} \int_{0}^{\varepsilon \zeta} z \left[ (z+h-c) + a \varepsilon \theta + (z+h-c) \varepsilon^2 \theta^2 + \frac{5}{6} a \varepsilon^3 \theta^3 \right] dzdy$$

$$= \int_{-w/2}^{w/2} \left\{ - \frac{\varepsilon^3}{3} (\zeta^+ - \zeta^-) - \frac{\varepsilon^2}{2} (h-c)(\zeta^+ - \zeta^-) + \frac{\varepsilon^3}{2} a \theta (\zeta^+ - \zeta^-) \right\} dy$$

$$\quad (A.3)$$

The free surface displacement $\zeta^\pm$ can be expanded for small $\varepsilon$

$$\zeta^\pm = \zeta^\pm(\zeta^\pm, y, t) = \zeta^\pm \left( \pm a - (\zeta^\pm + h-c) \varepsilon \theta \pm \frac{a}{2} \varepsilon^2 \theta^2, y, t \right) \quad (A.4)$$
and this expression can be repeatedly Taylor expanded to give

\[
\zeta^\pm = [\zeta^\pm]_{x=\pm a} + [\zeta^\pm]_{x=\pm a} \left( - (\zeta^\pm + h - c)\epsilon \theta \pm \frac{a}{2} \epsilon^2 \theta^2 \right) \\
+ [\zeta^\pm]_{x=\pm a} \frac{1}{2} \left( - (\zeta^\pm + h - c)\epsilon \theta \right)^2 + O(\epsilon^3) \\
= [\zeta^\pm]_{x=\pm a} + [\zeta^\pm]_{x=\pm a} \left( - (\zeta^\pm)_{x=\pm a} + [\zeta^\pm]_{x=\pm a} (- (h + c)\epsilon \theta) + h - c)\epsilon \theta \pm \frac{a}{2} \epsilon^2 \theta^2 \right) \\
+ [\zeta^\pm]_{x=\pm a} \frac{1}{2} \left( - ([\zeta^\pm]_{x=\pm a} + h - c)\epsilon \theta \right)^2 + O(\epsilon^3) \\
= [\zeta^\pm]_{x=\pm a} - \epsilon \left\{ (h - c) \theta [\zeta^\pm \xi^\pm]_{x=\pm a} \right\} + \epsilon^2 \left\{ \pm \frac{a}{2} \theta^2 [\zeta^\pm]_{x=\pm a} + (h - c) \theta^2 \right\} \\
[\zeta^\pm]_{x=\pm a} + \frac{\theta^2 (h - c)^2}{2} [\zeta^\pm]_{x=\pm a} + \theta^2 (h - c) [\zeta^\pm \xi^\pm]_{x=\pm a}. \\
(A.5)
\]

On substitution of this expression (A.5) into (A.3) yields

\[
T_{p2} = \int_{-w/2}^{w/2} \left\{ \frac{-e^3}{3} (\zeta^+ - \zeta^-) - \frac{e^2}{2} (h - c)(\zeta^+ - \zeta^-) + \frac{e^3}{2} a \theta (\zeta^+ - \zeta^-) \right\} dy \\
= \int_{-w/2}^{w/2} \left\{ \frac{-e^3}{3} ([\zeta^+]_{x=a} - [\zeta^-]_{x=-a}) - \frac{e^2}{2} (h - c) \left( ([\zeta^+]_{x=a} - [\zeta^-]_{x=-a}) \right) \\
- 2 \epsilon (h - c) \theta \left( [\zeta^+ \xi^+]_{x=a} - [\zeta^+ \xi^+]_{x=-a} \right) - 2 \epsilon \theta \left( [\zeta^+ \xi^+]_{x=a} \right) \\
- \left([\zeta^+ \xi^+]_{x=-a} \right) \right\} \frac{\epsilon^2}{2} a \theta \left( [\zeta^+]_{x=a} - [\zeta^-]_{x=-a} \right) \right\} dy. \\
(A.6)
\]

Using the following abbreviations

\[
\Delta(f) = [(f)^+]_{x=a} - [(f)^-]_{x=-a}, \\
\overline{(f)} = \frac{1}{2} \left\{ [(f)^+]_{x=a} + [(f)^-]_{x=-a} \right\}
\]

and neglecting the second last term within big brackets as \( \zeta \ll (h - c) \), (A.6) finally becomes

\[
T_{p2} = \int_{-w/2}^{w/2} \left\{ -\frac{e^3(h - c)}{2} - \frac{e^2}{2} (h - c)^2 \theta \Delta(\zeta \xi) \right\} dy. \\
(A.7)
\]
Torque due to the term $\varepsilon \Phi_i$

The term $T_{p1}^b$ in which the integration is performed between $-h + c$ to 0 can be written as

$$T_{p1}^b = -\int_{-w/2}^{w/2} \int_{-h+c}^{0} \varepsilon \Phi_i \left( (z + h - c) - a \varepsilon \theta + (z + h - c) \varepsilon^2 \theta^2 - \frac{5}{6} a \varepsilon^3 \theta^3 \right) dz dy$$

$$+ \int_{-w/2}^{w/2} \int_{-h+c}^{0} \varepsilon \Phi_i \left( (z + h - c) + a \varepsilon \theta + (z + h - c) \varepsilon^2 \theta^2 + \frac{5}{6} a \varepsilon^3 \theta^3 \right) dz dy$$

(A.8)

Taylor expansion of $\varepsilon \Phi_i$ about $x = \pm a$

$$\varepsilon \Phi_i^\pm = \varepsilon \left\{ \Phi_i^\pm |_{x=\pm a} + \Phi_i^\pm |_{x=\pm a} \left( -(z + h - c) \varepsilon \theta \pm \frac{a}{2} \varepsilon^2 \theta^2 \right) + \frac{1}{2} \Phi_{xx}^\pm |_{x=\pm a} -(z + h - c) \varepsilon \theta \right\}$$

$$= \varepsilon \Phi_i^\pm |_{x=\pm a} - \varepsilon^2 (z + h - c) \theta \Phi_{xx}^\pm |_{x=\pm a} + \varepsilon^3 \left\{ \pm \frac{a}{2} \varepsilon^2 \Phi_{xx}^\pm |_{x=\pm a} + \frac{(z + h - c)^2 \theta^2}{2} \Phi_{xx}^\pm |_{x=\pm a} \right\}$$

(A.9)

and substituting it in A.8 gives

$$T_{p1}^b = -\int_{-w/2}^{w/2} \int_{-h+c}^{0} \left\{ \varepsilon \Phi_i^+ |_{x=a} - \varepsilon^2 (z + h - c) \theta \Phi_{xx}^+ |_{x=a} + \varepsilon^3 \frac{a}{2} \varepsilon^2 \Phi_{xx}^+ |_{x=a} + \frac{\varepsilon^3 (z + h - c)^2 \theta^2}{2} \Phi_{xx}^+ |_{x=a} \right\} \times \left[ (z + h - c) - a \varepsilon \theta + (z + h - c) \varepsilon^2 \theta^2 \right] dz dy +$$

$$\int_{-w/2}^{w/2} \int_{-h+c}^{0} \left\{ \varepsilon \Phi_i^- |_{x=-a} - \varepsilon^2 (z + h - c) \theta \Phi_{xx}^- |_{x=-a} + \varepsilon^3 \frac{a}{2} \varepsilon^2 \Phi_{xx}^- |_{x=-a} + \frac{\varepsilon^3 (z + h - c)^2 \theta^2}{2} \Phi_{xx}^- |_{x=-a} \right\} \times \left[ (z + h - c) + a \varepsilon \theta + (z + h - c) \varepsilon^2 \theta^2 \right] dz dy$$

(A.10)

which on simplification finally becomes

$$T_{p1}^b = \int_{-w/2}^{w/2} \int_{-h+c}^{0} \left\{ \varepsilon \Delta \Phi_i (z + h - c) + \varepsilon^2 \left[ 2a \theta \Phi_i + (z + h - c)^2 \theta \Delta \Phi_{xx} \right] - \varepsilon^3 \left[ (z + h - c) \theta^2 \Delta \Phi_i + 3(z + h - c) a \theta^2 \Phi_{xx} + \frac{(z + h - c)^3}{2} \theta^2 \Delta \Phi_{xx} \right] \right\} dz dy$$

(A.11)
The other term $T_p^{b2}$, which involves integration between 0 to $\varepsilon \zeta^\pm$ is

$$
T_p^{b2} = -\int_{-w/2}^{w/2} \int_0^{\varepsilon \zeta^+} \varepsilon \Phi^+_x [(z + h - c) - a \varepsilon \theta + (z + h - c) \varepsilon^2 \theta^2] dz dy +
\int_{-w/2}^{w/2} \int_0^{\varepsilon \zeta^+} \varepsilon \Phi^-_x [(z + h - c) + a \varepsilon \theta + (z + h - c) \varepsilon^2 \theta^2] dz dy
$$

(A.12)

The limits of the above integral in $z$ are $O(\varepsilon)$, therefore it is sufficient to retain terms up to $O(\varepsilon^2)$ in the Taylor’s expansion of $\Phi_t$

$$
T_p^{b2} = -\int_{-w/2}^{w/2} \int_0^{\varepsilon \zeta^+} [\varepsilon [\Phi^+_t] x=a - \varepsilon^2 (z + h - c) \theta [\Phi^+_x] x=a] [(z + h - c)
- a \varepsilon \theta + (z + h - c) \varepsilon^2 \theta^2] dz dy +
\int_{-w/2}^{w/2} \int_0^{\varepsilon \zeta^+} [\varepsilon [\Phi^-_t] x=-a - \varepsilon^2 (z + h - c) \theta [\Phi^-_x] x=-a] [(z + h - c)
+ a \varepsilon \theta + (z + h - c) \varepsilon^2 \theta^2] dz dy
$$

(A.13)

Because of the $O(\varepsilon)$ dimension of the interval of integration in $z$, the terms $[\Phi^+_t] x=\pm a$ and $[\Phi^-_x] x=\pm a$ can in turn be Taylor expanded in powers of $z$ about $z = 0$ and then integrated from 0 to $\varepsilon \zeta^\pm$. Performing the Taylor’s expansion, neglecting the higher order terms and integrating in $z$ gives

$$
T_p^{b2} = -\int_{-w/2}^{w/2} \left\{ \frac{\varepsilon^3}{2} [\Phi^+_t]_0 [\zeta^+ x=a] + \varepsilon^2 [\Phi^+_t]_0 (h - c) [\zeta^+] x=a
- \varepsilon^3 [\Phi^+_t]_0 [\zeta^+ x=a] + \varepsilon^3 [\Phi^+_t]_0 [\zeta^+ x=a] (h - c) \theta
+ \varepsilon^3 [\Phi^+_t]_0 [\zeta^+ x=a] (h - c) - \varepsilon^3 [\Phi^+_t]_0 [\zeta^+ x=a] \theta - \varepsilon^3 [\Phi^+_t]_0 [\zeta^+ x=a] \theta (h - c)^2 \right\} dy
+ \int_{-w/2}^{w/2} \left\{ \frac{\varepsilon^3}{2} [\Phi^-_t]_0 [\zeta^- x=-a] + \varepsilon^2 [\Phi^-_t]_0 (h - c) [\zeta^- x=-a
- \varepsilon^3 [\Phi^-_t]_0 [\zeta^- x=-a] + \varepsilon^3 [\Phi^-_t]_0 [\zeta^- x=-a] (h - c) \theta
+ \varepsilon^3 [\Phi^-_t]_0 [\zeta^- x=-a] (h - c) + \varepsilon^3 [\Phi^-_t]_0 [\zeta^- x=-a] \theta
- \varepsilon^3 [\Phi^-_t]_0 [\zeta^- x=-a] (h - c)^2 \right\} dy
$$

(A.14)
which finally becomes

\[ T_p^{b2} = -\int_{-w/2}^{w/2} \left\{ -\varepsilon^2 (h - c) \Delta_0 (\Phi_t \zeta) + \varepsilon^3 \left[ -\frac{1}{2} \Delta_0 (\Phi_t \zeta^2) + (h - c)^2 \theta \Delta_0 (\Phi_t \zeta)_x + 2a \theta (\Phi_t \zeta)_0 - \frac{(h - c)}{2} \Delta_0 (\Phi_t \zeta^2) \right] \right\} dy \]  

(A.15)

**Torque due to the term**  \( \varepsilon^2 \frac{1}{2} |\nabla \Phi|^2 \)**

The expressions for \( T_p^c_1 \) and \( T_p^c_2 \) are given by

\[ T_p^{c1} = \int_{-w/2}^{w/2} \int_{-h+c}^{0} \left\{ -\varepsilon^2 \frac{1}{2} \Delta |\nabla \Phi|^2 (z + h - c) + \varepsilon^3 \left[ a \theta |\nabla \Phi|^2 + \frac{1}{2} (z + h - c)^2 \theta \Delta |\nabla \Phi|^2_{ix} \right] \right\} dz dy \]  

(A.16)

and

\[ T_p^{c2} = \int_{-w/2}^{w/2} \varepsilon^3 \frac{(h - c)}{2} \Delta_0 (|\nabla \Phi|^2 \zeta) dy \]  

(A.17)

respectively. The total torque due to fluid pressure (see equation A.1) is then expressed as

\[ T_p = -\varepsilon aw (h - c)^2 \theta - \varepsilon^3 \frac{5}{6} aw (h - c)^2 \theta^3 \]

\[ + \int_{-w/2}^{w/2} \left\{ -\varepsilon^2 \frac{(h - c)}{2} \Delta \zeta^2 - \varepsilon^3 \left( \frac{\Delta \zeta^3}{3} - a \theta \zeta^2 - (h - c)^2 \theta \Delta (\zeta \zeta_x) \right) \right\} dy \]

\[ + \int_{-w/2}^{w/2} \int_{-h+c}^{0} \left\{ -\varepsilon \Delta \Phi_t (z + h - c) + \varepsilon^2 \left( 2a \theta \Phi_t + (z + h - c)^2 \theta \Delta \Phi_t \right) \right. \]

\[ - \varepsilon^3 \left( (z + h - c) \theta^2 \Delta \Phi_t + 3 (z + h - c) a \theta^2 \Phi_t x + \frac{(z + h - c)^3}{2} \theta^2 \Delta \Phi_t \right) \right\} dz dy \]

\[ + \int_{-w/2}^{w/2} \left\{ -\varepsilon^2 (h - c) \Delta_0 (\Phi_t \zeta) \varepsilon^3 \left( -\frac{1}{2} \Delta_0 (\Phi_t \zeta^2) + (h - c)^2 \theta \Delta_0 (\Phi_t \zeta)_x \right. \]

\[ + 2a \theta (\Phi_t \zeta)_0 \frac{(h - c)}{2} \Delta_0 (\Phi_t \zeta^2) \right) \right\} dy + \int_{-w/2}^{w/2} \int_{-h+c}^{0} \left\{ -\varepsilon^2 \frac{1}{2} \Delta |\nabla \Phi|^2 (z + h - c) \right. \]

\[ + \varepsilon^3 \left( a \theta |\nabla \Phi|^2 + \frac{1}{2} (z + h - c)^2 \theta \Delta |\nabla \Phi|^2_{ix} \right) \right\} dz dy + \int_{-w/2}^{w/2} \varepsilon^3 \frac{(h - c)}{2} \Delta_0 (|\nabla \Phi|^2 \zeta) dy, \]

(A.18)
The procedure to obtain the solution to the 2D spatial diffraction and radiation potential is described in this section. Consider the 2D Green’s function
\[
G_n(x, y; \xi, \eta) = \frac{1}{4i} H_0^{(1)}(\kappa_n \sqrt{(x-\xi)^2 + (y-\eta)^2}),
\]
which satisfies the system of equations
\[
(\nabla^2 + \kappa_n^2)G_n = 0, \quad G_n = \frac{1}{2\pi} \ln r \quad \text{as} \quad r \to 0,
\]
where \( r = \sqrt{(x-\xi)^2 + (y-\eta)^2} \). Applying Green’s integral theorem to \( \phi_n \) and \( G_n \) for the whole fluid domain yields
\[
\varphi_n(x, y) = -\frac{i}{4} \sum_{m=1}^{M} \int_{y_m}^{y_m^H} \Delta \varphi_{nm} G_n^{(0)} \bigg|_{\xi=x_m} \, d\eta,
\]
where \( \Delta \varphi_{nm} \) is the difference in potential between the source and field points.
where $\Delta \phi_{nm} = \phi_n(x_m - \varepsilon, y) - \phi_n(x_m + \varepsilon, y)$ denotes the modal potential difference across the two sides of flap $m$. Applying the 2D spatial potential on the kinematic boundary conditions on the flaps, gives (see [55])

$$
\int_{y_n^A}^{y_n^B} \left\{ \frac{\Delta \phi_n^{(\beta)}}{\Delta \phi_n^{(\alpha)}} \right\} H_1^{(1)} \left( \kappa_n |y - \eta| \right) \frac{\kappa_n d\eta}{|y - \eta|} + \sum_{\gamma=1}^{M} \int_{y_n^\gamma}^{y_n^B} \left\{ \frac{\Delta \phi_n^{(\beta)}}{\Delta \phi_n^{(\gamma)}} \right\} \frac{-\kappa_n}{(x_\alpha - x_\gamma)^2 + (y - \eta)^2} \left[ \kappa_n (x_\alpha - x_\gamma)^2 \right] \left( H_2^{(1)} \left( \kappa_n \sqrt{(x_\alpha - x_\gamma)^2 + (y - \eta)^2} \right) - \frac{(y - \eta)^2}{\sqrt{(x_\alpha - x_\gamma)^2 + (y - \eta)^2}} \right) \frac{H_1^{(1)} \left( \kappa_n \sqrt{(x_\alpha - x_\gamma)^2 + (y - \eta)^2} \right)}{\kappa_n \sqrt{(x_\alpha - x_\gamma)^2 + (y - \eta)^2}} d\eta = 4i \left\{ \frac{f_{n\beta} \delta_{\alpha\beta}}{A_{n\alpha}^{(\alpha)} e^{iky \sin \psi}} \right\}, \tag{B.4}
$$

where $\delta$ is a Hadamard finite-part integral. Let $y_n^C = (y_n^A + y_n^B)/2$ denote the $y$ coordinate of the center of flap $m$, $m \in [1, M]$. Making the following change of variables

$$
\begin{align*}
\frac{u}{w_m} &= 2(\eta - y_n^C), & \frac{v}{w_m} &= 2(y - y_n^C), \\
\left\{ \begin{array}{c}
P_n^{(\beta)}(u) \\ Q_n^{(\alpha)}(u)
\end{array} \right\} &= \left\{ \begin{array}{c}
\Delta \phi_n^{(\beta)} \\ \Delta \phi_n^{(\alpha)}
\end{array} \right\}, \tag{B.5}
\end{align*}
$$

yields

$$
\int_{-1}^{1} \left\{ \begin{array}{c}
P_n^{(\beta)}(u) \\ Q_n^{(\alpha)}(u)
\end{array} \right\} H_1^{(1)} \left( \frac{\kappa_n w_{\alpha} v_{\alpha} - u}{2 |v_{\alpha} - u|} \right) \frac{\kappa_n du}{|v_{\alpha} - u|} + \sum_{\gamma=1}^{M} \int_{-1}^{1} \left\{ \begin{array}{c}
P_n^{(\gamma)}(u) \\ Q_n^{(\gamma)}(u)
\end{array} \right\} \frac{-\kappa_n 2w_{\gamma}}{4(x_\alpha - x_\gamma)^2 + w_{\gamma}(v_{\gamma} - u)^2} \left[ \kappa_n (x_\alpha - x_\gamma)^2 \right] \left( H_2^{(1)} \left( \frac{\kappa_n}{2} \sqrt{(x_\alpha - x_\gamma)^2 + w_{\gamma}(v_{\gamma} - u)^2} \right) - \frac{w_{\gamma}^2(v_{\gamma} - u)^2}{2 \sqrt{(x_\alpha - x_\gamma)^2 + w_{\gamma}^2(v_{\gamma} - u)^2}} \right) \times \frac{H_1^{(1)} \left( \frac{\kappa_n}{2} \sqrt{4(x_\alpha - x_\gamma)^2 + w_{\gamma}^2(v_{\gamma} - u)^2} \right)}{\kappa_n \sqrt{4(x_\alpha - x_\gamma)^2 + w_{\gamma}^2(v_{\gamma} - u)^2}} d\eta = 4i \left\{ \frac{f_{n\beta} \delta_{\alpha\beta}}{A_{n\alpha}^{(\alpha)} e^{iky \sin \psi}} \right\}, \tag{B.6}
$$

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Now
\[ H_1^{(1)} \left( \frac{\kappa_n w_\alpha |v_\alpha - u|}{2} \right) = \frac{4}{i\pi \kappa_n w_\alpha |v_\alpha - u|} + R_n \left( \frac{\kappa_n w_\alpha |v_\alpha - u|}{2} \right), \]  \hspace{1cm} (B.7)

where
\[ R_n(z) = J_1(z) \left[ 1 + \frac{2i}{\pi} \left( \ln \frac{z}{2} + \chi \right) \right] - \frac{i}{\pi} \left[ \frac{z}{2} + \sum_{j=2}^{\infty} (-1)^{j+1} (z/2)^{2j-1} \left( \frac{1}{j} + \sum_{q=1}^{j-1} \frac{1}{q} \right) \right] \]  \hspace{1cm} (B.8)
is the remainder, \( J_1 \) is the Bessel function of first kind and first order, and \( \chi = 0.577215 \cdots \) is the Euler constant [56]. Expanding the unknown jumps in potential across the two sides of the flap as
\[
\begin{bmatrix}
P_{nm}(u) \\
Q_{nm}(u)
\end{bmatrix} = (1 - u^2)^{1/2} \sum_{p=0}^{\infty} \begin{bmatrix}
a_{pnm}^{(\beta)} \\
b_{pnm}^{(\beta)}
\end{bmatrix} U_p(u),
\]  \hspace{1cm} (B.9)

where \( a_{pnm}^{(\beta)} \) and \( b_{pnm}^{(\beta)} \) are unknown complex coefficients to be determined and \( U_p(u) \) is the Chebyshev polynomial of second kind, finally gives
\[
\sum_{p=0}^{\infty} \begin{bmatrix}
a_{pnm}^{(\beta)} \\
b_{pnm}^{(\beta)}
\end{bmatrix} C_{np}(v_\alpha) + \sum_{\gamma=1, \gamma \neq \alpha}^{M} \begin{bmatrix}
a_{p\gamma\alpha}^{(\beta)} \\
b_{p\gamma\alpha}^{(\beta)}
\end{bmatrix} D_{np}(v_\alpha) = -i\pi w_\alpha \begin{bmatrix}
a_n^{(\alpha)} e^{ik(v_\alpha w_\alpha/2 + y_C)} \sin \psi
\end{bmatrix} f_{n\beta} \delta_{\alpha \beta}
\]  \hspace{1cm} (B.10)

where
\[ C_{pnm} = -\pi(p + 1) U_p(v_\alpha) + \frac{i\pi \kappa_n w_\alpha}{4} \int_{-1}^{1} (1 - u^2)^{1/2} U_p(u) \frac{R_n \left( \frac{1}{2} \kappa_n |v_\alpha - u| \right)}{|v_\alpha - u|} du \]  \hspace{1cm} (B.11)
\[ D_{\rho n\alpha\gamma} = -\frac{i\pi \kappa_n w_\alpha}{4} \int_{-1}^{1} \frac{(1 - u^2)^{1/2} U_p(u) 2w_\gamma}{4(x_\alpha - x_\gamma)^2 + (v_\alpha w_\alpha + 2y_\alpha^C - 2y_\gamma^C - w_\gamma u)^2} \]

\[ (x_\alpha - x_\gamma)^2 \left\{ \frac{\kappa_n H_2^{(1)} (\frac{\kappa_n}{2} \sqrt{4(x_\alpha - x_\gamma)^2 + (v_\alpha w_\alpha + 2y_\alpha^C - 2y_\gamma^C - w_\gamma u)^2})}{2H_1^{(1)} (\frac{\kappa_n}{2} \sqrt{4(x_\alpha - x_\gamma)^2 + (v_\alpha w_\alpha + 2y_\alpha^C - 2y_\gamma^C - w_\gamma u)^2})} \right\} \]

\[ \frac{2H_1^{(1)} (\frac{\kappa_n}{2} \sqrt{4(x_\alpha - x_\gamma)^2 + (v_\alpha w_\alpha + 2y_\alpha^C - 2y_\gamma^C - w_\gamma u)^2})}{2 \sqrt{4(x_\alpha - x_\gamma)^2 + (v_\alpha w_\alpha + 2y_\alpha^C - 2y_\gamma^C - w_\gamma u)^2}} \]

\[ H_1^{(1)} (\frac{\kappa_n}{2} \sqrt{4(x_\alpha - x_\gamma)^2 + (v_\alpha w_\alpha + 2y_\alpha^C - 2y_\gamma^C - w_\gamma u)^2}) \] \[ du \quad (B.12) \]

\[ v_\alpha = \frac{\cos(2q + 1)\pi}{2P + 2}, \quad q = 0, 1, 2, \ldots, P. \quad (B.13) \]

In the case of an inline array configuration, \( x_\alpha = x_\beta \) and the term \( D_{\rho n} \) indeed reduces to

\[ D_{\rho n\alpha\gamma} = \frac{i\pi \kappa_n w_\alpha w_\gamma}{4} \int_{-1}^{1} (1 - u^2)^{1/2} U_p(u) \frac{H_1^{(1)} (\frac{\kappa_n}{2} |v_\alpha w_\alpha + 2y_\alpha^C - 2y_\gamma^C - uw_\gamma|)}{|v_\alpha w_\alpha + 2y_\alpha^C - 2y_\gamma^C - uw_\gamma|} du \quad (B.14) \]

and correspond to the term (A.12) of [55]. Further generalising it for normal incidence reduces the system of equations (B.10) exactly to (A.10) of [55].
Consider the matching condition for the continuity of potential across the interface between the interior and exterior regions (4.41) and the boundary condition on the edge of the module for the scattering problem (4.43). Applying the orthogonality of $\cos(s\pi/c(z+h))$ in the domain $-h < z < -h + c$ and that of $e^{i\xi_0}$ in the domain $0 < \xi_0 < 2\pi$ yields

$$\gamma^{(D)}_{0ls} = \sum_{n=0}^{\infty} (\alpha_{0ln}^{(D)} + \beta_{0ln}^{(I)}) \left( \frac{2}{c} C_{sn} \right)$$  \hspace{1cm} (C.1)

where

$$C_{sn} = \begin{cases} \frac{\sqrt{2}}{(h + \omega^{-2} \sinh^2 \kappa_n h)^{1/2}} & s = 0, \\ \frac{\kappa_n \sinh(\kappa_n c)}{(h + \omega^{-2} \sinh^2 \kappa_n h)^{1/2}} & s \geq 0. \end{cases}$$  \hspace{1cm} (C.2)
Again using the orthogonality of \( Z_n(z) \) and \( e^{i\xi_0} \) on the matching condition for velocity at the interface (4.42), yields

\[
\alpha_{0l}^{(D)} \left( \frac{H_l^{(1)'}(\kappa_n a)}{H_l^{(1)}(\kappa_n a)} \kappa_n \right) + \beta_{0l}^{(l)} \left( \frac{J_l'(\kappa_n a)}{J_l(\kappa_n a)} \kappa_n \right) =
\]

\[
\sum_{s=1}^{\infty} \gamma_{j_{ij}}^{(D)} \left( \frac{|l|}{2\alpha} C_{0n} \right) + \sum_{s=1}^{\infty} \gamma_{j_{ij}}^{(D)} \left( \frac{I_l}{I_j} \frac{(as\pi)}{c} \frac{(s\pi)}{c} C_{sn} \right)
\]

(C.3)

The two above equations (C.1) and (E.3) can then be written in a matrix form as

\[
\begin{align*}
\gamma_{0l}^{(D)} &= M^1 \alpha_{0l}^{(D)} + M^1 \beta_{0l}^{(l)} \\
M^2_{ij} \alpha_{0l}^{(D)} + M^3_{ij} \beta_{0l}^{(l)} &= M^4_{ij} \gamma_{0l}^{(D)}
\end{align*}
\]

(C.4) (C.5)

where

\[
M^1(s, n) = \left( \frac{2}{c} C_{sn} \right)
\]

(C.6)

\[
M^2_{ij}(n_1, n_2) = \begin{cases} 
\frac{H_l^{(1)'}(\kappa_n a)}{H_l^{(1)}(\kappa_n a)} \kappa_n & \text{if } n_1 = n_2, \\
0 & \text{if } n_1 \neq n_2.
\end{cases}
\]

(C.7)

\[
M^3_{ij}(n_1, n_2) = \begin{cases} 
\frac{J_l'(\kappa_n a)}{J_l(\kappa_n a)} \kappa_n & \text{if } n_1 = n_2, \\
0 & \text{if } n_1 \neq n_2.
\end{cases}
\]

(C.8)
and

\[
M_I^1(n,s) = \begin{cases} 
    \frac{|l|}{2a} C_{\beta n} & \text{if } s = 0, \\
    \frac{I_l}{I_l} \left( \frac{d\pi}{c} \right) \left( \frac{s\pi}{c} \right) C_{\beta n} & \text{if } s > 0.
\end{cases} \tag{C.9}
\]

Solving the simultaneous equations we obtain the expressions for the unknown vectors \( \gamma^{(D)}_{0l} \) and \( \alpha^{(D)}_{0l} \) as follows,

\[
\gamma^{(D)}_{0l} = \mathcal{C}_l \beta^{(I)}_{0l}, \tag{C.10}
\]

\[
\alpha^{(D)}_{0l} = \mathcal{B}_l \beta^{(I)}_{0l} \tag{C.11}
\]

where

\[
\mathcal{C}_l = \left( I - M^1 (M^2 I)^{-1} (M^1 - M^1 (M^2 I)^{-1} M^4) \right), \tag{C.12}
\]

\[
\mathcal{B}_l = (M^2 I)^{-1} (M^3 \mathcal{C}_l - M^4). \tag{C.13}
\]

For the radiation problem, the incident wave field is absent and the matching conditions at the interface (4.41)-(4.42) and boundary condition (4.44) yield

\[
M^1 \alpha^{(R)}_{0l} = \gamma^{(R)}_{0l} + \gamma^*_0, \tag{C.14}
\]

\[
M^2 I \alpha^{(R)}_{0l} = M_I^r \gamma^{(R)}_{0l} + \alpha^*_0. \tag{C.15}
\]

where

\[
\gamma^*_0 = \begin{cases} 
    - \left( \frac{a^3}{6} - \frac{a}{8c} \right) \delta_{l|l|} & \text{if } s = 0, \\
    - \frac{ac}{\pi^2 s^2} (-1)^s \delta_{l|l|} & \text{if } s > 0,
\end{cases} \tag{C.16}
\]
and
\[
\alpha_{0l}^* = \left(-\frac{1}{4c} \int_0^{h+c} \left[ (z+h)^2 - \frac{3a^2}{4} \right] Z_n(z)dz + \frac{1}{2} \int_{-h+c}^0 (z+h-c)Z_n(z)dz \right) \delta_{|l|}.
\] (C.17)

Again solving the simultaneous equations, we obtain the unknown vectors

\[
\begin{align*}
\gamma_{0l}^{(R)} &= \mathcal{E}_l, \quad \text{(C.18)} \\
\alpha_{0l}^{(R)} &= \mathcal{D}_l \quad \text{(C.19)}
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{E}_l &= (I - M^1(M_1^2)M_1^3)^{-1}(M^1(M_1^2)^{-1} \alpha_{0l}^* - \gamma_{0l}^{(R)}), \quad \text{(C.20)} \\
\mathcal{D}_l &= (M_1^2)^{-1}(M_1^3 \mathcal{E}_l + \alpha_{0l}^{(R)}) \quad \text{(C.21)}
\end{align*}
\]
Using the solution to the isolated case, the solution to the system of multiple modules can be expressed in a matrix form as

\[
\begin{bmatrix}
\alpha^{(D)} \\
\gamma^{(D)}
\end{bmatrix} = \begin{bmatrix}
\mathbf{B}_{\text{array}}(\beta^{(D)} + \beta^{(I)}) \\
\mathbf{C}_{\text{array}}(\beta^{(D)} + \beta^{(I)})
\end{bmatrix}
\]  
(D.1)

for the diffraction problem and

\[
\begin{bmatrix}
\alpha^{(m)} \\
\gamma^{(m)}
\end{bmatrix} = \begin{bmatrix}
\mathbf{B}_{\text{array}}\beta^{(m)} + \mathbf{D}_{m} \\
\mathbf{C}_{\text{array}}\beta^{(m)} + \mathbf{E}_{m}
\end{bmatrix}
\]  
(D.2)

for the radiation problem. In (D.1) and (D.2), the matrices \(\mathbf{B}_{\text{array}}, \mathbf{C}_{\text{array}}, \mathbf{D}_{\text{array}}\) and \(\mathbf{E}_{\text{array}}\) are defined as follows:

\(\mathbf{B}_{\text{array}}\) is a square matrix of size \(M(2L+1)(N+1) \times M(2L+1)(N+1)\). If \(j_1, l_1, n_1\)
are the $j, l, n$ index of $\alpha_{jln}$ and $j_2, l_2, n_2$ are that of $\beta_{jln}$, then the matrix $\mathcal{B}_{array}$ can be expressed as

$$
\mathcal{B}_{array} = \begin{cases} 
\mathcal{B}_{l_1}(n_1, n_2) & \text{if } j_1 = j_2 \text{ and } l_1 = l_2, \\
0 & \text{else}.
\end{cases}
$$

(D.3)

$\mathcal{C}_{array}$ is a rectangular matrix of size $M(2L+1)(S+1) \times M(2L+1)(N+1)$. If $j_1, l_1, s_1$ are the $j, l, s$ index of $\gamma_{jls}$ and $j_2, l_2, n_2$ are that of $\beta_{jln}$, then the matrix $\mathcal{C}_{array}$ can be expressed as

$$
\mathcal{C}_{array} = \begin{cases} 
\mathcal{C}_{l_1}(s_1, n_2) & \text{if } j_1 = j_2 \text{ and } l_1 = l_2, \\
0 & \text{else}.
\end{cases}
$$

(D.4)

For the $m$th radiation problem, $\mathcal{D}_{m}$ is a column vector comprising of $M(2L+1)(N+1)$ elements and is given by

$$
\mathcal{D}_{m} = \begin{cases} 
\mathcal{D}_{l}(n) & \text{if } j = m, \\
0 & \text{else}.
\end{cases}
$$

(D.5)

While, $\mathcal{E}_{m}$ is a column vector comprising of $M(2L+1)(S+1)$ elements and is written as

$$
\mathcal{E}_{m} = \begin{cases} 
\mathcal{E}_{l}(s) & \text{if } j = m, \\
0 & \text{else},
\end{cases}
$$

where the infinite system of the modes in the interior region, $s$ is terminated at $S$. Solving the simultaneous equations (D.1) and (D.2), the expressions for the unknown coefficients are finally obtained as

$$
\begin{bmatrix}
\alpha^{(D)} \\
\gamma^{(D)}
\end{bmatrix}
= \begin{bmatrix}
(\mathcal{I} - \mathcal{B}_{array} \mathcal{T})^{-1}
\mathcal{B}_{array} \\
\mathcal{C}_{array} (\mathcal{T} (\mathcal{I} - \mathcal{B}_{array} \mathcal{T})^{-1} + \mathcal{I})
\end{bmatrix}
\mathcal{B}^{(l)}
$$

(D.7)
and

\[
\begin{pmatrix}
\alpha^{(m)} \\
\gamma^{(m)}
\end{pmatrix} = \begin{pmatrix}
(I - \mathcal{B}_{array} \mathcal{F})^{-1} \mathcal{D}^{array} \\
\mathcal{C}_{array} \mathcal{F} (I - \mathcal{B}_{array} \mathcal{F})^{-1} \mathcal{D}^{array} + \mathcal{C}^{array}
\end{pmatrix}
\]  
(D.8)

where \( \mathcal{F} \) is the coordinate transfer matrix while relates the incident wave coefficients \( \beta^{(D,m)} \) to \( \alpha^{(D,m)} \) using the relation (4.21). \( \mathcal{F} \) is a square matrix of size \( M(2L+1)(N+1) \times M(2L+1)(N+1) \). If \( j_1, l_1, n_1 \) are the \( j, l, n \) index of \( \beta_{jln} \) and \( j_2, l_2, n_2 \) are that of \( \alpha_{jln} \), then the matrix \( \mathcal{F} \) can be expressed as

\[
\mathcal{F} = \begin{cases}
T_{j_1,j_2,l_1,l_2,n_1} & \text{if } j_1 \neq j_2 \text{ and } n_1 = n_2, \\
0 & \text{else},
\end{cases}
\]  
(D.9)

where \( T_{j_1,j_2,l_1,l_2,n_1} \) is defined in (4.22).
The Green function for the semi infinite fluid domain $D = \{(x,y) : x > 0, -\infty < y < +\infty \}$ can be defined as

$$G_n(x,y;\xi,\eta) = -\frac{i}{4} \left[ G_n^{(0)}(x,y;\xi,\eta) + G_n^{(1)}(x,y;\xi,\eta) \right]$$ (E.1)

where

$$G_n^{(0)}(x,y;\xi,\eta) = H_0^{(1)}(\kappa_n \sqrt{(x-\xi)^2 + (y-\eta)^2})$$ (E.2)

and

$$G_n^{(1)}(x,y;\xi,\eta) = H_0^{(1)}(\kappa_n \sqrt{(x+\xi)^2 + (y-\eta)^2})$$ (E.3)
\[ \xi > 0, -\infty < \eta < +\infty. \] Application of the Green Integral theorem to the 2D radiation and diffraction potentials \( \varphi_n^{(R,D)} \) and \( G_n \) in the semi infinite fluid domain \( D \) [see 47] yields

\[
\varphi_n^{(R,D)}(x,y) = -\frac{i}{4} \int_{-\infty}^{\infty} \Delta \varphi_n^{(R,D)}[G_n^0(\xi) + G_n^1(\xi)] \left|_{\xi = d_c} \right. d\eta,
\] (E.4)

which is similar in form to expression (B7) of [56], but with different Green functions. Substituting (E.2) and (E.3) into (E.4), making the change of variables \( u = 2\eta, v = 2y, \)

\[ u, v \in (-1, 1), \]

and then decomposing the integral (E.4) into a Hadamard finite part integral [see 39] and a convergent integral gives

\[
\kappa_n \int P_n(u) \left( \frac{H_1^{(1)}(\frac{\kappa_n w}{|v-u|})}{|v-u|} - \frac{H_1^{(1)}(\frac{\kappa_n w}{|v-u|})}{\sqrt{16d_c^2 + (v-u)^2}} \right) du + 
\]

\[ \kappa_n \int_{-1}^{1} \left\{ \begin{array}{l}
P_n(u) \\
Q_n(u)
\end{array} \right\} L_n(u) du = 4i \left\{ \begin{array}{l}
V f_n \\
A_1 d_n
\end{array} \right\}. \] (E.6)

In the latter

\[
L_n(u) = \left( \frac{2}{16d_c^2 + (v-u)^2} \right) \times \left[ \frac{4d_c^2 \kappa_n H_1^{(1)}(\frac{\kappa_n}{2} \sqrt{16d_c^2 + (v-u)^2})}{\sqrt{16d_c^2 + (v-u)^2}} - \frac{8d_c^2 H_1^{(1)}(\frac{\kappa_n}{2} \sqrt{16d_c^2 + (v-u)^2})}{\sqrt{16d_c^2 + (v-u)^2}} - \frac{(v-u)^2}{2\sqrt{16d_c^2 + (v-u)^2}} H_1^{(1)}(\frac{\kappa_n}{2} \sqrt{16d_c^2 + (v-u)^2}) \right], \] (E.7)

where the property \( H_1^{(1)}(z) = -H_2^{(1)}(z) + H_1^{(1)}(z)/z \) has been also used [see 30]. Now expand the Hankel function in the Hadamard integral of (E.6) as

\[
H_1^{(1)}\left( \frac{1}{2} \frac{\kappa_n w}{|v-u|} \right) = \frac{4}{i\pi} \frac{1}{\kappa_n w |v-u|} + R_n\left( \frac{1}{2} \frac{\kappa_n w}{|v-u|} \right), \] (E.8)

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where

\[ R_n(\alpha) = J_1(\alpha) \left[ 1 + \frac{2i}{\pi} \left( \frac{\alpha}{2} + \gamma \right) \right] - \frac{i}{\pi} \left[ \frac{\alpha}{2} + \sum_{j=2}^{+\infty} \frac{(-1)^{j+1} (\alpha/2)^{2j-1}}{j!(j-1)!} \left( \frac{1}{j} + \sum_{q=1}^{j-2} \frac{2}{q} \right) \right] \]  

(E.9)

is the remainder, \( J_1(\alpha) \) is the Bessel function of first kind and first order, and \( \gamma = 0.577215 \ldots \) is the Euler constant [see 30]. Then substitute (E.8) into (E.6) to obtain

\[
\int \left\{ \frac{P_n(u)}{Q_n(u)} \right\} \frac{1}{\kappa_n w |v-u|^2} du + \frac{i\pi \kappa_n w}{4} \int_{-1}^{1} \left\{ \frac{P_n(u)}{Q_n(u)} \right\} \times \left[ L(u) + \frac{R_n(\frac{1}{2}\kappa_n w |v-u|)}{|v-u|} \right] du = -\pi w \left\{ \begin{array}{l} V f_n \\ A_I d_n \end{array} \right\},
\]

(E.10)

where \( P_n, Q_n \) are, respectively, the jump in radiation and diffraction potentials across the flap in the new variables. The structure of (E.10) suggests to seek a solution of the form

\[
\left\{ \begin{array}{l} P_n(u) \\ Q_n(u) \end{array} \right\} = \left\{ \begin{array}{l} V \\ A_I \end{array} \right\} (1-u^2)^{1/2} \sum_{p=0}^{+\infty} \left\{ \begin{array}{l} \alpha_{pn} \\ \beta_{pn} \end{array} \right\} U_p(u),
\]

(E.11)

where the \( \alpha_{pn} \) and \( \beta_{pn} \) are unknown complex constants [56]. Substituting (E.11) into the hypersingular integral equation (E.10) yields

\[
\sum_{p=0}^{+\infty} \left\{ \begin{array}{l} \alpha_{pn} \\ \beta_{pn} \end{array} \right\} C_{pn}(v) = -\pi \left\{ \begin{array}{l} f_n \\ d_n \end{array} \right\}
\]

(E.12)

where

\[
C_{pn} = -\pi (p+1) U_p(v) + \frac{i\pi \kappa_n}{4} \int_{-1}^{1} (1-u^2)^{1/2} U_p(u) \left[ \frac{R_n(\frac{1}{2}\kappa_n |v-u|)}{|v-u|} + L_n(u) \right] du,
\]

(E.13)

with \( v \in (-1,1) \). In order to solve it numerically, the linear system (E.12) can be truncated to a finite number of terms, evaluated at

\[
v = v_{0j} = \cos \left( \frac{2j+1}{2P+2} \pi \right), \quad j = 0, 1, \ldots, P,
\]

(E.14)
for fast numerical convergence. This produces two \((P + 1) \times (P + 1)\) truncated algebraic systems for each \(n\), which are finally solved to determine \(\alpha_{pn}\) and \(\beta_{pn}\). This, together with (E.11), (E.5), (E.4) and (5.15) allows to calculate the complex spatial potentials (5.21) and (5.22).
The 2D Green’s function

\[ G_n(x, y; \xi, \eta) = \frac{1}{4i} H_0^{(1)}(\kappa_n \sqrt{(x - \xi)^2 + (y - \eta)^2}), \]  

satisfies the system of equations

\[ (\nabla^2 + \kappa_n^2)G_n = 0, \quad G_n = \frac{1}{2\pi} \ln r \quad \text{as} \quad r \to 0, \]

where \( r = \sqrt{(x - \xi)^2 + (y - \eta)^2} \). Application of Green’s integral theorem to \( \varphi_n \) and \( G_n \) for the whole fluid domain yields

\[ \varphi_n(x, y) = \int_{-w/2}^{w/2} \Delta \varphi_{n1} G_{n, \xi} \bigg|_{\xi=0} \, d\eta + \int_{-b/2}^{b/2} \Delta \varphi_{n2} G_{n, \xi} \bigg|_{\xi=-d} \, d\eta, \]

(F.1)
where $\Delta \phi_{n1} = \phi_n(0 - \varepsilon, y) - \phi_n(0 + \varepsilon, y)$ and $\Delta \phi_{n2} = \phi_n(-d - \varepsilon, y) - \phi_n(-d + \varepsilon, y)$ denotes the modal potential difference across the two sides of flap and breakwater respectively. Application of the 2D spatial potential on the kinematic boundary conditions on the flaps gives

$$
\int_{-w/2}^{w/2} \left\{ \frac{\Delta \phi_{Rn1}}{\Delta \phi_{Dn1}} \right\} \frac{H_1^{(1)}(\kappa_n|y-\eta|)}{|y-\eta|} \kappa_n d\eta + \int_{-b/2}^{b/2} \left\{ \frac{\Delta \phi_{Rn2}}{\Delta \phi_{Dn2}} \right\} \frac{-\kappa_n}{d^2 + (y-\eta)^2} d\eta
$$

$$
\kappa_n d^2 \left\{ H_2^{(1)}(\kappa_n \sqrt{d^2 + (y-\eta)^2}) - \frac{H_1^{(1)}(\kappa_n \sqrt{d^2 + (y-\eta)^2})}{\kappa_n \sqrt{d^2 + (y-\eta)^2}} \right\} - \frac{(y-\eta)^2}{\sqrt{d^2 + (y-\eta)^2}}
$$

$$
\kappa_n d^2 \left\{ \frac{H_1^{(1)}(\kappa_n \sqrt{d^2 + (y-\eta)^2})}{\kappa_n \sqrt{d^2 + (y-\eta)^2}} - \frac{H_1^{(1)}(\kappa_n \sqrt{d^2 + (y-\eta)^2})}{\kappa_n \sqrt{d^2 + (y-\eta)^2}} \right\}
$$

$$
= 4i \left\{ \begin{array}{c} V_f n \ A_I d_n \\ \end{array} \right\}
$$

\[ \text{(F.4)} \]

and that on the breakwater to be

$$
\int_{-w/2}^{w/2} \left\{ \frac{\Delta \phi_{Rn1}}{\Delta \phi_{Dn1}} \right\} \frac{-\kappa_n}{d^2 + (y-\eta)^2} \kappa_n d\eta
$$

$$
\kappa_n d^2 \left\{ H_2^{(1)}(\kappa_n \sqrt{d^2 + (y-\eta)^2}) - \frac{H_1^{(1)}(\kappa_n \sqrt{d^2 + (y-\eta)^2})}{\kappa_n \sqrt{d^2 + (y-\eta)^2}} \right\}
$$

$$
- \frac{(y-\eta)^2}{\sqrt{d^2 + (y-\eta)^2}} H_1^{(1)}(\kappa_n \sqrt{d^2 + (y-\eta)^2}) d\eta + \int_{-b/2}^{b/2} \left\{ \frac{\Delta \phi_{Rn2}}{\Delta \phi_{Dn2}} \right\} \frac{-\kappa_n}{|y-\eta|} \kappa_n d\eta
$$

$$
= 4i \left\{ \begin{array}{c} 0 \\ \end{array} \right\}
$$

\[ \text{(F.5)} \]

Making the following change of variables

$$
u_1 = \frac{2\eta}{w}, \quad \nu_2 = \frac{2\eta}{b}, \quad \nu_1 = \frac{2y}{w}, \quad \nu_2 = \frac{2y}{b},
$$

\[ \text{(F.6)} \]
and

\[
\begin{pmatrix}
P_{n1}(u_1) \\
Q_{n1}(u_1) \\
P_{n2}(u_2) \\
Q_{n2}(u_2)
\end{pmatrix} = \begin{pmatrix}
\Delta \phi_{n1}^R(\eta) \\
\Delta \phi_{n1}^D(\eta) \\
\Delta \phi_{n2}^R(\eta) \\
\Delta \phi_{n2}^D(\eta)
\end{pmatrix},
\]

(F.7)
yields

\[
\int_{p}^{1} \begin{pmatrix}
P_{n1}(u_1) \\
Q_{n1}(u_1)
\end{pmatrix} \frac{H_1^{(1)}(\frac{\kappa w}{2}|v_1 - u_1|)}{|v_1 - u_1|} \kappa d\eta_1 + \int_{-1}^{1} \begin{pmatrix}
P_{n2}(u_2) \\
Q_{n2}(u_2)
\end{pmatrix} \frac{-\kappa_2 b}{4d^2 + b^2(v_2 - u_2)^2} \kappa d\eta_2
\]

\[
= 4i \int_{-1}^{1} \begin{pmatrix}
f_n \\
A_1 d_n^{(1)}
\end{pmatrix}
\]

(F.8)
for the flap and

\[
\int_{-1}^{1} \begin{pmatrix}
P_{n1}(u_1) \\
Q_{n1}(u_1)
\end{pmatrix} \frac{-\kappa_2 w}{4d^2 + w^2(v_1 - u_1)^2} \kappa d\eta_1 + \frac{2H_1^{(1)}(\frac{\kappa n}{2} \sqrt{d^2 + w^2(v_1 - u_1)^2})}{\kappa n \sqrt{d^2 + w^2(v_1 - u_1)^2}} - \frac{w^2(v_1 - u_1)^2}{2\sqrt{d^2 + w^2(v_1 - u_1)^2}}
\]

\[
= 4i \int_{-1}^{1} \begin{pmatrix}
P_{n2}(u_2) \\
Q_{n2}(u_2)
\end{pmatrix} \frac{H_1^{(1)}(\frac{\kappa b}{2}|v_2 - u_2|)}{|v_2 - u_2|} \kappa d\eta_2
\]

(F.9)
for the breakwater. Now,

\[
H_1^{(1)}(\frac{\kappa}{2}w|v - u|) = \frac{1}{i\pi \kappa w|v - u|} + R_n\left(\frac{\kappa}{2}w|v - u|\right),
\]

(F.10)
where

$$R_n(z) = J_1(z) \left[ 1 + \frac{2i}{\pi} (\ln \frac{z}{2} + \chi) \right] - \frac{i}{\pi} \left[ \frac{z}{2} + \sum_{j=2}^{\infty} \frac{(-1)^{j+1}(z/2)^{2j-1}}{j!} \left( \frac{1}{j} + \sum_{q=1}^{\infty} \frac{2}{q} \right) \right] \tag{F.11}$$

is the remainder, $J_1$ is the Bessel function of first kind and first order, and $\chi = 0.577215$... is the Euler constant. Expanding the unknown jumps in potential across the two sides of the flap as

$$\begin{align*}
\begin{cases}
P_{n1}(u_1) \\
Q_{n1}(u_1)
\end{cases} &= (1 - u_1^2)^{1/2} \sum_{p=0}^{+\infty} \begin{cases}
V_{a_{pn1}} \\
A_I b_{pn1}
\end{cases} U_p(u_1),
\end{align*} \tag{F.12}$$

where $a_{pn}$ and $b_{pn}$ are unknown complex coefficients to be determined and $U_p(u_1)$ is the Chebyshev polynomial of second kind. Similarly expanding the difference in potential on the two sides of the breakwater to be

$$\begin{align*}
\begin{cases}
P_{n2}(u_2) \\
Q_{n2}(u_2)
\end{cases} &= (1 - u_2^2)^{1/2} \sum_{p=0}^{+\infty} \begin{cases}
V_{a_{pn2}} \\
A_I b_{pn2}
\end{cases} U_p(u_2),
\end{align*} \tag{F.13}$$

gives

$$\sum_{p=0}^{\infty} \begin{cases}
a_{pn1} \\
b_{pn1}
\end{cases} \begin{cases}
C_{np1}(v_1) \\
D_{np2}(v_1)
\end{cases} + \begin{cases}
a_{pn2} \\
b_{pn2}
\end{cases} \begin{cases}
D_{np1}(v_2) \\
C_{np2}(v_2)
\end{cases} = -\pi w \begin{cases}
f_n \\
d_{n1}
\end{cases} \tag{F.14}$$

for the OWSC and

$$\sum_{p=0}^{\infty} \begin{cases}
a_{pn1} \\
b_{pn1}
\end{cases} \begin{cases}
D_{np1}(v_2) \\
C_{np2}(v_2)
\end{cases} + \begin{cases}
a_{pn2} \\
b_{pn2}
\end{cases} \begin{cases}
D_{np1}(v_2) \\
C_{np2}(v_2)
\end{cases} = -\pi b \begin{cases}
0 \\
d_{n2}
\end{cases} \tag{F.15}$$
for the breakwater. Note, in Eq. F.14 and Eq. F.15

\[ C_{pn1} = -\pi(p + 1)U_p(u_1) + \frac{i\pi \kappa_n w}{4} \int_{-1}^{1} (1 - u_1^2)^{1/2}U_p(u_1) \frac{R_n(\frac{1}{2}, \kappa_n|v_1 - u_1|)}{|v_1 - u_1|} du_1, \]

(F.16)

\[ D_{pn2} = -\frac{i\pi \kappa_n wb}{2} \int_{-1}^{1} \frac{(1 - u_2^2)^{1/2}U_p(u_2)}{4d^2 + (v_1w - bu_2)^2} \left[ d^2 \left\{ \kappa_n H_2^{(1)}(\frac{\kappa_n}{2}\sqrt{4d^2 + (v_1w - bu_2)^2}) - \frac{2H_1^{(1)}(\frac{\kappa_n}{2}\sqrt{4d^2 + (v_1w - bu_2)^2})}{\sqrt{4d^2 + (v_1w - bu_2)^2}} \right\} \right] - \frac{(v_1w - bu_2)^2}{2\sqrt{4d^2 + (v_1w - bu_2)^2}} \times H_1^{(1)}(\frac{\kappa_n}{2}\sqrt{4d^2 + (v_1w - bu_2)^2}) du_2, \]

(F.17)

\[ D_{pn1} = -\frac{i\pi \kappa_n wb}{2} \int_{-1}^{1} \frac{(1 - u_1^2)^{1/2}U_p(u_1)}{4d^2 + (v_2b - wu_1)^2} \left[ d^2 \left\{ \kappa_n H_2^{(1)}(\frac{\kappa_n}{2}\sqrt{4d^2 + (v_2b - wu_1)^2}) - \frac{2H_1^{(1)}(\frac{\kappa_n}{2}\sqrt{4d^2 + (v_2b - wu_1)^2})}{\sqrt{4d^2 + (v_2b - wu_1)^2}} \right\} \right] - \frac{(v_2b - wu_1)^2}{2\sqrt{4d^2 + (v_2b - wu_1)^2}} \times H_1^{(1)}(\frac{\kappa_n}{2}\sqrt{4d^2 + (v_2b - wu_1)^2}) du_1 \]

(F.18)

and

\[ C_{pn2} = -\pi(p + 1)U_p(u_2) + \frac{i\pi \kappa_n b}{4} \int_{-1}^{1} (1 - u_2^2)^{1/2}U_p(u_2) \frac{R_n(\frac{1}{2}, \kappa_n b|v_2 - u_2|)}{|v_2 - u_2|} du_2, \]

(F.19)

The two system of equations Eq. F.14 and Eq. F.15 are finally solved using a numerical collocation scheme as outlined in previous studies [see e.g. 56, 57].