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Based on research conducted in the School of Mathematics and Statistics

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To my family.
ABSTRACT

The spectral properties of nonnegative matrices have intrigued pure and applied mathematicians alike, beginning with the classical works of Oskar Perron and Georg Frobenius at the start of the twentieth century. One question which stems naturally from this area of research is that of the Nonnegative Inverse Eigenvalue Problem, or NIEP. This is the problem of characterising those lists of complex numbers which are realisable as the spectrum of some entrywise nonnegative matrix. This thesis explores the NIEP, as well as one of its variants, the Symmetric Nonnegative Inverse Eigenvalue Problem, or SNIEP, which considers realisability by a symmetric nonnegative matrix.

The question of determining which operations on lists preserve realisability is pertinent in the NIEP, since such operations can allow us to construct more complicated lists from simple building blocks. We present some new results along these lines. In particular, we discuss how to replace parts of realisable lists by longer lists, while preserving realisability.

In those cases where a realising matrix is known to exist, one can consider studying the properties of this matrix. We focus our attention on the problem of characterising the diagonal elements of the realising matrix and achieve a complete solution in the case where every entry in the list (apart from the Perron eigenvalue) has nonpositive real part. In order to prove this result, we derived complex analogues of Newton’s inequalities, which are of independent interest.

In the context of the SNIEP, we unify a large body of research by presenting a recursive method for constructing symmetrically realisable lists and showing that essentially all previously known sufficient conditions are either contained in, or equivalent to the family we introduce. Our construction also reveals several interesting properties of the family in question and allows for an explicit algorithmic characterisation of the lists that lie within it.

Finally, we construct families of symmetrically realisable lists which do not satisfy any previously known sufficient conditions.
Some of the results in this thesis have previously appeared in the following publications:


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ACRONYMS

NIEP  Nonnegative Inverse Eigenvalue Problem
SNIEP  Symmetric Nonnegative Inverse Eigenvalue Problem
RNIEP  Real Nonnegative Inverse Eigenvalue Problem
DNIEP  Diagonalisable Nonnegative Inverse Eigenvalue Problem
NNIEP  Normal Nonnegative Inverse Eigenvalue Problem
INTRODUCTION

Over a hundred years ago, Oskar Perron [49] and Georg Frobenius [18] laid the foundations of the theory of nonnegative matrices by discovering several fundamental facts about their spectra. These results have proven very important in diverse areas, including economics, finance, machine learning and signal processing, and nonnegative matrix theory has been an active area of research ever since. In this thesis, we make advances in some of the problems in this area.

1.1 A BRIEF WORD ON NOTATION

We denote the spectrum of a matrix $A$ by $\sigma(A)$. We say that $A$ is nonnegative (positive) if it is entrywise nonnegative (positive) and in this case we write $A \succeq 0$ ($A > 0$). In general, for $A, B \in \mathbb{R}^{n \times n}$ or $y, z \in \mathbb{R}^n$, we use notation such as $A \succeq B$ or $y \succeq z$ if the inequalities hold entrywise. Unless otherwise stated, we will assume all nonnegative matrices are square. We denote the $n \times n$ identity matrix by $I_n$.

The characteristic polynomial of $A$ is defined as

$$f(x) = \text{Det}(xI_n - A)$$

and, for a polynomial

$$g(x) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n,$$

the companion matrix of $g$ is

$$C(g) := \begin{bmatrix}
0 & 1 & & \\
& 0 & 1 & \\
& & \ddots & \ddots \\
& & & 0 & 1 \\
-c_n & -c_{n-1} & \cdots & -c_2 & -c_1
\end{bmatrix}.$$

Recall that the characteristic polynomial of $C(g)$ is $g$.

We will use the notation $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ to refer to an unordered list (multiset) of elements $\lambda_1, \lambda_2, \ldots, \lambda_n$. If the elements are to be ordered in some way, then we will say so explicitly. Similarly, if we state that $A$ has diagonal elements $(a_1, a_2, \ldots, a_n)$, then the numbers $a_1, a_2, \ldots, a_n$ may appear in any order on the diagonal of $A$. We may also use parentheses to enclose the elements of a sequence, but this will either be stated explicitly, or be clear from context. For a list of complex numbers $\sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n)$,

$$s_k(\sigma) := \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k$$
shall denote the k-th power sum of σ.

### 1.2 NONNEGATIVE MATRICES

In this section, we will give the briefest of summaries of the Perron-Frobenius theory of nonnegative matrices.

**Definition 1.2.1.** Matrices X and Y are said to be *permutationally similar* if there exists a permutation matrix P such that $X = P^TYP$.

**Definition 1.2.2.** A nonnegative $n \times n$ matrix $A$ ($n \geq 2$) is called *reducible* if it is permutationally similar to a matrix of the form

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where B and D are square matrices. If $A$ is not reducible, it is called *irreducible*.

The irreducibility of an $n \times n$ nonnegative matrix $A = (a_{ij})$ is directly connected with the digraph associated with $A$; specifically, consider the directed graph $G = (V, E)$, where $V := \{1, 2, \ldots, n\}$ and

$$E := \{(i, j) : a_{ij} > 0\}.$$

It is well-known\(^1\) that $A$ is irreducible if and only if for each pair of distinct vertices $i$ and $j$, there exits a path from $i$ to $j$.

The Perron-Frobenius theory of irreducible matrices can be summarised as follows:

**Theorem 1.2.3 (Perron-Frobenius theory of irreducible matrices).** Let $A$ be an $n \times n$ irreducible nonnegative matrix. Then

(i) $A$ has a positive real eigenvalue $\rho$—called the Perron eigenvalue of $A$—such that

$$\rho \geq |\lambda_i|$$

for every eigenvalue $\lambda_i$ of $A$;

(ii) there exists a positive eigenvector—called the Perron eigenvector of $A$—corresponding to $\rho$;

(iii) $\rho$ is an algebraically (and hence geometrically) simple eigenvalue of $A$;

(iv) $A$ has no eigenvector in the set

$$E^n = \left\{0 \leq x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1\right\},$$

other than the Perron eigenvector;

---

\(^1\) See, for example, [44, Chapter 4, Theorem 3.2].
For a not-necessarily-irreducible nonnegative matrix, a weaker version of Theorem 1.2.3 holds.

**Theorem 1.2.4 (Perron-Frobenius theory of nonnegative matrices).** Let \( A \) be an \( n \times n \) nonnegative matrix. Then

(i) \( A \) has a nonnegative real eigenvalue \( \rho \)—called the Perron eigenvalue of \( A \)—such that

\[
\rho \geq |\lambda_i|
\]

for every eigenvalue \( \lambda_i \) of \( A \);

(ii) there exists a nonnegative eigenvector—called the Perron eigenvector of \( A \)—corresponding to \( \rho \);

(iii) \( \rho \) is at least as large as the Perron eigenvalue of any principal submatrix of \( A \).

A stronger condition than that of irreducibility (while still being weaker than positivity) is the notion of primitivity:

**Definition 1.2.5.** A nonnegative matrix \( A \) is said to be primitive if there exists a positive integer \( k \) such that \( A^k \) is positive.

If an irreducible matrix \( A \) has Perron eigenvalue \( \rho \), then Frobenius [18] showed that \( A \) is primitive if and only if every other eigenvalue of \( A \) has modulus strictly less than \( \rho \). In fact, the latter characterisation is often taken to be the definition of primitivity.

Excellent accounts of the theory of nonnegative matrices can be found in Berman and Plemmons [2], Minc [44] and Bapat and Raghavan [1].

### 1.3 The Nonnegative Inverse Eigenvalue Problem

In 1949, Suleimanova [60] considered the question of characterising the spectra of nonnegative matrices; put another way, she asked if it might be possible to write down a set of necessary and sufficient conditions which determine whether a given list of \( n \) complex numbers can arise as the spectrum of some \( n \times n \) nonnegative matrix. This has become known as the Nonnegative Inverse Eigenvalue Problem (NIEP) and its exploration is the primary focus of this thesis.

We say a list \( \sigma \) of complex numbers is realisable if there exists a nonnegative matrix \( A \) with spectrum \( \sigma \), and in this case, we say that \( A \) realises \( \sigma \).
1.3.1 Necessary conditions

We begin by stating some well-known conditions that are necessary for a list to be realisable.

**Theorem 1.3.1 (Necessary conditions in the NIEP).** Suppose \( \sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n) \) is the spectrum of a nonnegative matrix \( A \). Then

(i) \( \sigma \) is self-conjugate, i.e. \( \overline{\sigma} := (\overline{\lambda_1}, \overline{\lambda_2}, \ldots, \overline{\lambda_n}) = \sigma \);

(ii) \( \max_{i} |\lambda_i| \in \sigma \);

(iii) \( s_m(\sigma) \geq 0 \) for every positive integer \( m \);

(iv) \( s_k(\sigma)^m \leq n^{m-1}s_{km}(\sigma) \) for all positive integers \( k \) and \( m \).

Condition (i) follows from the fact that the characteristic polynomial of \( A \) has real coefficients. Condition (ii) (existence of the Perron eigenvalue) is already familiar from Theorem 1.2.4. We will usually write the Perron eigenvalue as the first entry in a realisable list. Condition (iii) follows from the fact that \( s_m(\sigma) \) is the trace of \( A^m \). The inequalities in (iv) are called the JLL conditions. They were proved by Loewy and London [39] and independently by Johnson [27].

We also mention a necessary condition due to Holtz [24] expressed in terms of M-matrices. An M-matrix is a matrix of the form \( M = \alpha I_n - A \), where \( A \geq 0 \) and \( \alpha \) is greater than or equal to the Perron eigenvalue of \( A \). Holtz showed that if \( M \) is an M-Matrix, then the coefficients of its characteristic polynomial must satisfy Newton’s inequalities, i.e. if

\[
\text{Det}(xI_n - M) = x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_n,
\]

then

\[
\left( \frac{c_k}{\binom{n}{k}} \right)^2 \leq \frac{c_{k-1}}{\binom{n}{k-1}} \cdot \frac{c_{k+1}}{\binom{n}{k+1}} : \quad k = 1, 2, \ldots, n - 1. \tag{1}
\]

Since the \( c_k \) depend only on the spectrum of \( M \), which, in turn, depends on the spectrum of \( A \), (1) gives an additional set of conditions which a realisable list must satisfy.

For an extensive treatment of Newton’s inequalities, see Chapter 3. There, we derive complex analogues of Newton’s inequalities, which we apply to the NIEP in Chapter 5.

1.3.2 Solutions in particular cases

The NIEP has been solved for lists of length \( n \leq 4 \).

If \( n = 2 \), note that, assuming \( \lambda_1 \geq \lambda_2 \) and \( \lambda_1 + \lambda_2 \geq 0 \), the matrix

\[
A = \frac{1}{2} \begin{bmatrix}
\lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\
\lambda_1 - \lambda_2 & \lambda_1 + \lambda_2
\end{bmatrix}
\]

...
is nonnegative and has spectrum $(\lambda_1, \lambda_2)$. Hence, in this case, the condition $s_1(\sigma) \geq 0$ is both necessary and sufficient.

If $n = 3$, Loewy and London [39] showed that an even weaker set of conditions than those of Theorem 1.3.1 are sufficient:

**Theorem 1.3.2.** [39] Let $\sigma := (\lambda_1, \lambda_2, \lambda_3)$ and suppose:

(i) $\sigma$ is self-conjugate;

(ii) $\lambda_1 \geq |\lambda_2|, |\lambda_3|$;

(iii) $s_1(\sigma) \geq 0$;

(iv) $s_1(\sigma)^2 \leq 3s_2(\sigma)$.

Then $\sigma$ is realisable.

In Section 5.4, we show that if $\sigma$ obeys (i)–(iv) above, then $\sigma$ may always be realised by a $3 \times 3$ nonnegative matrix which is the sum of a companion matrix and a diagonal matrix. Moreover, we give necessary and sufficient conditions for a list of nonnegative numbers $(a_1, a_2, a_3)$ to arise as the diagonal elements of a matrix which realises $\sigma$.

A complete solution to the NIEP for $n = 4$ has been given by Laffey and Meehan. The result, which appears in Meehan’s PhD thesis [42], is phrased in terms of the power sums of $\sigma$. Independently, Torre-Mayo et al. [61] gave an equivalent solution in terms of the coefficients of the polynomial

$$f(x) = \prod_{i=1}^{n}(x - \lambda_i).$$

Both solutions are quite technical, so we will not present them here.

Laffey and Meehan have also given a complete solution for lists of five elements whose sum is zero:

**Theorem 1.3.3.** [33] Let $\sigma := (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ and suppose $s_1(\sigma) = 0$. Then $\sigma$ is the spectrum of a nonnegative matrix if and only if the following conditions are satisfied:

(i) $s_k(\sigma) \geq 0 : k = 2, 3, 4, 5$;

(ii) $s_2(\sigma)^2 \leq 4s_4(\sigma)$;

(iii) $12s_5(\sigma) - 5s_2(\sigma)s_3(\sigma) + 5s_3(\sigma)s_2(\sigma)\sqrt{4s_4(\sigma) - s_2(\sigma)^2} \geq 0$.

For general lists of five or more elements, the Nonnegative Inverse Eigenvalue Problem remains open; however, numerous authors have made significant progress by restricting their attention to lists of a certain type. The first result of this kind was given by Suleimanova [60] when she proved:
Theorem 1.3.4. [60] Let \( \sigma := (\rho, \lambda_2, \lambda_3, \ldots, \lambda_n) \), where \( \rho \geq 0 \) and \( \lambda_i \leq 0 : i = 2, 3, \ldots, n \). Then \( \sigma \) is the spectrum of a nonnegative matrix if and only if

\[
\rho + \lambda_2 + \lambda_3 + \cdots + \lambda_n \geq 0.
\]

Perhaps the most elegant proof of Sule\'imanova's result is due to Perfect [48], who showed that, under the assumptions of the theorem, the companion matrix of the polynomial

\[
f(x) = (x - \rho) \prod_{i=2}^{n} (x - \lambda_i)
\]

is nonnegative.

Laffey and Šmigoc [34] generalised Sule\'imanova's theorem to complex lists in which every element (save for the Perron eigenvalue) has real part less than or equal to zero:

Theorem 1.3.5. [34] Let \( \rho \geq 0 \) and let \( \lambda_2, \lambda_3, \ldots, \lambda_n \) be complex numbers such that \( \text{Re} \lambda_i \leq 0 \) for all \( i = 2, 3, \ldots, n \). Then the list \( \sigma := (\rho, \lambda_2, \lambda_3, \ldots, \lambda_n) \) is the spectrum of a nonnegative matrix if and only if the following conditions hold:

(i) \( \sigma \) is self-conjugate;

(ii) \( s_1(\sigma) \geq 0 \);

(iii) \( s_1(\sigma)^2 \leq ns_2(\sigma) \).

Furthermore, when the above conditions are satisfied, \( \sigma \) may be realised by a matrix of the form \( C + \alpha I_n \), where \( C \) is a nonnegative companion matrix with trace zero and \( \alpha \) is a nonnegative scalar.

Observation 1.3.6. The condition that \( \text{Re} \lambda_i \leq 0 \) for all \( i = 2, 3, \ldots, n \) in Theorem 1.3.5 can be relaxed to \( \text{Re} \lambda_i \leq s_1(\sigma)/n \).

Proof. The quantity

\[ ns_2(\sigma) - s_1(\sigma)^2 \]

is unchanged by subtracting a scalar from \( \sigma \), i.e.

\[ ns_2(\rho - \delta, \lambda_2 - \delta, \ldots, \lambda_n - \delta) - s_1(\rho - \delta, \lambda_2 - \delta, \ldots, \lambda_n - \delta)^2 = ns_2(\rho, \lambda_2, \ldots, \lambda_n) - s_1(\rho, \lambda_2, \ldots, \lambda_n)^2 \]

for all \( \delta \in \mathbb{C} \). Hence, if \( (\rho, \lambda_2, \ldots, \lambda_n) \) satisfies (i)-(iii), then so does \( (\rho - s_1(\sigma)/n, \lambda_2 - s_1(\sigma)/n, \ldots, \lambda_n - s_1(\sigma)/n) \). \( \square \)

The crucial ingredient in Laffey and Šmigoc's result is the following lemma (also proved by the authors).
Lemma 1.3.7. [34] Let \((\lambda_2, \lambda_3, \ldots, \lambda_n)\) be a self-conjugate list of complex numbers with nonpositive real parts, let \(\rho \geq 0\) and let

\[
f(x) := (x - \rho) \prod_{i=2}^{n} (x - \lambda_i) = x^n - b_1 x^{n-1} - b_2 x^{n-2} - \cdots - b_n.
\]

If \(b_1, b_2 \geq 0\), then \(b_i \geq 0 : i = 3, 4, \ldots, n\).

In Section 2.4, we will generalise the above Lemma, which, in turn, will allow us to prove a generalisation\(^3\) of Theorem 1.3.5.

There are many results in the NIEP which allow us to construct new realisable lists from known realisable lists. In Section 4.2, we mention several of these results, and in the remainder of Chapter 4, we give some new results of this type.

We finish this overview with a fundamental result due to Guo [21], which states that, as long as \((\lambda_2, \lambda_3, \ldots, \lambda_n)\) is self-conjugate, the list \((\rho, \lambda_2, \lambda_3, \ldots, \lambda_n)\) will be realisable for all sufficiently large \(\rho\).

Theorem 1.3.8. [21] Let \((\lambda_2, \lambda_3, \ldots, \lambda_n)\) be self-conjugate. Then there exists a real number \(\rho_0\), with

\[
\max_{2 \leq i \leq n} |\lambda_i| \leq \rho_0 \leq 2n \max_{2 \leq i \leq n} |\lambda_i|,
\]

such that \((\rho, \lambda_2, \ldots, \lambda_n)\) is realisable if and only if \(\rho \geq \rho_0\).

Hence, the NIEP is equivalent to the problem of determining the critical value \(\rho_0\) in Theorem 1.3.8.

In those cases where a realising matrix is known to exist, it is of interest to look at the properties of this matrix, for example, one can consider its sparsity, its rank, or the possible set of diagonal elements it may possess. In Chapter 5, we examine the problem of characterising the possible diagonal elements of the realising matrix.

1.3.3 Adding zeros

Understanding how the addition of zeros to a list affects realisability is crucial to understanding the NIEP as a whole.

Definition 1.3.9. We say a list of nonzero complex numbers \(\sigma\) is the nonzero spectrum of a nonnegative (primitive) matrix if there exists a nonnegative integer \(N\) such that \(\sigma\) with \(N\) zeros appended is the spectrum of a nonnegative (primitive) matrix.

One of the most celebrated results in the NIEP is due to Boyle and Handelman [5]. They showed that, provided \(\sigma\) satisfies some remarkably simple conditions, there exists a nonnegative integer \(N\) such that \(\sigma\) with \(N\) zeros added is realisable.

\(^2\)See Lemma 2.4.1.

\(^3\)See Theorem 5.5.1.
Theorem 1.3.10. [5] A list of nonzero complex numbers $\sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the nonzero spectrum of some primitive matrix if and only if the following conditions hold:

(i) $\sigma$ is self-conjugate;

(ii) $\sigma$ contains a positive number strictly greater than the modulus of any other entry in $\sigma$;

(iii) for all positive integers $n$ and $k$, $s_n(\sigma) \geq 0$ and $s_n(\sigma) > 0$ implies $s_{nk}(\sigma) > 0$.

The proof of Theorem 1.3.10 uses Ergodic theory and symbolic dynamics and does not yield a usable algorithm to construct the realising matrix. Furthermore, no bound is given on the number of zeros which must be added to achieve realisability. In 2012, these issues were addressed by Laffey [32]:

Theorem 1.3.11. [32] Let $\sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a list of complex numbers and let $N$ be a natural number. Define

\[
f(x) := \prod_{i=1}^{n} (x - \lambda_i) = x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n,
\]

\[
q_i := \frac{p_i}{(n+N)(n+N-1)\cdots(n+N-i+1)} : i = 1, 2, \ldots, n,
\]

\[
q(x) := x^n + q_1x^{n-1} + q_2x^{n-2} + \cdots + q_n.
\]

Let $\mu_1, \mu_2, \ldots, \mu_n$ be the roots of $q$ and let

\[
x_k := \sum_{i=1}^{n} \mu_i^k : k = 1, 2, \ldots, n + N
\]

and

\[
X := \begin{bmatrix}
  x_1 & 1 \\
  x_2 & x_1 & 2 \\
  x_3 & x_2 & x_1 & 3 \\
  \vdots & \vdots & \ddots & \ddots & \ddots \\
  x_{n+N-1} & x_{n+N-2} & \cdots & x_2 & x_1 & n - 1 \\
  x_{n+N} & x_{n+N-1} & x_{n+N-2} & \cdots & x_2 & x_1
\end{bmatrix}.
\]

Then $X$ has characteristic polynomial $x^N f(x)$. Furthermore, $X$ is nonnegative for some $N$, provided the following conditions hold:

(i) $\sigma$ is self-conjugate;

(ii) $\sigma$ contains a positive number strictly greater than the modulus of any other entry in $\sigma$;

(iii) $s_1(\sigma) \geq 0$ and $s_k(\sigma) > 0$ for $k = 2, 3, \ldots$

Laffey also gives a bound on the number of zeros which much be added to guarantee realisability.
1.3.4 A classical example

We mention the following classical example in the NIEP:

Example 1.3.12. Consider the list
\[ \tau(t) := (3 + t, 3 - t, -2, -2, -2). \]

We first note that \( \tau(0) \) is not realisable. To see this, suppose there exists a nonnegative matrix \( A \) with spectrum \( \tau(0) \). Since the Perron eigenvalue of \( \tau(0) \) is repeated, Theorem 1.2.3 implies \( A \) must be reducible; however, there is no way to partition \( \tau(0) \) into two lists, in such a way that the sum of the entries in each list is nonnegative.

On the other hand, if \( \tau(t_0) \) is realisable, then so is \( \tau(t) \) for all \( t \geq t_0 \) (see Theorem 4.2.2), so it is logical to ask for the minimum \( t \) for which \( \tau(t) \) is realisable. From Laffey and Meehan’s solution to the NIEP for lists with five elements and trace zero (see Theorem 1.3.3), we know that the answer to this question is \( t = \sqrt{16\sqrt{6} - 39} \approx 0.437991 \).

1.3.5 Variants of the NIEP

The NIEP is a rich and complex problem and has spawned several variants over the years, including

- the Real Nonnegative Inverse Eigenvalue Problem (RNIEP), where we pose no additional conditions on the realising matrix, but we assume \( \sigma \) is a list of real numbers;
- the Normal Nonnegative Inverse Eigenvalue Problem (NNIEP), where we require the realising matrix to be normal;
- the Diagonalisable Nonnegative Inverse Eigenvalue Problem (DNIEP), where we require the realising matrix to be diagonalisable;

and, most notably,

- the Symmetric Nonnegative Inverse Eigenvalue Problem (SNIEP), where we require the realising matrix to be symmetric.

Since the spectrum of a symmetric matrix is necessarily real, we need only concern ourselves with real lists in the context of the SNIEP. Hence, in 1978, Hershkowitz [23] asked if the RNIEP and SNIEP might in fact be equivalent. The answer to this question remained unknown until 1996, when Johnson, Laffey and Loewy [28] showed that there exist real lists which are realisable as the spectrum of a nonnegative matrix, but not as the spectrum of symmetric nonnegative matrix. Interestingly, the authors did not give an example of such a list, rather they showed that, after a certain point, adding zeros to a real list does not aid symmetric realisability, but that this cannot be said for realisability by a not-necessarily-symmetric matrix.

For an extensive survey and comparison of various known results in the RNIEP and SNIEP, see Chapter 6.
1.4 ABOUT THIS THESIS

The main body of this thesis is logically divided into three parts.

In Part I, we study some basic properties of polynomials and give some inequalities involving their coefficients. In particular, in Chapter 2, we give a lower bound on the number of real roots of certain real polynomials and use this information to generalise Lemma 1.3.7. In Chapter 3, we derive families of “Newton-like” inequalities involving the elementary symmetric functions of a self-conjugate list of complex numbers. Some of these inequalities rely on the facts about polynomial roots discovered in Chapter 2. Although the research in Part I is motivated by the Nonnegative Inverse Eigenvalue Problem, many of the results are of independent interest.

In Part II, we study the Nonnegative Inverse Eigenvalue Problem. In Chapter 4, we demonstrate some constructive techniques whereby new realisable lists can be built from known realisable lists. In Chapter 5, motivated by certain constructive methods, we consider the problem of characterising the possible diagonal elements of a matrix which realises $\sigma$. If $n \leq 3$, or if every entry in the list (apart from the Perron eigenvalue) has nonpositive real part, a complete solution is achieved. The proof requires several results from Part I.

Part III is devoted to the study of the Symmetric Nonnegative Inverse Eigenvalue Problem. The theme of diagonal elements is continued in Chapter 6, where they are used to recursively construct symmetrically realisable lists. The properties of the realisable family we obtain allow us to show that essentially all previously known sufficient conditions, beginning with the work of Fiedler in 1974, are either contained in, or equivalent to, the family we introduce. In Chapter 7, we construct symmetrically realisable lists which do not satisfy any previously known sufficient conditions.
Part I

POLYNOMIALS AND INEQUALITIES
ON THE REAL ROOTS OF CERTAIN REAL POLYNOMIALS

2.1 INTRODUCTION

In this chapter, we find lower bounds on the number of real roots of certain real polynomials and these bounds are used to generalise Lemma 1.3.7. Since several results in Chapters 3 and 5 rely on these results, this chapter is the foundation of much of the thesis.

In Section 2.2, we will discuss several known results related to polynomial roots. In Section 2.3, we will use these results to show that if \( f(x) = P(x) + iQ(x) \) is a complex polynomial and the signs of the imaginary parts of the roots of \( f \) are known, then it is possible to give a lower bound on the number of real roots of \( P \) and \( Q \). Finally, in Section 2.4, we will generalise Lemma 1.3.7.

2.2 PRELIMINARIES

2.2.1 Polynomials with real roots

We begin by giving three lemmas, which essentially appear in [15] as Problems 581, 727 and 736, respectively.

**Lemma 2.2.1.** [15] Let \( f : \mathbb{C} \to \mathbb{C} \) be a rational function

\[
f(z) = \frac{P(z)}{Q(z)} : Q \neq 0,
\]

where \( P \) and \( Q \) are real polynomials. If \( a \) is a root of \( f(z) - f(a) \) of multiplicity \( k \), then for sufficiently small \( \epsilon \), the circle \( |z - a| = \epsilon \) contains precisely \( 2k \) points at which \( \text{Re}(f(z)) = \text{Re}(f(a)) \) and precisely \( 2k \) points at which \( \text{Im}(f(z)) = \text{Im}(f(a)) \).

**Definition 2.2.2.** Let \( X \) and \( Z \) be sequences of real numbers. We say \( X \) and \( Z \) interlace if the following two conditions hold:

(i) if \( x_i \) and \( x_j \) are two distinct elements of \( X \) with \( x_i < x_j \), then there exists an element \( z_k \) of \( Z \) such that \( x_i \leq z_k \leq x_j \) (and vice versa);

(ii) if \( x_i \) appears in \( X \) with multiplicity \( m \), then \( x_i \) appears in \( Z \) with multiplicity at least \( m - 1 \) (and vice versa).

We say \( X \) and \( Z \) strictly interlace if every element of \( X \) and \( Z \) occurs with multiplicity 1, \( X \) and \( Z \) have no element in common and whenever \( x_i \) and \( x_j \) are two distinct elements of \( X \) with \( x_i < x_j \), there exists an element \( z_k \) of \( Z \) such that \( x_i < z_k < x_j \) (and vice versa).
The following two lemmas (and their proofs) have been modified only slightly from [15], in order, for example, to explicitly take into account multiple roots.

**Lemma 2.2.3.** [15] Let $P(x)$ and $Q(x)$ be real polynomials and suppose that for arbitrary $\alpha, \beta \in \mathbb{R}$ (not both zero), every root of the polynomial $F(x) := \alpha P(x) + \beta Q(x)$ is real. Then the roots of $P$ and $Q$ interlace. Moreover, if $P$ and $Q$ have no roots in common, then the interlacing is strict.

**Proof.** By setting $\alpha = 0, \beta = 1$ or $\alpha = 1, \beta = 0$, we see at once that the roots of $P$ and $Q$ are real.

Suppose $x_1$ and $x_2$ are two distinct roots of $P$ with $x_1 < x_2$ and that $Q(x) \neq 0$ for all $x \in [x_1, x_2]$. Then $\psi(x) := \frac{P(x)}{Q(x)}$ is continuous and differentiable on $[x_1, x_2]$ and vanishes at the endpoints of this interval. Hence, by Rolle’s theorem, there exists $x_0 \in (x_1, x_2)$ such that $\psi'(x_0) = 0$. Then $x_0$ is a root of $\psi(x) - \psi(x_0)$ of multiplicity at least 2 and so, by Lemma 2.2.1, there exists $\epsilon > 0$ such that the circle $|z - a| = \epsilon$ contains at least 4 points at which $\text{Im}(\psi(z)) = \text{Im}(\psi(x_0)) = 0$.

Let $z_0$ be one of the nonreal points with this property. Since $\psi(z_0)$ is real, setting $\alpha_0 := -1, \beta_0 := \psi(z_0)$, we find that $z_0$ is a nonreal root of $F(x) = \alpha_0 P(x) + \beta_0 Q(x)$, contradicting our hypothesis.

Now suppose $x_1$ is a root of $P$ of multiplicity $k$ and $x_1$ is a root of $Q$ of multiplicity $l \leq k - 2$. Let us write $P(x) = (x - x_1)^l \hat{P}(x)$ and $Q(x) = (x - x_1)^l \hat{Q}(x)$. Set

$$\hat{\psi}(x) := \frac{\hat{P}(x)}{\hat{Q}(x)}.$$

Since $x_1$ is root of $\hat{P}$ of multiplicity at least 2 and $\hat{Q}(x_1) \neq 0$, we must have $\hat{\psi}'(x_1) = 0$. Hence (as above), there exists a nonreal $z_0$ such that $\hat{\psi}(z_0)$ is real and choosing $\alpha_0 := -1, \beta_0 := \hat{\psi}(z_0)$, we see that $F(z_0) = \alpha_0 P(z_0) + \beta_0 Q(z_0) = 0$, again contradicting our hypothesis. Therefore, we have shown that the roots of $P$ and $Q$ interlace. It is easy to see that if if $P$ and $Q$ have no roots in common, then this interlacing must be strict. 

**Lemma 2.2.4.** [15] Let

$$f(x) := (a_0 + ib_0)x^n + (a_1 + ib_1)x^{n-1} + \cdots + (a_n + ib_n),$$

where the $a_i$ and $b_i$ are real and $a_0$ and $b_0$ are not both zero. If $f$ has at least one nonreal root and every root of $f$ has nonnegative (alternatively, nonpositive) imaginary part, then the roots of the polynomials

$$P(x) := a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

and

$$Q(x) := b_0 x^n + b_1 x^{n-1} + \cdots + b_n$$

are real and interlace. If, in addition, $f$ has no real roots, then the interlacing is strict.
Proof. First assume that $f$ has no real roots. In this case, we note that $P$ and $Q$ can have no roots in common. To see this, note that if $P(\alpha) = Q(\alpha) = 0$ for some real $\alpha$, it would follow that $f(\alpha) = P(\alpha) + iQ(\alpha) = 0$. Similarly, if $P(\alpha \pm i\beta) = Q(\alpha \pm i\beta) = 0$ for some $\alpha \in \mathbb{R}$, $\beta > 0$, it would follow that $\alpha + i\beta$ and $\alpha - i\beta$ are both roots of $f$, contradicting the hypothesis that all roots of $f$ have nonnegative real part.

Let us label the roots of $f(x) = P(x) + iQ(x)$ as $x_1, x_2, \ldots, x_n$. Consider the polynomial $\hat{f}(x) = P(x) - iQ(x)$, whose roots are clearly $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$, and define

$$\Phi(x) := \frac{f(x)}{\hat{f}(x)} = \frac{a_0 + ib_0}{a_0 - ib_0} \prod_{j=1}^n \frac{x - x_j}{x - \overline{x}_j}.$$ 

Let $x^*$ be a root of $P$. Since $x^*$ is not also a root of $Q$, we have $\Phi(x^*) = -1$ and hence

$$|\Phi(x^*)| = \prod_{j=1}^n \frac{|x^* - x_j|}{|x^* - \overline{x}_j|} = 1. \quad (3)$$

Now, let us assume that $\text{Im}(x_j) > 0$ for all $j$. In this case, it is easy to see geometrically that if $\text{Im}(x^*) > 0$, then $\frac{|x^* - x_j|}{|x^* - \overline{x}_j|} < 1$ for all $j$, and if $\text{Im}(x^*) < 0$, then $\frac{|x^* - x_j|}{|x^* - \overline{x}_j|} > 1$ for all $j$. Alternatively, if $\text{Im}(x_j) < 0$ for all $j$, then the reverse is true. In either case, it is clear that (3) is only possible if $x^*$ is real. Hence we have shown that all roots of $P$ are real.

Finally, let $\alpha$ and $\beta$ be arbitrary real numbers (not both zero) and consider the polynomial

$$g(x) := (\alpha - i\beta)f(x) = \alpha P(x) + \beta Q(x) + i(\alpha Q(x) - \beta P(x)).$$

Since the roots of $g$ do not differ from those of $f$, it follows from the above that the roots of $\alpha P(x) + \beta Q(x)$ are real. Hence, by Lemma 2.2.3, the roots of $P$ and $Q$ are real and strictly interlace.

Now suppose that some of the roots of $f$ are real. Let us label these real roots $\mu_1, \mu_2, \ldots, \mu_l$ and suppose they have multiplicities $k_1, k_2, \ldots, k_l$, respectively. Then we may write

$$P(x) = \left( \prod_{j=1}^l (x - \mu_j)^{k_j} \right) \hat{P}(x), \quad Q(x) = \left( \prod_{j=1}^l (x - \mu_j)^{k_j} \right) \hat{Q}(x).$$

Applying the argument above to the polynomial $\hat{P}(x) + i\hat{Q}(x)$, we see that the roots of $\hat{P}$ and $\hat{Q}$ are real and strictly interlace. Therefore, the roots of $P$ and $Q$ are real and interlace (but not strictly).

An immediate consequence of Lemma 2.2.4 is the following:

Corollary 2.2.5. [15] Suppose that the real parts of all roots of the real polynomial $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$ are nonnegative (alternatively...
nonpositive) and that $f$ has at least one root which is not purely imaginary. Then the roots of the polynomials
\[ x^n - a_2x^{n-2} + a_4x^{n-4} - \ldots \]
and
\[ a_1x^{n-1} - a_3x^{n-3} + a_5x^{n-5} - \ldots \]
are real and interlace. If, in addition, $f$ has no purely imaginary roots, then the interlacing is strict.

**Proof.** The real parts of the roots of $f$ correspond to the imaginary parts of the roots of the polynomial
\[ g(x) := i^n f(-ix) = x^n + i a_1 x^{n-1} - a_2 x^{n-2} - i a_3 x^{n-3} + \ldots. \]
The result follows from Lemma 2.2.4.

### 2.2.2 The Cauchy index of a rational function

**Definition 2.2.6.** Let $f(x)$ be a real rational function and let $\theta, \phi \in \mathbb{R} \cup \{-\infty, \infty\}$, with $\theta < \phi$. The **Cauchy index** of $f(x)$ between the limits $\theta$ and $\phi$—written $I^\phi_\theta f(x)$—is defined as the number of times $f(x)$ jumps from $-\infty$ to $\infty$, minus the number of times $f(x)$ jumps from $\infty$ to $-\infty$, as $x$ moves from $\theta$ to $\phi$.

**Example 2.2.7.** If
\[ f(x) = \frac{1}{(x+1)(x-1)}, \]
then $I^-_\infty f(x) = -1$, $I^\infty_0 f(x) = 1$ and $I^\infty_\infty f(x) = 0$.

We introduce some additional notation: if $f(x)$ is a complex-valued function and $C$ is a contour in the complex plane, let $\Delta_C f(x)$ denote the total increase in $\arg f(x)$ as $x$ traverses the contour $C$. If $C$ is the line segment from $\theta$ to $\phi$, then we write $\Delta^\phi_\theta f(x)$.

The following result (and its proof) essentially appears in [19, Chapter 15, §3]. The proof is included for completeness.

**Theorem 2.2.8 (See [19]).** Let $f(x) := P(x) + iQ(x)$, where
\[ P(x) := x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n \]
and
\[ Q(x) := b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n \]
are real polynomials. Suppose $f$ has $n_+$ roots with positive imaginary part, $n_-$ roots with negative imaginary part and $n_0$ real roots ($n_+ + n_- + n_0 = n$). Then
\[ I^-_\infty \frac{Q(x)}{P(x)} = n_- - n_+. \]
Proof. We first consider the case when $n_0 = 0$. Define the closed contour $C = C_1 + C_2$ (shown in Figure 1), where $C_1$ is the line segment from $-R$ to $R$ and $C_2$ is the semicircle

$$x(t) = \text{Re}^{it} : 0 \leq t \leq \pi.$$ 

Assume $R$ is large enough so that all of the roots of $f$ with positive imaginary part lie within the region enclosed by $C$.

![Figure 1: Contours $C_1$ and $C_2$](image)

Denote the roots of $f$ by $x_1, x_2, \ldots, x_n$. For each $j = 1, 2, \ldots, n$, if $\text{Re}(x_j) > 0$, then $\Delta_C(x - x_j) = 2\pi$. Otherwise, $\Delta_C(x - x_j) = 0$. Therefore

$$\Delta_C f(x) = \Delta_C \left( (a_0 + ib_0) \prod_{j=1}^{n} (x - x_j) \right) = \sum_{j=1}^{n} \Delta_C(x - x_j) = 2n_+ \pi.$$ 

Similarly,

$$\lim_{R \to \infty} \Delta_C f(x) = n\pi.$$ 

Hence

$$\Delta_{-\infty} f(x) = (2n_+ - n)\pi;$$

(4)

however, since

$$\arg f(x) = \tan^{-1} \frac{Q(x)}{P(x)}$$

and

$$\lim_{x \to \pm\infty} \frac{Q(x)}{P(x)} = 0,$$

it follows that

$$\frac{1}{\pi} \Delta_{-\infty} f(x) = -\int_{-\infty}^{\infty} \frac{Q(x)}{P(x)}.$$ 

Combining (4) and (5) gives

$$\int_{-\infty}^{\infty} \frac{Q(x)}{P(x)} = n - 2n_+ = n_+ - n_-,$$
as required.

Now consider the case when \( n_0 > 0 \). Let us label the real roots of \( f \) as \( \eta_1, \eta_2, \ldots, \eta_{n_0} \). Writing

\[
f(x) = \prod_{j=1}^{n_0} (x - \eta_j) f(x),
\]

\[
\tilde{f}(x) = \tilde{P}(x) + i \tilde{Q}(x),
\]

we note that the polynomial \( \tilde{f} \) has \( n_+ \) roots with positive imaginary part, \( n_- \) roots with negative imaginary part and no real roots. Hence, from the above,

\[
I_{-\infty}^{\infty} \frac{\tilde{Q}(x)}{\tilde{P}(x)} = n_- - n_+.
\]

We note, however, that

\[
P(x) = \prod_{j=1}^{n_0} (x - \eta_j) \tilde{P}(x),
\]

\[
Q(x) = \prod_{j=1}^{n_0} (x - \eta_j) \tilde{Q}(x)
\]

and for all \( j = 1, 2, \ldots, n_0 \),

\[
\lim_{x \to \eta_j} \frac{Q(x)}{P(x)} = \lim_{x \to \eta_j} \frac{\tilde{Q}(x)}{\tilde{P}(x)}.
\]

Therefore

\[
I_{-\infty}^{\infty} \frac{Q(x)}{P(x)} = I_{-\infty}^{\infty} \frac{\tilde{Q}(x)}{\tilde{P}(x)}.
\]

In the literature, Theorem 2.2.8 is often used to compute the number of roots of a real polynomial which have positive real part. In this scenario, the Cauchy index may be calculated by Routh’s algorithm. See, for example, [19, Chapter 15].

1 An important special case is when all roots have negative real part (the “stability” case).

### 2.3 Polynomials with at least \( d \) real roots

Using Theorem 2.2.8, it is possible to generalise Lemma 2.2.4; if the signs of the imaginary parts of the roots of \( f(x) = P(x) + iQ(x) \) are known, then it is possible to give a lower bound on the number of real roots of \( P \) and \( Q \).

**Theorem 2.3.1.** Consider the polynomial

\[
f(x) := x^n + (a_1 + i b_1) x^{n-1} + \cdots + (a_n + i b_n),
\]
where the \( a_i \) and \( b_i \) are real. Suppose \( f \) has \( n_+ \) roots with positive imaginary part, \( n_- \) roots with negative imaginary part and \( n_0 < n \) real roots \((n_+ + n_- + n_0 = n)\). Let
\[
    P(x) := x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n,
    \]
\[
    Q(x) := b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n
\]
and \( d := n - 2 \min(n_+, n_-) \). Then (counting multiplicities) there exist at least \( d \) real roots of \( P \) (say \( \mu_1, \mu_2, \ldots, \mu_d \)) and at least \( d - 1 \) real roots of \( Q \) (say \( \nu_1, \nu_2, \ldots, \nu_{d-1} \)) such that
\[
    \mu_1 \leq \nu_1 \leq \mu_2 \leq \nu_2 \leq \cdots \leq \nu_{d-1} \leq \mu_d.
\]
If \( n_0 = 0 \), then the inequalities in (6) may be assumed to be strict.

Proof. As in the proof of Theorem 2.2.8, we first consider the case when \( n_0 = 0 \). In this case, \( P \) and \( Q \) can have no real root in common, since if \( x_0 \) were a real root of both \( P \) and \( Q \), then \( x_0 \) would also be a real root of \( f \). Suppose also that \( n_- > n_+ \).

Let \( p_1 < p_2 < \cdots < p_s \) be the points on the real line at which \( Q(x)/P(x) \) jumps from \(-\infty\) to \( \infty \) and let \( q_1 < q_2 < \cdots < q_t \) be the points on the real line at which \( Q(x)/P(x) \) jumps from \( \infty \) to \(-\infty \). Clearly, the \( p_i \) and \( q_i \) are roots of \( P \). Suppose they are arranged as follows:

\[
    \cdot \cdot \cdot < p_{kj} < p_{kj+1} < \cdots < p_{kj+1-1} < q_{lj} < q_{lj+1} < \cdots < q_{lj+1-1} < \cdots
\]

Now consider the interval \( R := (p_{r-1+kj}, p_{r+kj}) \), where \( 1 \leq r \leq k_j+1 - k_j - 1 \). By definition of the \( p_i \),
\[
    \lim_{x \to p_{r-1+kj}} \frac{Q(x)}{P(x)} = \infty, \quad \lim_{x \to p_{r+kj}} \frac{Q(x)}{P(x)} = -\infty.
\]

Furthermore, although \( Q(x)/P(x) \) may have discontinuities in \( R \) (at points where \( P \) has a root of even multiplicity), \( Q(x)/P(x) \) does not change sign at these discontinuities. Hence \( Q(x)/P(x) \) has a root, say \( w_{jr} \), in \( R \). Obviously, \( w_{jr} \) is also a root of \( Q \).

Let us now consider the sequence
\[
    \mathcal{T} := (\ldots, p_{kj}, w_{jr}, p_{kj+1}, w_{jr+1}, p_{kj+1-1}, \ldots, p_{kj+1}, w_{j+1,1}, p_{kj+1+1}, p_{kj+2-1}, \ldots, p_{kj+2-1}, \ldots).
\]

This sequence consists of strictly interlacing roots of \( P \) and \( Q \), apart from certain pairs of adjacent roots of \( P \) of the form \((p_{kj+1-1}, p_{kj+1})\).

Hence, we form a new sequence \( \mathcal{T}' \) from \( \mathcal{T} \) by deleting either \( p_{kj+1-1} \) or \( p_{kj+1} \) for each \( j \). Since \( \mathcal{T}' \) is a strictly interlacing sequence of real
roots of $P$ and $Q$, whose first and last entries are roots of $P$, it is sufficient to check that $T'$ is sufficiently long.

Let $h$ be the number of subsequences $(q_{l_1} < q_{l_1+1} < \cdots < q_{l_1+l-1})$ which lie between $p_1$ and $p_s$. We note that $T$ has length $2s - h - 1$. Since $T'$ was formed by deleting $h$ elements from $T$, it follows that $T'$ has length

$$2(s - h) - 1 \geq 2(s - s') - 1 = 2I_{\infty} \frac{Q(x)}{P(x)} - 1.$$

By Theorem 2.2.8, it follows that $T'$ has at least

$$2(n_- - n_-) - 1 = 2(n - 2n_+) = 2d - 1$$

elements, as required.

We have yet to consider $n_+ > n_-$ or $n_0 > 0$. If $n_0 = 0$ and $n_+ = n_-$, then the statement says nothing; hence we may ignore this case. If $n_0 = 0$ and $n_+ > n_-$, then the proof is analogous to the above.

Finally, suppose $n_0 \geq 0$. Let us label the real roots of $f$ as $\eta_1, \eta_2, \ldots, \eta_n_0$. Writing

$$f(x) = \prod_{j=1}^{n_0} (x - \eta_j) \left( \tilde{P}(x) + i\tilde{Q}(x) \right),$$

we note that the polynomial $\tilde{P}(x) + i\tilde{Q}(x)$ has $n_+$ roots with positive imaginary part, $n_-$ roots with negative imaginary part and no real roots. Hence, from the above, there exist $d - n_0$ real roots of $\tilde{P}$ (say $\mu_1, \mu_2, \ldots, \mu_{d-n_0}$) and $d - n_0 - 1$ real roots of $\tilde{Q}$ (say $\nu_1, \nu_2, \ldots, \nu_{d-n_0-1}$) such that

$$\mu_1 < \nu_1 < \mu_2 < \nu_2 < \cdots < \nu_{d-n_0-1} < \mu_{d-n_0}.$$ 

All that remains is to note that the sequences

$$(\mu_1, \mu_2, \ldots, \mu_{d-n_0}, \eta_1, \eta_2, \ldots, \eta_n_0)$$

and

$$(\nu_1, \nu_2, \ldots, \nu_{d-n_0-1}, \eta_1, \eta_2, \ldots, \eta_n_0)$$

interlace (though not strictly).

$\square$

**Example 2.3.2.** The polynomial

$$f(x) = x^5 - ix^4 - 3x^3 - 4x + i$$

has $n_+ = 4$ roots with positive imaginary part and $n_- = 1$ root with negative imaginary part. Taking the real part of $f(x)$ gives the real polynomial $P(x) = x^5 - 3x^3 - 4x$ with roots $-2, 0, 2, i, -i$ and taking the imaginary part gives the polynomial $Q(x) = -x^4 + 1$ with roots $-1, 1, i, -i$. In this example, the bound given in Theorem 2.3.1 on the number of real roots of $P$ and $Q$ is achieved.
Note that it is also possible to prove Theorem 2.3.1 directly (without reference to Cauchy indices). In Appendix A, we give an inductive proof using Lemma 2.2.4 as the base case. This proof reveals additional information about the structure of the real roots of $P$ and $Q$, and together with this additional information, Theorem 2.3.1 becomes equivalent to Theorem 2.2.8. Hence, Appendix A may also be seen as an alternate proof of Theorem 2.2.8.

As a consequence of Theorem 2.3.1, we obtain the following generalisation of Corollary 2.2.5.

**Corollary 2.3.3.** Consider the real polynomial

$$f(x) := x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n.$$  

Suppose $f$ has $n_+$ roots with positive real part, $n_-$ roots with negative real part and $n_0 < n$ purely imaginary roots ($n_+ + n_- + n_0 = n$). Let

$$P(x) := x^n - a_2 x^{n-2} + a_4 x^{n-4} - \cdots,$$

$$Q(x) := a_1 x^{n-1} - a_3 x^{n-3} + a_5 x^{n-5} - \cdots$$  

(7)

and $d := n - 2 \min(n_+, n_-)$. Then (counting multiplicities) there exist at least $d$ real roots of $P$ (say $\nu_1, \nu_2, \ldots, \nu_d$) and at least $d - 1$ real roots of $Q$ (say $\nu_1, \nu_2, \ldots, \nu_{d-1}$) such that

$$\nu_1 \leq \nu_2 \leq \nu_3 \leq \cdots \leq \nu_{d-1} \leq \nu_d.$$  

(8)

If $n_0 = 0$, then the inequalities in (217) may be assumed to be strict.

**Proof.** The real parts of the roots of $f$ correspond to the imaginary parts of the roots of the polynomial

$$g(x) := i^n f(-ix) = x^n + i a_1 x^{n-1} - a_2 x^{n-2} - i a_3 x^{n-3} + \cdots.$$  

The result follows from Theorem 2.3.1. \qed

It turns out that, under certain circumstances, we can infer the existence of an additional two real roots of the polynomial $Q$ given in (7). We will need these additional roots in the next section.

**Observation 2.3.4.** Assume the hypotheses and conclusion of Theorem 2.3.1 (alternatively Corollary 2.3.3).

(i) If $n_+ > n_-$ and $\lim_{x \to -\infty} (P(x)/Q(x)) = \infty$, or alternatively if $n_- < n_+$ and $\lim_{x \to -\infty} (P(x)/Q(x)) = -\infty$, then there exists an additional real root $\nu_0$ of $Q$ such that $\nu_0 \leq \nu_1$.

(ii) If $n_+ > n_-$ and $\lim_{x \to \infty} (P(x)/Q(x)) = -\infty$, or alternatively if $n_- < n_+$ and $\lim_{x \to \infty} (P(x)/Q(x)) = \infty$, then there exists an additional real root $\nu_d$ of $Q$ such that $\nu_d \geq \nu_d$.

If $n_0 = 0$, then $\nu_0 < \nu_1$ and $\nu_d > \nu_d$. 

Proof. Assume the hypotheses and conclusion of Theorem 2.3.1 (those of Corollary 2.3.3 are equivalent). First suppose \( n_- > n_+ \) and

\[
\lim_{x \to -\infty} \frac{P(x)}{Q(x)} = \infty. \tag{9}
\]

In the proof of Theorem 2.3.1, the first element \( p_1 \) of \( T' \) was chosen such that

\[
\lim_{x \to p_1^-} \frac{Q(x)}{P(x)} = -\infty.
\]

Hence, in this case, (9) implies the existence of an additional real root \( w_0 \) of \( Q(x)/P(x) \) such that \( w_0 < p_1 \). It follows that there exists an additional real root \( \nu_0 \) of \( Q \) such that \( \nu_0 \leq \mu_1 \).

The remaining cases are dealt with similarly. \( \square \)

2.4 An Application

Recall that the NIEP has been solved for lists of the form \((\rho, \lambda_2, \lambda_3, \ldots, \lambda_n)\), where \( \rho \geq 0 \) and \( \text{Re}(\lambda_i) \leq 0 \) : \( i = 2, 3, \ldots, n^2 \) and that Lemma 1.3.7 was the key ingredient in this result. Using the results in this chapter, it is possible to generalise Lemma 1.3.7 whilst simultaneously giving a more compact proof. This more general lemma will be applied to the NIEP in Chapter 5.

Lemma 2.4.1. Consider the real polynomial

\[
f(x) := x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n.
\]

Suppose \( f \) has roots \( r, x_2, x_3, \ldots, x_n \), where \( r \) is real and \( \text{Re}(x_j) \leq 0 \) : \( j = 2, 3, \ldots, n \). Then the sequence \( a_1, a_2, \ldots, a_n \) satisfies the following conditions:

(i) Let \( t \) be the largest integer such that \( a_{2t} \neq 0 \). Then either \( a_{2j} > 0 \) for all \( j = 1, 2, \ldots, t \), or there exists \( s \in \{1, 2, \ldots, t\} \) such that

\[
\begin{align*}
    a_{2j} &> 0 : \quad j = 1, 2, \ldots, s-1, \\
    a_{2s} &\leq 0, \\
    a_{2j} &< 0 : \quad j = s+1, s+2, \ldots, t.
\end{align*}
\]

(ii) Let \( t' \) be the largest integer such that \( a_{2t'-1} \neq 0 \). Then either \( a_{2j-1} > 0 \) for all \( j = 1, 2, \ldots, t' \), or there exists \( s' \in \{1, 2, \ldots, t'\} \) such that

\[
\begin{align*}
    a_{2j-1} &> 0 : \quad j = 1, 2, \ldots, s'-1, \\
    a_{2s'-1} &\leq 0, \\
    a_{2j-1} &< 0 : \quad j = s'+1, s'+2, \ldots, t'.
\end{align*}
\]

\(^2\) See Theorem 1.3.5.
Proof. First suppose \( n \) is even and write \( n = 2m \). The polynomial
\[
f(x) = x^{2m} + a_1x^{2m-1} + a_2x^{2m-2} + \cdots + a_{2m}
\]
has at most one root with positive real part. Therefore, by Corollary 2.3.3, the polynomial
\[
x^{2m} - a_2x^{2m-2} + a_4x^{2m-4} - \cdots + (-1)^ma_{2m}
\]
has at least \( 2m - 2 \) real roots. It follows that the polynomial
\[
y^m - a_2y^{m-1} + a_4y^{m-2} - \cdots + (-1)^ma_{2m}
\]
has at least \( m - 1 \) nonnegative roots. Let \( t \) be the largest integer such that \( a_{2t} \neq 0 \). Then the polynomial
\[
y^t - a_2y^{t-1} + a_4y^{t-2} - \cdots + (-1)^{t}a_{2t}
\]
has at least \( t - 1 \) positive roots. Therefore, by Descartes’ rule of signs, the number of sign changes which occur between consecutive nonzero terms of the sequence
\[
T := (1, -a_2, a_4, -a_6, \ldots, (-1)^ta_{2t})
\]
is at least \( t - 1 \). In particular, since \( T \) contains \( t + 1 \) elements, this implies at most one of the elements in \( T \) is zero. There are now three cases to consider:

Case 1: If every element in \( T \) is nonzero and \( T \) has \( t \) sign changes, then \( a_{2j} > 0 \) for each \( j = 1, 2, \ldots, t \).

Case 2: If every element in \( T \) is nonzero and \( T \) has \( t - 1 \) sign changes, then the sequence
\[
(1, a_2, a_4, \ldots, a_{2t})
\]
has precisely one sign change.

Case 3: Suppose there exists \( s \in \{1, 2, \ldots, t\} \) such that \( a_{2s} = 0 \). Then, removing \( a_{2s} \) from \( T \), we obtain a sequence
\[
T_0 := (1, -a_2, a_4, \ldots, (-1)^{s-1}a_{2s-2}, (-1)^{s-1}a_{2s+2}, \ldots, (-1)^ta_{2t})
\]
with \( t \) elements (each nonzero) and \( t - 1 \) sign changes. It follows that
\[
a_{2j} > 0 : \quad j = 1, 2, \ldots, s - 1,
\]
\[
a_{2j} < 0 : \quad j = s + 1, s + 2, \ldots, t.
\]
We have now shown that the sequence \( a_2, a_4, \ldots \) satisfies condition (i).

Similarly, by Corollary 2.3.3, the polynomial
\[
a_1x^{2m-1} - a_3x^{2m-3} + a_5x^{2m-5} - \cdots + (-1)^{m-1}a_{2m-1}x \quad (10)
\]
has at least \( 2m - 3 \) real roots, one of which is zero. It follows that the polynomial
\[
a_1y^{m-1} - a_3y^{m-2} + a_5y^{m-3} - \cdots + (-1)^{m-1}a_{2m-1}
\]
has at least \( m - 2 \) nonnegative roots. Let \( t' \) be the largest integer such that \( a_{2t'-1} \neq 0 \). Then the polynomial
\[
a_{1}y^{t'-1} - a_{3}y^{t'-2} + a_{5}y^{t'-3} - \cdots + (-1)^{t'-1}a_{2t'-1}
\]
has at least \( t' - 2 \) positive roots. Therefore, by Descartes’ rule of signs, the number of sign changes which occur between consecutive nonzero terms of the sequence
\[
\mathcal{T}' := (a_{1}, -a_{3}, a_{5}, \ldots, (-1)^{t'-1}a_{2t'-1})
\]
is at least \( t' - 2 \). As above, this implies at most one of the elements in \( \mathcal{T}' \) is zero.

If \( a_1 > 0 \), then the sequences \( \mathcal{T} \) and \( \mathcal{T}' \) have the same properties. In this case, it follows from the above that the sequence \( a_{1}, a_{3}, \ldots \) satisfies condition (ii).

If \( a_1 < 0 \), then for \( P(x) \) and \( Q(x) \) defined as in (7), we see that
\[
\lim_{x \to -\infty}(P(x)/Q(x)) = \infty \quad \text{and} \quad \lim_{x \to \infty}(P(x)/Q(x)) = -\infty.
\]
Hence, by Observation 2.3.4, every root of (10) is real. It follows that (11) has \( t' - 1 \) positive roots and \( \mathcal{T}' \) has \( t' - 1 \) sign changes. Therefore \( a_{2j-1} < 0 \) for all \( j = 1, 2, \ldots, t' \).

Finally, if \( a_1 = 0 \), then consider the polynomial
\[
f_\epsilon(x) := (x - r - \epsilon) \prod_{j=2}^{n}(x - x_j) = x^n - \epsilon x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \cdots + b_n,
\]
where \( \epsilon > 0 \). From the above, we see that \( b_{2j-1} \leq 0 \): \( j = 2, 3, \ldots, \lfloor n/2 \rfloor \). Furthermore, since each \( b_j \) depends continuously on \( \epsilon \) and
\[
\lim_{\epsilon \to 0} f_\epsilon(x) = f(x),
\]
it follows that \( a_{2j-1} \leq 0 \): \( j = 2, 3, \ldots, \lfloor n/2 \rfloor \). Since at most one of the elements in \( \mathcal{T}' \) is zero, we conclude that \( a_{2j-1} < 0 \) for all \( j = 2, 3, \ldots, t' \). We have now shown that the sequence \( a_{1}, a_{3}, \ldots \) satisfies condition (ii).

The proof for odd \( n \) is similar. \( \square \)

**Example 2.4.2.** Consider the polynomial
\[
f(x) = x^{12} + a_1x^{11} + a_2x^{10} + \cdots + a_{12}
= x^{12} + x^{11} + 9x^{10} - x^9 + 14x^8 - 56x^7 - 64x^6 - 144x^5 - 160x^4
\]
with roots \( (2, -1 + 2i, -1 - 2i, 2i, -2i, 2i, -2i, -1, 0, 0, 0, 0) \). The sequences
\[
(a_{2j})_{j=1}^{6} = (9, 14, -64, -160, 0, 0)
\]
and
\[
(a_{2j-1})_{j=1}^{6} = (1, -1, -56, -144, 0, 0)
\]
correspond to \( (s, t) = (3, 4) \) and \( (s', t') = (2, 4) \) in Lemma 2.4.1, respectively.
NEWTON-LIKE INEQUALITIES FOR SETS OF COMPLEX NUMBERS

3.1 INTRODUCTION

The \( k \)-th elementary symmetric function of the variables \( x_1, x_2, \ldots, x_n \) is defined by

\[
e_0(x_1, x_2, \ldots, x_n) := 1, \quad e_k(x_1, x_2, \ldots, x_n) := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} : \quad k = 1, 2, \ldots, n.
\]

It will also be convenient to define \( e_k(x_1, x_2, \ldots, x_n) = 0 \) if \( k < 0 \) or \( k > n \).

In this chapter, we give families of inequalities involving the elementary symmetric functions of a set of self-conjugate complex numbers with nonnegative real parts and show that the given inequalities are optimal. In Chapter 5, we will apply these inequalities to the NIEP, but they are also of general interest.

3.2 PRELIMINARIES

3.2.1 Newton’s inequalities for real numbers

In order to state the celebrated Newton’s inequalities, it is more convenient to consider the \( k \)-th elementary symmetric mean

\[
E_k(x_1, x_2, \ldots, x_n) := \binom{n}{k}^{-1} e_k(x_1, x_2, \ldots, x_n) : \quad k = 0, 1, \ldots, n.
\]

For brevity, we will often write simply \( e_k \) or \( E_k \) when there is no confusion as to the variables involved.

**Theorem 3.2.1. (Newton’s Inequalities)** If \( X := (x_1, x_2, \ldots, x_n) \) is a list of real numbers, then

\[
E_k(X)^2 \geq E_{k-1}(X) E_{k+1}(X) : \quad k = 1, 2, \ldots, n - 1,
\]

with equality if and only if all of the \( x_i \) coincide or both sides vanish.

Theorem 3.2.1 is a consequence of a rule stated (without proof) by Newton: Given a polynomial

\[
p(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n, \quad a_0 \neq 0,
\]
the number of its nonreal roots is greater than or equal to the number of sign changes that occurs in the sequence

\[
a_0^2 \frac{a_1}{(n_1)^2} - a_0^2 \frac{a_2}{(n_2)^2} - a_1 \frac{a_3}{(n_3)^2} - \cdots - a_{n-1} \frac{a_n}{(n_{n-1})^2} = \frac{a_{n-2}}{n_{n-2}} \frac{a_n}{n} a^2_n.
\]

If the roots of \( p \) are real, then all terms in the above sequence must be nonnegative, thus yielding Newton’s inequalities; however, since Newton did not give a proof of his rule, the proof of Theorem 3.2.1 is due to MacLaurin. For an inductive proof of Theorem 3.2.1 in the case where \( x_1, x_2, \ldots, x_n \) are nonnegative, see [22, §2.22]. For a proof by differential calculus in the case where \( x_1, x_2, \ldots, x_n \) are real, see [22, §4.3], or alternatively [51]. In general, a sequence of nonnegative numbers \((E_k)\) is said to be \textit{log-concave} if \( E_k^2 \geq E_{k-1}E_{k+1} \) for all \( k \). Therefore, the study of Newton-like inequalities for sets of complex numbers is further motivated by the literature on log-concave sequences.\(^1\)

Note that (13) is equivalent to

\[
e_k(x)^2 \geq \frac{k + 1}{k} \frac{n - k + 1}{n - k} e_{k-1}(x)e_{k+1}(x),
\]

which is stronger than

\[
e_k(x)^2 \geq e_{k-1}(x)e_{k+1}(x).
\]

It is well-known that (13) is equivalent to

\[
E_k E_l \geq E_{k-1}E_{l+1} : \quad 1 \leq k < l \leq n - 1,
\]

provided \( E_1, E_2, \ldots, E_n \geq 0 \) and the sequence \( E_1, E_2, \ldots, E_n \) has no internal zeros, namely if \( k < l \), then \( E_k, E_l > 0 \) implies \( E_i > 0 \) for all \( k < i < l \). This follows from the fact that

\[
E_k^2 E_{k+1}^2 \cdots E_l^2 \geq (E_{k-1}E_{k+1})(E_kE_{k+2}) \cdots (E_{l-1}E_{l+1}).
\]

In particular, if the \( x_i \) are nonnegative, then (14) holds.

Several reformulations and generalisations of Newton’s inequalities have been given over the years, for example in [43, 51, 62] and more recently in [47, 54]. The relationship between Newton’s inequalities and matrix spectra have been studied in [24, 26]. Newton-like inequalities for certain families of complex numbers have been studied in [45, 63, 64].

3.2.2 \textit{Newton-like inequalities for families of complex numbers}

Now suppose \( X := (x_1, x_2, \ldots, x_n) \) is a list of complex numbers. It is natural to assume that \( X \) is self-conjugate, since this ensures that each \( e_i(X) \) is a real number. We will also assume that the \( x_i \) have

\(^1\) See [6, 59].
nonnegative real parts, since this guarantees that $e_i(\mathcal{X}) \geq 0 : i = 0, 1, \ldots, n$.

In general, Newton’s inequalities do not hold under these assumptions; however, Monov [45] showed that a weaker version of Theorem 3.2.1 does hold. For $0 \leq \lambda \leq 1$, define the wedge

$$\Omega := \{ z \in \mathbb{C} : |\arg(z)| \leq \cos^{-1} \sqrt{\lambda} \}.$$

**Theorem 3.2.2.** [45] Let $\mathcal{X} := (x_1, x_2, \ldots, x_n)$ be a list of self-conjugate variables in $\Omega$. Then

$$E_k(\mathcal{X})^2 \geq \lambda E_{k-1}(\mathcal{X}) E_{k+1}(\mathcal{X}) : \ k = 1, 2, \ldots, n - 1. \quad (15)$$

Theorem 3.2.2 was generalised by Xu [63, 64]:

**Theorem 3.2.3.** [63, 64] Let $\mathcal{X} := (x_1, x_2, \ldots, x_n)$ be a list of self-conjugate variables in $\Omega$. Then

$$E_k(\mathcal{X}) E_l(\mathcal{X}) \geq \lambda E_{k-1}(\mathcal{X}) E_{l+1}(\mathcal{X}) : \ 1 \leq k \leq l \leq n - 1. \quad (16)$$

The inequalities in (15) are known as the $\lambda$-Newton inequalities and those in (16) are known as the generalised $\lambda$-Newton inequalities.

### 3.3 New Newton-like Inequalities for Complex Numbers

Note that the strength of the inequalities in (16) depends on the proximity of the $x_i$ to the real axis, via the parameter $\lambda$. In particular, if the $x_i$ are all real, then (16) reduces to Newton’s inequalities. On the other hand, if any of the $x_i$ are purely imaginary, then (16) reduces to the trivial inequality $E_k E_l \geq 0$. In this section, we develop inequalities of the form

$$e_k(\mathcal{X}) e_l(\mathcal{X}) \geq C e_{k-h}(\mathcal{X}) e_{l+h}(\mathcal{X}), \quad (17)$$

where the constant $C$ is independent of $x_1, x_2, \ldots, x_n$.

The following simple example illustrates that, in some cases, the best possible constant in (17) is $C = 0$:

**Example 3.3.1.** Consider the list $\mathcal{X} := (i, -i, i, -i, \ldots, i, -i)$ of length $2m$. We have

$$e_{2i}(\mathcal{X}) = \binom{m}{i} : \ i = 0, 1, \ldots, m,$$

$$e_{2i+1}(\mathcal{X}) = 0 : \ i = 0, 1, \ldots, m - 1.$$

This example shows us that if $k, l$ and $h$ are all odd, then we are forced to choose $C = 0$ in (17).

If $k$ and $l$ have the same parity and $h$ is even, the constant $C$ is best expressed by normalising the elementary symmetric functions in a new way: let us define

$$P_{2k}(\mathcal{X}) := \binom{[n/2]}{k}^{-1} e_{2k}(\mathcal{X}) : \ k = 0, 1, \ldots, [n/2],$$
\[ P_{2k+1}(x) := \left(\frac{n}{2} - 1\right)^{-1} e_{2k+1}(x) : \quad k = 0, 1, \ldots, \lfloor n/2 \rfloor - 1. \]

We then have:

**Theorem 3.3.2.** Let \( X := (x_1, x_2, \ldots, x_n) \) be a self-conjugate list of complex numbers with nonnegative real parts. Then

\[ P_{2k}(X)P_{2l}(X) \geq P_{2k-2}(X)P_{2l+2}(X) : \quad 1 \leq k \leq l \leq \lfloor n/2 \rfloor - 1 \quad (18) \]

and

\[ P_{2k+1}(X)P_{2l+1}(X) \geq P_{2k-1}(X)P_{2l+3}(X) : \quad 1 \leq k \leq l \leq \lfloor n/2 \rfloor - 2. \quad (19) \]

**Proof.** First suppose \( n \) is even and write \( n = 2m \). The polynomial

\[ x^{2m} + e_1(X)x^{2m-1} + e_2(X)x^{2m-2} + \cdots + e_{2m}(X) \]

has roots \( -x_1, -x_2, \ldots, -x_{2m} \). Therefore, by Corollary 2.2.5, the polynomial

\[ x^{2m} - e_2(X)x^{2m-2} + e_4(X)x^{2m-4} - \cdots + (-1)^me_{2m}(X) \]

has real roots. Hence the roots of the polynomial

\[ w^m - e_2(X)w^{m-1} + e_4(X)w^{m-2} - \cdots + (-1)^me_{2m}(X), \]

say \( w_1, w_2, \ldots, w_m \), are real and nonnegative. Setting \( W := (w_1, w_2, \ldots, w_m) \), we note that \( e_k(W) = e_{2k}(X) : k = 0, 1, \ldots, m \), and hence, applying Newton’s inequalities (14) to \( W \) gives

\[ \frac{e_{2k}(X)}{e_{2k-2}(X)} \frac{e_{2l+2}(X)}{e_{2l}(X)} \geq \frac{m}{m-1} \]

or

\[ P_{2k}(X)P_{2l}(X) \geq P_{2k-2}(X)P_{2l+2}(X) : \quad 1 \leq k \leq l \leq m - 1. \]

Similarly, by Corollary 2.2.5, the polynomial

\[ e_1(X)x^{2m-1} - e_3(X)x^{2m-3} + e_5(X)x^{2m-5} - \cdots + (-1)^{m-1}e_{2m-1}(X)x \]

has real roots. If \( e_1(X) = 0 \), then \( \text{Re}(x_i) = 0 \) for all \( i \). This would imply that \( e_{2k+1}(X) = 0 \) for all \( 0 \leq k \leq m - 1 \), in which case (19) holds trivially. If \( e_1(X) > 0 \), it follows that the roots of the polynomial

\[ w^{m-1} - \frac{e_3(X)}{e_1(X)}w^{m-2} + \frac{e_5(X)}{e_1(X)}w^{m-3} - \cdots + (-1)^{m-1}\frac{e_{2m-1}(X)}{e_1(X)}, \]

say \( w_1, w_2, \ldots, w_{m-1} \), are real and nonnegative. We note that

\[ e_k(W) = \frac{e_{2k+1}(X)}{e_1(X)} : \quad k = 0, 1, \ldots, m - 1, \]
and hence, applying Newton’s inequalities \((14)\) to \(W\) gives
\[
\frac{e_{2k+1}(X)}{(m-1)_k} \frac{e_{2l+1}(X)}{(m-1)_l} \geq \frac{e_{2k-1}(X)}{(m-1)_{k-1}} \frac{e_{2l+3}(X)}{(m-1)_{l+1}}
\]
or
\[
P_{2k+1}(X)P_{2l+1}(X) \geq P_{2k-1}(X)P_{2l+3}(X) : \quad 1 \leq k \leq l \leq m - 2.
\]
The proof for odd \(n\) is similar.

As with Newton’s inequalities, we note that \((18)\) and \((19)\) are stronger than \(e_{2k}e_{2l} \geq e_{2k-2}e_{2l+2}\) and \(e_{2k+1}e_{2l+1} \geq e_{2k-1}e_{2l+3}\), respectively.

If \(k\) and \(l\) have different parity and \(h = 1\), it turns out that the best possible constant in \((17)\) is \(C = 1\):

**Theorem 3.3.3.** Let \(X := (x_1, x_2, \ldots, x_n)\) be a list of self-conjugate variables with nonnegative real parts. If \(k\) and \(l\) have different parity, \(1 \leq k < l \leq n - 1\), then
\[
e_k(X)e_1(X) \geq e_{k-1}(X)e_{l+1}(X).
\]

*Proof.* Let us write
\[
X = (a_1 \pm ib_1, a_2 \pm ib_2, \ldots, a_m \pm ib_m, \mu_1, \mu_2, \ldots, \mu_s),
\]
where \(n = 2m + s\) and the \(a_i, b_i\) and \(\mu_i\) are nonnegative. Consider the functions
\[
f(a_1, \ldots, a_m, b_1, \ldots, b_m, \mu_1, \ldots, \mu_s) = e_k(X)e_1(X) - e_{k-1}(X)e_{l+1}(X)
\]
and
\[
g(a_1, \ldots, a_m, b_1, \ldots, b_m, \mu_1, \ldots, \mu_s) = e_k(X)e_1(X) - e_{k-2}(X)e_{l+2}(X)
\]
as multivariable polynomials in \(a_1, \ldots, a_m, b_1, \ldots, b_m, \mu_1, \ldots, \mu_s\). We claim that

(i) for all \(1 \leq k < l \leq n - 1\), where \(k\) and \(l\) have different parity, the coefficient of every term in \(f\) is positive and

(ii) for all \(2 \leq k \leq l \leq n - 2\), where \(k\) and \(l\) have the same parity, the coefficient of every term in \(g\) is positive.

The proof is by induction on \(n\). If \(n = 1\) or \(n = 2\), then there is nothing to prove. Now assume that (i) and (ii) hold for all lists of length strictly less than \(n\). If \(s > 0\), we note that, for \(i = 0, 1, \ldots, n,\)
\[
e_i(X) = e_i(X') + \mu_se_{i-1}(X'),
\]
where
\[
X' := (a_1 \pm ib_1, a_2 \pm ib_2, \ldots, a_m \pm ib_m, \mu_1, \mu_2, \ldots, \mu_{s-1}).
\]
Therefore, we may write
\[ e_k(x)e_t(x) - e_{k-1}(x)e_{t+1}(x) = \Lambda \mu^2 + B \mu_s + C, \]
where
\[ A := e_{k-2}(x')e_{t-2}(x') - e_{k-3}(x')e_{t-1}(x'), \]
\[ B := e_k(x')e_{t-1}(x') - e_{k-2}(x')e_{t+1}(x'), \]
\[ C := e_k(x')e_t(x') - e_{k-1}(x')e_{t+1}(x'). \]

If \( k \) and \( l \) have different parity, \( 1 \leq k < l \leq n - 1 \), then the inductive hypothesis guarantees that \( A, B \) and \( C \) consist entirely of positive terms. Hence every term in \( f \) is positive.

Similarly, we may write
\[ e_k(x)e_t(x) - e_{k-2}(x)e_{t+2}(x) = \Lambda \mu^2 + B \mu_s + C, \]
where
\[ A := e_{k-2}(x')e_{t-1}(x') - e_{k-3}(x')e_{t+1}(x'), \]
\[ B := e_k(x')e_{t-1}(x') - e_{k-2}(x')e_{t+2}(x') \]
\[ + e_k(x')e_{t-1}(x') - e_{k-2}(x')e_{t+1}(x'), \]
\[ C := e_k(x')e_{t-1}(x') - e_{k-2}(x')e_{t+2}(x'). \]

If \( k \) and \( l \) have the same parity, \( 2 \leq k < l \leq n - 2 \), then the inductive hypothesis again guarantees that \( A, B \) and \( C \) consist entirely of positive terms. Hence every term in \( g \) is positive.

On the other hand, if \( s = 0 \), we note that for \( i = 0, 1, \ldots, n \),
\[ e_i(x) = e_i(x') + 2a_m e_{i-1}(x') + (a_m^2 + b_m^2) \ e_{i-2}(x'), \]
where
\[ x' := (a_1 \pm ib_1, a_2 \pm ib_2, \ldots, a_{m-1} \pm ib_{m-1}). \]

Hence, we may write
\[
e_k(x)e_t(x) - e_{k-1}(x)e_{t+1}(x)
= \Lambda (a_m^2 + b_m^2)^2 + (B + 2Xa_m)(a_m^2 + b_m^2) + 4Ya_m^2 + 2Za_m + C,
\]
where
\[ A := e_{k-2}(x')e_{t-2}(x') - e_{k-3}(x')e_{t-1}(x'), \]
\[ B := e_k(x')e_{t-1}(x') - e_{k-2}(x')e_{t+1}(x') \]
\[ + e_k(x')e_{t-2}(x') - e_{k-1}(x')e_{t-1}(x'), \]
\[ C := e_k(x')e_{t-1}(x') - e_{k-1}(x')e_{t+1}(x'), \]
\[ X := e_k(x')e_{t-2}(x') - e_{k-3}(x')e_{t-1}(x'), \]
\[ Y := e_k(x')e_{t-1}(x') - e_{k-2}(x')e_{t+1}(x'), \]
\[ Z := e_k(x')e_{t-1}(x') - e_{k-2}(x')e_{t+1}(x'). \]
If $k$ and $l$ have different parity, $1 \leq k < l \leq n - 1$, then the inductive hypothesis guarantees that $A, B, C, X, Y$ and $Z$ consist entirely of positive terms. Hence every term in $f$ is positive, as before.

Similarly, we may write
\[
e_k(x)e_l(x) - e_{k-2}(x)e_{l+2}(x)
= A(a_m^2 + b_m^2) + (B + 2Xa_m)(a_m^2 + b_m^2) + 4Ya_m^2 + 2Za_m + C,
\]
where
\[
A := e_{k-2}(x')e_{l-2}(x') - e_{k-4}(x')e_{l}(x'),
B := e_k(x')e_{l-2}(x') - e_{k-4}(x')e_{l+2}(x'),
C := e_k(x')e_{l}(x') - e_{k-2}(x')e_{l+2}(x'),
X := e_{k-2}(x')e_{l-1}(x') - e_{k-4}(x')e_{l+1}(x')
+ e_{k-1}(x')e_{l-2}(x') - e_{k-3}(x')e_{l}(x'),
Y := e_{k-1}(x')e_{l-1}(x') - e_{k-3}(x')e_{l+1}(x'),
Z := e_{k-1}(x')e_{l}(x') - e_{k-3}(x')e_{l+2}(x')
+ e_k(x')e_{l-1}(x') - e_{k-2}(x')e_{l+1}(x').
\]

If $k$ and $l$ have the same parity, $2 \leq k < l \leq n - 2$, then the inductive hypothesis guarantees that $A, B, C, X, Y$ and $Z$ consist entirely of positive terms. Hence every term in $g$ is positive, as before.

**Remark.** In the proof of Theorem 3.3.3, we saw that if $k$ and $l$ have the same parity, then $e_k e_l \geq e_{k-2} e_{l+2}$ and the difference $e_k e_l - e_{k-2} e_{l+2}$ is a multivariable polynomial in $a_1, \ldots, a_m, b_1, \ldots, b_m, m_1, \ldots, m_t$ consisting entirely of positive terms. This inequality is weaker than the inequality $P_k P_l \geq P_{k-2} P_{l+2}$, obtained from Theorem 3.3.2, but the difference $P_k P_l - P_{k-2} P_{l+2}$ does not consist entirely of positive terms.

It is clear that if $k$ and $l$ have different parity, then Theorem 3.3.3 implies
\[
e_k(x)e_l(x) \geq e_{k-h}(x)e_{l+h}(x) : h = 2, 3, \ldots;
\]
however, such inequalities may always be strengthened by combining Theorems 3.3.2 and 3.3.3. For example, if $n$ is odd, then it is clear from the definition of $P_t$ that
\[
\frac{P_{2k-1} P_{2k+2}}{P_{2k-2} P_{2k+3}} = \frac{e_{2k-1} e_{2k+2}}{e_{2k-2} e_{2k+3}}
\]
and in this case,
\[
P_{2k} P_{2k+1} \geq \sqrt{P_{2k-2} P_{2k-1} P_{2k+2} P_{2k+3}} \geq P_{2k-2} P_{2k+3},
\]
where the first inequality follows from Theorem 3.3.2 and the second follows from Theorem 3.3.3 and (21). This is stronger than the inequality $e_{2k} e_{2k+1} \geq e_{2k-2} e_{2k+3}$, which would be obtained from Theorem 3.3.3 alone.
We have yet to consider the case when \( k \) and \( l \) are both even in (17), but \( h \) is odd. Specifically, we ask if it is possible to derive inequalities of form
\[
e_{2k}(X)e_{2l}(X) \geq Ce_{2k-1}(X)e_{2l+1}(X) : \quad 1 \leq k \leq l \leq (n-1)/2,
\]
where \( C > 0 \). It turns out that if we allow \( X \) to contain unpaired real numbers, then the answer is negative, as the following example illustrates.

**Example 3.3.4.** Consider the list
\[
X := (\epsilon_i, -\epsilon_i, \epsilon_i, -\epsilon_i, \ldots, \epsilon_i, -\epsilon_i, 1)
\]
of length \( n = 2m + 1 \). We have
\[
e_1(X) = 1,
\]
\[
e_{2i}(X) = e_{2(l+1)}(X) = \binom{m}{i}e^l : \quad i = 1, 2, \ldots, m.
\]
Hence, for all \( 1 \leq k \leq l \leq m \),
\[
\frac{e_{2k}(X)e_{2l}(X)}{\sqrt{e_{2k-1}(X)e_{2l+1}(X)}} = \left( \frac{m+1}{k} - 1 \right) e.
\]
This example shows that, given any \( k, l \) and \( n \), it is always possible to find a list \( X \) of length \( n \), such that \( e_{2k}(X)e_{2l}(X) \) is arbitrarily small compared to \( e_{2k-1}(X)e_{2l+1}(X) \).

Surprisingly, if we insist that \( X \) contain only complex-conjugate pairs (all real numbers in \( X \) appear with even multiplicity), it turns out that
\[
e_{2k}(X)^2 \geq e_{2k-1}(X)e_{2k+1}(X) : \quad k = 1, 2, \ldots, m - 1. \tag{22}
\]
We note the similarity of (22) to Newton’s inequalities (13). To prove (22), we first require a technical lemma.

**Lemma 3.3.5.** Let \( X := (a_1 \pm ib_1, a_2 \pm ib_2, \ldots, a_m \pm ib_m) \), where \( a_i, b_i \geq 0 : i = 1, 2, \ldots, m \). Let \( U := \{1, 2, \ldots, m\} \) and for each \( S \subseteq U \), let \( W_S := (a_i^2 + b_i^2 : i \in S) \). Then for \( 0 \leq k \leq m \),
\[
e_{2k}(X) = \sum_{r=0}^{k} 2^{2r} \sum_{S \subseteq U \atop |S| = 2r} \left( \prod_{i \in S} a_i \right) e_{k-r}(W_U \setminus S) \tag{23}
\]
and for \( 1 \leq k \leq m \),
\[
e_{2k-1}(X) = \sum_{r=0}^{k-1} 2^{2r+1} \sum_{S \subseteq U \atop |S| = 2r+1} \left( \prod_{i \in S} a_i \right) e_{k-r-1}(W_U \setminus S). \tag{24}
\]
Proof. The proof is by induction on \( m \). If \( m = 1 \), then (23) and (24) give \( e_0(\mathcal{X}) = 1, e_1(\mathcal{X}) = 2a_1 \) and \( e_2(\mathcal{X}) = a_1^2 + b_1^2 \), as required. Now assume the statement holds for lists with \( m - 1 \) complex-conjugate pairs.

We note that, for \( i = 0, 1, \ldots, 2m \),

\[
e_i(\mathcal{X}) = e_i(\mathcal{X}') + 2a_m e_{i-1}(\mathcal{X}') + (a_m^2 + b_m^2) e_{i-2}(\mathcal{X}'),
\]

where \( \mathcal{X}' := (a_1 \pm ib_1, a_2 \pm ib_2, \ldots, a_{m-1} \pm ib_{m-1}) \). Hence, by (25) and the inductive hypothesis,

\[
e_{2k}(\mathcal{X}) = \sum_{r=0}^{k} 2^{2r} \sum_{S \subseteq U' \atop |S| = 2r} \left( \prod_{i \in S} a_i \right) e_{k-r}(\mathcal{W}_{U \setminus S})
\]

\[
+ a_m \sum_{r=1}^{k} 2^{2r} \sum_{S \subseteq U' \atop |S| = 2r-1} \left( \prod_{i \in S} a_i \right) e_{k-r}(\mathcal{W}_{U \setminus S}) \tag{26}
\]

\[
+ (a_m^2 + b_m^2) \sum_{r=0}^{k-1} 2^{2r} \sum_{S \subseteq U' \atop |S| = 2r} \left( \prod_{i \in S} a_i \right) e_{k-r-1}(\mathcal{W}_{U \setminus S}),
\]

where \( U' := \{1, 2, \ldots, m\} \); however, since

\[
\sum_{S \subseteq U \atop |S| = 2r} \left( \prod_{i \in S} a_i \right) e_{k-r}(\mathcal{W}_{U \setminus S}) =
\]

\[
\sum_{S \subseteq U' \atop |S| = 2r} \left( \prod_{i \in S} a_i \right) e_{k-r}(\mathcal{W}_{U \setminus S} | \cup(m))
\]

\[
+ a_m \sum_{S \subseteq U' \atop |S| = 2r-1} \left( \prod_{i \in S} a_i \right) e_{k-r}(\mathcal{W}_{U \setminus S})
\]

and

\[
e_{k-r}(\mathcal{W}_{U \setminus S} | \cup(m)) = e_{k-r}(\mathcal{W}_{U \setminus S}) + (a_m^2 + b_m^2) e_{k-r-1}(\mathcal{W}_{U \setminus S}),
\]

it follows that the right hand side of (26) equals

\[
\sum_{r=0}^{k} 2^{2r} \sum_{S \subseteq U \atop |S| = 2r} \left( \prod_{i \in S} a_i \right) e_{k-r}(\mathcal{W}_{U \setminus S}).
\]

This establishes (23).

The proof of (24) is similar. \( \square \)

Theorem 3.3.6. Let \( \mathcal{X} := (a_1 \pm ib_1, a_2 \pm ib_2, \ldots, a_m \pm ib_m) \), where \( a_i, b_i \geq 0 : i = 1, 2, \ldots, m \). Then

\[
e_{2k}(\mathcal{X})^2 \geq e_{2k-1}(\mathcal{X}) e_{2k+1}(\mathcal{X}) : \quad k = 1, 2, \ldots, m - 1. \tag{27}
\]
Proof. Let $\mathcal{U} := \{1, 2, \ldots, m\}$ and for each $S \subseteq \mathcal{U}$, let $\forall_S := (a_i^2 - b_i^2 : i \in S)$. We will show that

$$e_{2k}(x)^2 - e_{2k-1}(x)e_{2k+1}(x) \geq \Theta,$$

where

$$\Theta := \sum_{r=0}^{k-1} 2^{2r} \sum_{S \subseteq \mathcal{U}} \left( \prod_{i \in S} a_i^2 b_i^2 \right) e_{k-r}(\forall_{\mathcal{U} \setminus S})^2.$$

More specifically, consider the function

$$f(a_1, \ldots, a_m, b_1, \ldots, b_m) = e_{2k}(x)^2 - e_{2k-1}(x)e_{2k+1}(x) - \Theta$$

as a multivariable polynomial in $a_1, \ldots, a_m, b_1, \ldots, b_m$. We will prove:

**Claim 1**: The coefficient of every term in $f$ is positive.

Ultimately, the proof of Claim 1 will be by induction on $m$; however, before we begin, there is a term in $f$ whose coefficient we must explicitly compute. Consider

$$T := \prod_{i=1}^{k} a_i^2 b_i^2.$$

**Claim 2**: The coefficient of $T$ in $f$ is $2^{2k}$.

In order to prove Claim 2, we will determine the coefficients of $T$ in $e_{2k}(x)^2$, $e_{2k-1}(x)e_{2k+1}(x)$ and $\Theta$ separately. First, recall that, by Lemma 3.3.5, $e_{2k}(x)$ may be written in the form (23). The coefficient of $T$ in $e_{2k}(x)^2$ is calculated by considering the sum $\sum T_1 T_2$, where the sum is over all appropriately chosen terms $T_1$ and $T_2$ in (23). Suppose $T_1$ and $T_2$ correspond to choices $S = S_1$ and $S = S_2$ in (23), respectively. It is clear that since each $a_i$ in $T$ has exponent 2, the only contributions to the coefficient of $T$ in $e_{2k}(x)$ come from choosing $S_1 = S_2 \subseteq \{1, 2, \ldots, k\}$. In fact, we must choose $S_1 = S_2 = \emptyset$, since $i \in S$ implies $e_{k-r}(\forall_{\mathcal{U} \setminus S})$ is independent of $b_i$. Hence, the only contributions to the coefficient of $T$ come from setting $r = 0$ in (23), i.e., the coefficient of $T$ in $e_{2k}(x)^2$ is precisely the coefficient of $T$ in $e_k(\forall_{\mathcal{U}})^2$. This is the same as the coefficient of $T$ in $\prod_{i=1}^{k} (a_i^2 + b_i^2)^2$, which equals $2^k$.

Similarly, we note that $e_{2k-1}(x)$ may be written in the form (24) and that

$$e_{2k+1}(x) = \sum_{r=0}^{k} 2^{2r+1} \sum_{S \subseteq \mathcal{U}} \left( \prod_{i \in S} a_i \right) e_{k-r}(\forall_{\mathcal{U} \setminus S}). \quad (28)$$

Since it is not possible to choose $S = \emptyset$ in (24) or (28), we conclude that the coefficient of $T$ in $e_{2k-1}(x)e_{2k+1}(x)$ is zero.

To compute the coefficient of $T$ in $\Theta$, we note that for any set of integers $i_1, i_2, \ldots, i_{k-r}$ satisfying $1 \leq i_1 < i_2 < \cdots < i_{k-r} \leq k$ and
which equals \((-2)^{k-r}\). Hence, the coefficient of \(T\) in \(\Theta\) is

\[
(−2)^k \sum_{r=0}^{k-1} (-2)^r \binom{k}{r} = 2^k(1 - 2^k).
\]

This establishes Claim 2.

We are now ready to prove Claim 1 by induction. If \(m = 2\), we need only check the claim holds for \(k = 1\). Setting \(X = (a_1 \pm ib_1, a_2 \pm ib_2)\),

\[
e_2(X^2) - e_1(X)e_3(X) - e_1(a_1^2 - b_1^2, a_2^2 - b_2^2) = 4a_1^2a_2 + 8a_1^2b_2^2 + 4a_1a_2^3 + 4a_1^2b_1^2 + 4a_1a_2b_1^2 + 4a_1^2b_2^2 + 4a_2^2b_1^2 + 4a_2b_1^2b_2 + 4a_2^2b_2^2.
\]

Now assume the claim holds for lists with \(m - 1\) complex-conjugate pairs and all \(1 \leq k \leq m - 2\). Note that for \(i = 0, 1, \ldots, 2m\),

\[
e_i(X) = e_i(X') + 2a_me_{i-1}(X') + (a_m^2 + b_m^2)e_{i-2}(X'),
\]

where \(X' := (a_1 \pm ib_1, a_2 \pm ib_2, \ldots, a_{m-1} \pm ib_{m-1})\). Hence, for \(k = 1, 2, \ldots, m - 1\),

\[
e_{2k}(X^2) - e_{2k-1}(X)e_{2k+1}(X) = A(a_m^2 + b_m^2)^2 + (B + 2Xa_m)(a_m^2 + b_m^2) + 4Y a_m^2 + 2Z a_m + C,
\]

where

\[
A := e_{2k-2}(X')^2 - e_{2k-3}(X')e_{2k-1}(X'),
\]

\[
B := -e_{2k-1}(X')^2 + 2e_{2k-2}(X')e_{2k}(X') - e_{2k-3}(X')e_{2k+1}(X'),
\]

\[
C := e_{2k}(X')^2 - e_{2k-1}(X')e_{2k+1}(X'),
\]

\[
X := e_{2k-2}(X')e_{2k-1}(X') - e_{2k-3}(X')e_{2k}(X'),
\]

\[
Y := e_{2k-1}(X')^2 - e_{2k-2}(X')e_{2k}(X'),
\]

\[
Z := e_{2k-1}(X')e_{2k}(X') - e_{2k-2}(X')e_{2k+1}(X').
\]

Similarly, we may expand \(\Theta\) in terms of \(a_m\) and \(b_m\):

\[
\Theta = \alpha(a_m^4 + b_m^4) + 2\beta a_m^2 b_m^2 + 2\gamma(a_m^2 - b_m^2) + \delta,
\]

where

\[
U' := \{1, 2, \ldots, m - 1\},
\]

\[
\alpha := \sum_{r=0}^{k-1} 2^{2r} \sum_{S \subseteq U'} \prod_{i \in S} a_i^2 b_i^2 e_{k-r-1}(\mathcal{V}_{U' \setminus S})^2,
\]

\[
\beta := \frac{1}{2} \sum_{S \subseteq U'} \prod_{i \in S} a_i^2 b_i^2 e_{k-r}(\mathcal{V}_{U' \setminus S})^2,
\]

\[
\gamma := -\frac{1}{2} \sum_{S \subseteq U'} \prod_{i \in S} a_i^2 b_i^2 e_{k-r-1}(\mathcal{V}_{U' \setminus S})^2,
\]

\[
\delta := -\frac{1}{2} \sum_{S \subseteq U'} \prod_{i \in S} a_i^2 b_i^2 e_{k-r}(\mathcal{V}_{U' \setminus S})^2.
\]
\[ \beta := \sum_{r=0}^{k-2} 2^{2r} \sum_{S \subseteq U', \ |S| = r} \left( \prod_{i \in S} a_i^2 b_i^2 \right) e_{k-r-1} (\mathcal{V}_{U \setminus S})^2 \]

\[ -2^{2(k-1)} \sum_{S \subseteq U', \ |S| = k-1} \prod_{i \in S} a_i^2 b_i^2, \]

\[ \gamma := \sum_{r=0}^{k-1} 2^{2r} \sum_{S \subseteq U', \ |S| = r} \left( \prod_{i \in S} a_i^2 b_i^2 \right) e_{k-r} (\mathcal{V}_{U \setminus S}) e_{k-r-1} (\mathcal{V}_{U \setminus S}), \]

\[ \delta := \sum_{r=0}^{k-1} 2^{2r} \sum_{S \subseteq U', \ |S| = r} \left( \prod_{i \in S} a_i^2 b_i^2 \right) e_{k-r} (\mathcal{V}_{U \setminus S})^2. \]

Let us first consider the terms in \( f \) which are independent of \( a_m \) and \( b_m \). By (29) and (30), the sum of all such terms is given by \( C - \delta \). Hence, the inductive hypothesis guarantees that every such term is positive.

Next, let us consider the terms in \( f \) which depend on either \( a_m^4 \) or \( b_m^4 \). The sum of all terms in \( f \) which depend on \( a_m^4 \) is given by \( (A - \alpha)a_m^4 \) and the sum of all terms in \( f \) which depend on \( b_m^4 \) is given by \( (A - \alpha)b_m^4 \). Observe that \( \alpha \) may be written as

\[ \alpha = \sum_{r=0}^{k-2} 2^{2r} \sum_{S \subseteq U', \ |S| = r} \left( \prod_{i \in S} a_i^2 b_i^2 \right) e_{k-r-1} (\mathcal{V}_{U \setminus S})^2 \]

\[ + 2^{2(k-1)} \sum_{S \subseteq U', \ |S| = k-1} \prod_{i \in S} a_i^2 b_i^2. \]

Hence, by Claim 2, for any \( S \subseteq U' \) with \( |S| = k - 1 \), the term \( \prod_{i \in S} a_i^2 b_i^2 \) in \( (A - \alpha) \) vanishes. The inductive hypothesis guarantees that the coefficients of all other terms in \( (A - \alpha) \) are positive. Hence, every term in \( f \) which depends on \( a_m^4 \) or \( b_m^4 \) is positive.

Similarly, the sum of all terms in \( f \) which depend on \( a_m^2 b_m^2 \) is given by \( 2(A - \beta) \), but since

\[ A - \beta = A - \alpha + 2^{2k-1} \sum_{S \subseteq U', \ |S| = k-1} \prod_{i \in S} a_i^2 b_i^2, \]

we see that the coefficient of every such term in \( f \) is positive.

Now consider those terms in \( f \) in which the exponent of \( a_m \) is 1 or 3. The sum of all such terms is given by \( 2Xa_m(a_m^2 + b_m^2) + 2Za_m \). It follows from the proof of Theorem 3.3.3 that every term in \( X \) and \( Z \) is positive. Hence every term in \( f \) in which the exponent of \( a_m \) is 1 or 3 is positive.

By symmetry, we have shown that a given term in \( f \) is positive if any of the following conditions are satisfied for any \( i = 1, 2, \ldots, m \):
(i) it is independent of \( a_i \) and \( b_i \);  
(ii) it is of the form \( Da_i^1 \) or \( Db_i^4 \), where \( D \) is independent of \( a_i \) and \( b_i \);  
(iii) it is of the form \( Da_i^2b_i^2 \), where \( D \) is independent of \( a_i \) and \( b_i \);  
(iv) it is a term in which the exponent of \( a_i \) is 1 or 3.

From the expansions given in (29) and (30), we see that a general term in \( f \) has the form

\[
a_1^{\zeta_1}a_2^{\zeta_2}\cdots a_m^{\zeta_m}b_1^{\eta_1}b_2^{\eta_2}\cdots b_m^{\eta_m},
\]

where for each \( i = 1, 2, \ldots, m \), \( \zeta_i \in \{0, 1, 2, 3, 4\} \), \( \eta_i \in \{0, 2, 4\} \) and \( \zeta_i + \eta_i \leq 4 \). If (31) does not satisfy any of the conditions (i)–(iv) above for any \( i \), then

\[
(\zeta_i, \eta_i) = (0, 2) \text{ or } (2, 0) : \quad i = 1, 2, \ldots, m.
\]

In particular, the degree of such a term is equal to \( 2m \); however, since every term in \( f \) has degree \( 4k \), we conclude the following: if \( m \neq 2k \), then every term in \( f \) must satisfy one of the above conditions for some \( i \) and if \( m = 2k \), then any term which does not satisfy any of the above conditions for any \( i \), can, up to relabelling the \( (a_i, b_i) \), be written in the form

\[
T^* := \left( \prod_{i=1}^{p} a_i^2 \right) \left( \prod_{i=p+1}^{2k} b_i^2 \right)
\]

for some \( p = 0, 1, \ldots, 2k \). Therefore, it suffices to show:

**Claim 3:** If \( m = 2k \), the coefficient of \( T^* \) in \( f \) is nonnegative.

In order to prove Claim 3, we will compute the coefficients of \( T^* \) in \( e_1^{2k}(x) \), \( e_{2k-1}(x)e_{2k+1}(x) \) and \( \emptyset \) separately. Our logic will be similar to that used in the proof of Claim 2.

Using (23) and the fact that the exponent of each \( a_i \) in \( T^* \) is 2, we conclude that the coefficient of \( T^* \) in \( e_{2k}(x)^2 \) is the same as its coefficient in

\[
\sum_{r=0}^{\lfloor p/2 \rfloor} 2^{4r} \sum_{S \subseteq \{1, \ldots, p\} \atop |S| = 2r} \left( \prod_{i \in S} a_i^2 \right) e_{k-r} \left( W_{\{1, \ldots, 2k\}\setminus S} \right)^2.
\]

In addition, for any \( S \subseteq \{1, \ldots, p\} \) with \( |S| = 2r \), the coefficient of

\[
\left( \prod_{i \in \{1, \ldots, p\} \setminus S} a_i^2 \right) \left( \prod_{i=p+1}^{2k} b_i^2 \right)
\]

in \( e_{k-r} \left( W_{\{1, \ldots, 2k\}\setminus S} \right)^2 \) is \( \left( \binom{2(k-r)}{k-r} \right) \). To see this, note that for arbitrary subsets \( S_1 \subseteq \{1, \ldots, p\} \setminus S \) and \( S_2 \subseteq \{p+1, p+2, \ldots, 2k\} \), the coefficient of \( \left( \prod_{i \in S_1} a_i^2 \right) \left( \prod_{i \in S_2} b_i^2 \right) \) in \( e_{k-r} \left( W_{\{1, \ldots, 2k\}\setminus S} \right) \) is 1 if \( |S_1| + |S_2| = |S| = 2r \).
\[ |S_2| = k - r \text{ and zero otherwise. Furthermore, there are } \binom{2(k-r)}{k-r} \text{ ways of choosing } S_1 \text{ and } S_2 \text{ subject to } |S_1| + |S_2| = k - r. \text{ It follows that the coefficient of } T^* \text{ in } e_{2k}(X)^2 \text{ is given by}
\]

\[
\sum_{r=0}^{\lfloor p/2 \rfloor} 2^{4r} \binom{p}{2r} \binom{2(k-r)}{k-r}.
\]

Similarly, using (24) and (28), one can show that the coefficient of \( T^* \) in \( e_{2k-1}(X) e_{2k+1}(X) \) is given by

\[
4 \sum_{r=0}^{\lfloor (p-1)/2 \rfloor} 2^{4r} \binom{p}{2r+1} \binom{2(k-r)-1}{k-r}.
\]

Next, note that the coefficient of \( T^* \) in \( \Theta \) is precisely its coefficient in \( e_k \left( V_{\{1, \ldots, 2k\}} \right)^2 \). Consider arbitrary subsets \( S_1 \subseteq \{1, 2, \ldots, p\}, S_2 \subseteq \{p+1, p+2, \ldots, 2k\} \). If \( |S_1| + |S_2| \neq k \), then the coefficient of

\[
\left( \prod_{i \in S_1} a_i^2 \right) \left( \prod_{i \in S_2} b_i^2 \right)
\]

in \( e_k \left( V_{\{1, \ldots, 2k\}} \right) \) is zero. If \( |S_1| + |S_2| = k \), then the coefficient of (32) in \( e_k \left( V_{\{1, \ldots, 2k\}} \right) \) is equal to its coefficient in \( \prod_{i \in S_1 \cup S_2} (a_i^2 - b_i^2) \), which equals \((-1)^{|S_2|} \). Since there are \( \binom{2k}{k} \) ways of choosing \( S_1, S_2 \) subject to \( |S_1| + |S_2| = k \), it follows that the coefficient of \( T^* \) in \( \Theta \) is \((-1)^p \binom{2k}{k} \).

Therefore, we have shown that the coefficient of \( T^* \) in \( f \) is

\[
\sum_{r=0}^{\lfloor p/2 \rfloor} 2^{4r} \binom{p}{2r} \binom{2(k-r)}{k-r} - 4 \sum_{r=0}^{\lfloor (p-1)/2 \rfloor} 2^{4r} \binom{p}{2r+1} \binom{2(k-r)-1}{k-r} - (-1)^p \binom{2k}{k}.
\]

The remainder of the proof is devoted to showing that the quantity given in (33) is nonnegative. First suppose that \( p \) is odd and write \( p = 2q + 1 \). In this case, noting that \( \binom{2(k-r)}{k-r} = 2 \binom{2(k-r)-1}{k-r} \), we may write (33) as

\[
\sum_{r=0}^{q} 2^{4r} \binom{2(k-r)}{k-r} \left( \binom{2q+1}{2r} - 2 \binom{2q+1}{2r+1} \right) + \binom{2k}{k}.
\]

If \( q = 0 \), then (34) vanishes, so assume \( q \geq 1 \). At this point, it is helpful to consider two related sums which are explicitly summable:

\[
\sum_{r=0}^{q} 2^{2r} \binom{2q+1}{2r} = \frac{1}{2} \left( 3^{2q+1} - 1 \right),
\]

\[
\sum_{r=0}^{q} 2^{2r} \binom{2q+1}{2r+1} = \frac{1}{4} \left( 3^{2q+1} + 1 \right).
\]
Bearing in mind (35) and (36), it is convenient to rewrite (34) as
\[
\binom{2k}{k} \left[ -4q\omega(0) + \sum_{r=1}^{q} 2^{2r} \omega(r) \left( \binom{2q+1}{2r} - 2 \binom{2q+1}{2r+1} \right) \right],
\tag{37}
\]
where
\[
\omega(r) := 2^{2r} \binom{2(k-r)}{k-r} \binom{2k}{k}^{-1} : \ r = 0, 1, \ldots, q.
\]
Note that
\[
\frac{\omega(r+1)}{\omega(r)} = 1 + \frac{1}{2(k-r)-1} > 1 : \ r = 0, 1, \ldots, q-1,
\]
i.e. \(\omega(r)\) is a strictly increasing function of \(r\). In order to determine which terms in (37) are negative and which are positive, we compute
\[
\frac{\binom{2q+1}{2r+1}}{2\binom{2q+1}{2r}} = \frac{1}{2} + \frac{1 + q}{1 + 2(q-r)}.
\]
Therefore, defining \(r_0 := \lfloor (4q+1)/6 \rfloor\), we see that the summand in (37) is strictly negative when \(r \leq r_0\) and strictly positive when \(r > r_0\).

Since \(\omega(r)\) is a strictly increasing function of \(r\), it follows that the expression in (37) is strictly greater than
\[
\omega(r_0) \binom{2k}{k} \left[ -4q + \sum_{r=1}^{q} 2^{2r} \left( \binom{2q+1}{2r} - 2 \binom{2q+1}{2r+1} \right) \right]
= \omega(r_0) \binom{2k}{k} \left[ 1 + \sum_{r=0}^{q} 2^{2r} \left( \binom{2q+1}{2r} - 2 \binom{2q+1}{2r+1} \right) \right],
\]
which, by (35) and (36), equals zero.

Similarly, if \(p\) is even, then, writing \(p = 2q\), (33) becomes
\[
\sum_{r=0}^{q} 2^{2r} \binom{2(k-r)}{k-r} \binom{2q}{2r} - 2 \sum_{r=0}^{q-1} 2^{2r} \binom{2(k-r)}{k-r} \binom{2q}{2r+1} - \binom{2k}{k}.
\tag{38}
\]
If \(q = 0\), then (38) vanishes. If \(q = 1\), then (38) equals
\[
16 \binom{2(k-1)}{k-1} - 4 \binom{2k}{k} = 4 \binom{2k}{k} (\omega(1) - \omega(0)) > 0.
\]
Hence, assume \(q \geq 2\). Then we may express (38) as
\[
\binom{2k}{k} \left[ -4q\omega(0) + \sum_{r=1}^{q-1} 2^{2r} \omega(r) \left( \binom{2q}{2r} - 2 \binom{2q}{2r+1} \right) + 2^{2q} \omega(q) \right].
\tag{39}
\]
Since
\[
\frac{\binom{2q}{2r}}{2\binom{2q}{2r+1}} = \frac{1}{2} + \frac{1 + 2q}{4(q-r)},
\]

we see that, for \( r_0 := [(4q - 1)/6] \), the summand in (39) is strictly negative when \( r \leq r_0 \) and strictly positive when \( r > r_0 \). It follows that (39) is strictly greater than

\[
\omega(r_0) \binom{2k}{k} \left[ -4q + \sum_{r=1}^{q-1} 2^{2r} \left( \frac{2q}{2r} - 2 \frac{2q}{2r+1} \right) + 2^q \right] = \omega(r_0) \binom{2k}{k} \left[ -1 + \sum_{r=0}^{q} 2^{2r} \left( \frac{2q}{2r} \right) - 2 \sum_{r=0}^{q-1} 2^{2r} \left( \frac{2q}{2r+1} \right) \right]. \tag{40}
\]

Finally, since

\[
\sum_{r=0}^{q} 2^{2r} \left( \frac{2q}{2r} \right) = \frac{1}{2} \left( 3^{2q} + 1 \right),
\]

\[
\sum_{r=0}^{q-1} 2^{2r} \left( \frac{2q}{2r+1} \right) = \frac{1}{4} \left( 3^{2q} - 1 \right),
\]

we see that the expression given in (40) equals zero. \( \square \)

**Corollary 3.3.7.** Let \( X := \{a_1 \pm ib_1, a_2 \pm ib_2, \ldots, a_m \pm ib_m\} \), where \( a_i, b_i \geq 0 : i = 1, 2, \ldots, m \). Then for \( 1 \leq k \leq l \leq m - 1 \),

\[
e_{2k}(X)e_{2l}(X) \geq \sqrt{l(m-k)} e_{2k-1}(X)e_{2l+1}(X).
\]

**Proof.** If \( k = l \), then the statement reduces to Theorem 3.3.6. If \( k < l \), then by Theorems 3.3.6 and 3.3.2,

\[
e_{2k}(X)e_{2l}(X) \geq \sqrt{e_{2k-1}(X)e_{2k+1}(X)e_{2l-1}(X)e_{2l+1}(X)}
\]

\[
\geq \sqrt{l(m-k)} e_{2k-1}(X)e_{2l+1}(X). \quad \square
\]

The inequalities developed in this section are summarised in the following theorem:

**Theorem 3.3.8.** Let \( X := \{x_1, x_2, \ldots, x_n\} \) be a self-conjugate list of complex numbers with nonnegative real parts. Then for all \( k \leq l \), the following inequalities hold:

(i) \( e_{2k}(X)e_{2l}(X) \geq \frac{(1 + 1)([n/2] - k + 1)}{k([n/2] - 1)} e_{2k-2}(X)e_{2l+2}(X) \);

(ii) \( e_{2k+1}(X)e_{2l+1}(X) \geq \frac{(1 + 1)([n/2] - k)}{k([n/2] - 1 - 1)} e_{2k-1}(X)e_{2l+3}(X) \);

(iii) \( e_{2k-1}(X)e_{2l+1}(X) \geq e_{2k-2}(X)e_{2l+1}(X) \);

(iv) \( e_{2k}(X)e_{2l+1}(X) \geq e_{2k-1}(X)e_{2l+2}(X) \).

Furthermore, if all real numbers in \( X \) appear with even multiplicity, then

(v) \( e_{2k}(X)e_{2l}(X) \geq \frac{l(n-2k)}{k(n-2l)} e_{2k-1}(X)e_{2l+1}(X) \).
3.4 Optimality and Comparison to the Generalised \( \lambda \)-Newton Inequalities

In this section, we will show by example that Theorems 3.3.2, 3.3.3 and 3.3.6 are optimal. We will also compare our results to the corresponding generalised \( \lambda \)-Newton inequalities.

Example 3.4.1. Let us reconsider the list \( X := (i, -i, i, -i, \ldots, i, -i) \) of length \( 2m \), given in Example 3.3.1. We have \( P_{2i}(X) = 1 \) for all \( i = 0, 1, \ldots, m \), which gives equality in (18).

Example 3.4.2. Consider the list

\[
X := (i, -i, i, -i, \ldots, i, -i, t, t),
\]

where \( t \) is real. We have

\[
e_{2i}(X) = \binom{m-1}{i} + t^2 \binom{m-1}{i-1} : \quad i = 0, 1, \ldots, m-1,
\]

\[
e_{2i+1}(X) = 2t \binom{m-1}{i} : \quad i = 0, 1, \ldots, m-1,
\]

\[
e_{2m}(X) = t^2.
\]

Note that (41) is equivalent to \( P_{2i+1}(X) = 2t : \quad i = 0, 1, \ldots, m-1 \), which gives equality in (19). Furthermore, for all \( 1 \leq k' \leq l' \leq m-1 \),

\[
\lim_{t \to 0} \frac{e_{2k'-1}(X)e_{2l'}(X)}{e_{2k-1}(X)e_{2l'+1}(X)} = 1
\]

and

\[
\lim_{t \to \infty} \frac{e_{2k'+1}(X)e_{2l'+1}(X)}{e_{2k-1}(X)e_{2l'+2}(X)} = 1,
\]

which shows that (20) is optimal. Finally, if \( t = \sqrt{\frac{m}{k} - 1} \), then for all \( 1 \leq k \leq m-1 \),

\[
e_{2k}(X)^2 = e_{2k-1}(X)e_{2k+1}(X) = 4 \binom{m-1}{k}^2,
\]

giving equality in (27).

Let us now compare the inequalities developed in Section 3.3 to the corresponding generalised \( \lambda \)-Newton inequalities (16). Suppose, for example, that \( X \) consists of 8 complex-conjugate pairs. By Theorem 3.3.6,

\[
e_8(X)^2 \geq e_7(X)e_9(X).
\]

This is equivalent to

\[
E_8(X)^2 \geq \left( \frac{8}{9} \right)^2 E_7(X)E_9(X).
\]
Hence, if it is known that each \( x_i \) lies in the wedge
\[ \Omega = \{ z \in \mathbb{C} : |\arg(z)| \leq \cos^{-1}(8/9) \}, \]
then the corresponding \( \lambda \)-Newton inequality is stronger than (42). Otherwise, (42) is stronger. This wedge is shown in Figure 2 (left).

Note that as the values of \( m \) and \( k \) in Theorem 3.3.6 grow larger, this critical wedge grows narrower. For example, if \( X \) consists of 100 complex-conjugate pairs, then
\[ e_{100}(X)^2 \geq e_{99}(X)e_{101}(X) \]
is equivalent to
\[ E_{100}(X)^2 \geq \left( \frac{100}{101} \right)^2 E_{99}(X)E_{101}(X). \]
The corresponding wedge is shown in Figure 2 (right).

Figure 2: Critical wedges

In general, it is clear that for any inequality given in Theorem 3.3.8, there is a critical value of \( \lambda \) (and an associated wedge \( \Omega \)) such that, if each \( x_i \) lies in \( \Omega \), the associated generalised \( \lambda \)-Newton inequality gives a stronger result; however, if any of the \( x_i \) lie outside of \( \Omega \) (or the \( x_i \) are unknown), then Theorem 3.3.8 will yield the stronger result. Furthermore, it is always possible to choose values of \( k \), \( l \) and \( n \) in Theorem 3.3.8 such that this critical value of \( \lambda \) is arbitrarily close to 1 and the corresponding wedge \( \Omega \) is arbitrarily narrow.

### 3.5 Application-Specific Inequalities

In this section, we will derive a family of inequalities which take a similar form to the others given in this chapter. The study of these inequalities is motivated by the NIEP and they will find their application in Chapter 5.
Let \( \mathcal{X} := (x_1, x_2, \ldots, x_n) \) be a list of self-conjugate complex numbers with nonnegative real parts and let \( r \) be real. We define
\[
Q_{2k}(r; \mathcal{X}) := \frac{e_{2k}(\mathcal{X}) - re_{2k-1}(\mathcal{X})}{\binom{n/2}{k}} : \quad k = 0, 1, \ldots, \left\lfloor n/2 \right\rfloor,
\]
\[
Q_{2k+1}(r; \mathcal{X}) := \frac{e_{2k+1}(\mathcal{X}) - re_{2k}(\mathcal{X})}{\binom{n/2}{k}} : \quad k = 0, 1, \ldots, \left\lfloor n/2 \right\rfloor.
\]

Suppose
\[
g(x) = x^n + e_1(\mathcal{X})x^{n-1} + e_2(\mathcal{X})x^{n-2} + \cdots + e_n(\mathcal{X})
\]
is the monic polynomial with roots \(-x_1, -x_2, \ldots, -x_n\). Then, if \( n = 2m \) is even, the polynomial
\[
(x - r)g(x) = x^{2m+1} + Q_1(r; \mathcal{X})x^{2m} + \left(\frac{m-1}{1}\right)Q_2(r; \mathcal{X})x^{2m-1} + \left(\frac{m-1}{1}\right)Q_3(r; \mathcal{X})x^{2m-2} + \cdots + \left(\frac{m}{m}\right)Q_{2m}(r; \mathcal{X})x + \left(\frac{m}{m}\right)Q_{2m+1}(r; \mathcal{X})
\]
has roots \( r, -x_1, -x_2, \ldots, -x_n \). Similarly, if \( n = 2m - 1 \) is odd, the polynomial
\[
(x - r)g(x) = x^{2m} + Q_1(r; \mathcal{X})x^{2m-1} + \left(\frac{m}{m}\right)Q_2(r; \mathcal{X})x^{2m-2} + \left(\frac{m-1}{1}\right)Q_3(r; \mathcal{X})x^{2m-3} + \cdots + \left(\frac{m}{m}\right)Q_{2m}(r; \mathcal{X})x + \left(\frac{m}{m}\right)Q_{2m+1}(r; \mathcal{X})
\]
has roots \( r, -x_1, -x_2, \ldots, -x_n \).

**Lemma 3.5.1.** Let \( \mathcal{X} := (x_1, x_2, \ldots, x_n) \) be a self-conjugate list of complex numbers with nonnegative real parts and let \( r \in \mathbb{R} \). If \( Q_{2s}(r; \mathcal{X}) > 0 \), then \( Q_{2l}(r; \mathcal{X}) > 0 \) : \( i = 0, 1, \ldots, s \) and
\[
Q_{2k}(r; \mathcal{X})Q_{2l}(r; \mathcal{X}) \geq Q_{2k-2}(r; \mathcal{X})Q_{2l+2}(r; \mathcal{X}) : \quad 1 \leq k \leq l \leq s - 1.
\]

**Similarly, if** \( Q_{2l+1}(r; \mathcal{X}) > 0 \), **then** \( Q_{2l+1}(r; \mathcal{X}) > 0 \) : \( i = 0, 1, \ldots, t \) and
\[
Q_{2k+1}(r; \mathcal{X})Q_{2l+1}(r; \mathcal{X}) \geq Q_{2k-1}(r; \mathcal{X})Q_{2l+3}(r; \mathcal{X}) : \quad 1 \leq k \leq l \leq t - 1.
\]

**Proof.** The proof is similar to that of Theorem 3.3.2.

Let \( Q_{2s}(r; \mathcal{X}) \) and \( Q_{2l+1}(r; \mathcal{X}) \) be positive. First suppose \( n \) is even and write \( n = 2m \). Let
\[
g(x) := x^{2m} + e_1(\mathcal{X})x^{2m-1} + e_2(\mathcal{X})x^{2m-2} + \cdots + e_{2m}(\mathcal{X}).
\]
The polynomial \((43)\) has roots \(r, -x_1, -x_2, \ldots, -x_n\) and therefore, by Lemma 2.4.1, \(Q_{2i}(r; x) > 0\) for each \(i = 0, 1, \ldots, s\) and \(Q_{2i+1}(r; x) > 0\) for each \(i = 0, 1, \ldots, t\).

Furthermore, by Corollary 2.3.3, the polynomial
\[
\chi^{2m+1} - \binom{m}{1} Q_2(r; x) \chi^{2m-1} + \cdots + (-1)^m \binom{m}{m} Q_{2m}(r; x) \chi
\]
has at least \(2m - 1\) real roots, one of which is zero. Hence the polynomial
\[
g_1(w) := w^m - \binom{m}{1} Q_2(r; x) w^{m-1} + \cdots + (-1)^m \binom{m}{m} Q_{2m}(r; x)
\]
has at least \(m - 1\) nonnegative roots. In particular, this implies all of the roots of \(g_1\), say \(w_1, w_2, \ldots, w_m\), are real. Setting \(W := (w_1, w_2, \ldots, w_m)\), we note that
\[
e_k(W) = \binom{m}{k} Q_{2k}(r; x) : \quad k = 0, 1, \ldots, m,
\]
and hence, applying Newton’s inequalities \((13)\) to \(W\) gives
\[
Q_{2k}(r; x)^2 \geq Q_{2k-2}(r; x) Q_{2k+2}(r; x) : \quad 1 \leq k \leq m - 1.
\]

The inequalities given in \((44)\) then follow from the fact that
\[
Q_2^2 Q_{2k+2}^2 \cdots Q_{2l}^2 \geq (Q_{2k-2} Q_{2k+2})(Q_{2k} Q_{2k+4}) \cdots (Q_{2l-2} Q_{2l+2}).
\]

Similarly, by Corollary 2.3.3, the polynomial
\[
Q_1(r; x) \chi^{2m} - \binom{m}{1} Q_3(r; x) \chi^{2m-2} + \cdots + (-1)^m \binom{m}{m} Q_{2m+1}(r; x)
\]
has at least \(2m - 2\) real roots. Hence the polynomial
\[
g_2(w) := w^m - \binom{m}{1} Q_3(r; x) w^{m-1} + \cdots + (-1)^m \binom{m}{m} Q_{2m+1}(r; x) \frac{w}{Q_1(r; x)}
\]
has at least \(m - 1\) nonnegative roots. In particular, this implies all of the roots of \(g_2\), say \(w_1, w_2, \ldots, w_m\), are real. Setting \(W := (w_1, w_2, \ldots, w_m)\), we note that
\[
e_k(W) = \binom{m}{k} Q_{2k+1}(r; x) \frac{Q_{2k+1}(r; x)}{Q_1(r; x)} : \quad k = 0, 1, \ldots, m,
\]
and hence, applying Newton’s inequalities \((13)\) to \(W\) gives
\[
Q_{2k+1}(r; x)^2 \geq Q_{2k-1}(r; x) Q_{2k+3}(r; x) : \quad 1 \leq k \leq m - 1.
\]

The inequalities given in \((45)\) then follow from the fact that
\[
Q_2^2 Q_{2k+3}^2 \cdots Q_{2l+1}^2 \geq
(Q_{2k-1} Q_{2k+3})(Q_{2k+1} Q_{2k+5}) \cdots (Q_{2l-1} Q_{2l+3}).
\]

The proof for odd \(n\) is similar. \(\square\)
Part II

THE NONNEGATIVE INVERSE EIGENVALUE PROBLEM
NEW LISTS FROM OLD

4.1 INTRODUCTION

In this chapter, we present our first results on the Nonnegative Inverse Eigenvalue Problem. Given a realisable list \((\rho, \lambda_2, \lambda_3, \ldots, \lambda_m)\), where \(\rho\) is the Perron eigenvalue and \(\lambda_2\) is real, we find families of lists \((\mu_1, \mu_2, \ldots, \mu_n)\), for which

\[
(\mu_1, \mu_2, \ldots, \mu_n, \lambda_3, \lambda_4, \ldots, \lambda_m)
\]

is realisable.

In addition, given a realisable list \((\rho, \alpha + i\beta, \alpha - i\beta, \lambda_4, \lambda_5, \ldots, \lambda_m)\), where \(\rho\) is the Perron eigenvalue and \(\alpha\) and \(\beta\) are real, we find families of lists \((\mu_1, \mu_2, \mu_3, \mu_4)\), for which \((\mu_1, \mu_2, \mu_3, \mu_4, \lambda_4, \lambda_5, \ldots, \lambda_m)\) is realisable.

4.2 PRELIMINARIES

We denote by \(e\) the vector of appropriate size with every entry equal to 1, i.e. \(e := [1 \ 1 \ \ldots \ 1]^T\). The following useful result allows us to assume without loss of generality that the Perron eigenvector of a realising matrix is \(e\).

**Lemma 4.2.1.** [27] Let \(A\) be a nonnegative matrix with Perron eigenvalue \(\rho\). Then there exists a nonnegative matrix \(B\), cospectral with \(A\), satisfying \(Be = \rho e\).

The results in this chapter fall into the category of constructing new realisable lists from known realisable lists. We give some earlier results of this type below.

Guo [21] gave the following theorem regarding the perturbation of a realisable list.

**Theorem 4.2.2.** [21] If \((\rho, \lambda_2, \lambda_3, \ldots, \lambda_n)\) is realisable, where \(\rho\) is the Perron eigenvalue and \(\lambda_2\) is real, then

\[
(\rho + \epsilon, \lambda_2 \pm \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_n)
\]

is realisable for all \(\epsilon \geq 0\).

To generalise Theorem 4.2.2 to the perturbation of nonreal eigenvalues, we have the following theorem. Result (46) is due to Laffey [31] and an alternative proof can be found in [20]. Result (47) is due to Guo and Guo [20].
Theorem 4.2.3. [20, 31] If \((\rho, \alpha + i\beta, \alpha - i\beta, \lambda_4, \lambda_5, \ldots, \lambda_n)\) is realisable, where \(\rho\) is the Perron eigenvalue and \(\alpha\) and \(\beta\) are real, then for all \(\epsilon \geq 0\), the lists

\[
(\rho + 2\epsilon, \alpha - \epsilon + i\beta, \alpha - \epsilon - i\beta, \lambda_4, \lambda_5, \ldots, \lambda_n)
\]

and

\[
(\rho + 4\epsilon, \alpha + \epsilon + i\beta, \alpha + \epsilon - i\beta, \lambda_4, \lambda_5, \ldots, \lambda_n)
\]

are realisable.

In [65], Šmigoc gives a construction by which the Perron eigenvalue of a realisable list may be replaced by a new list:

Theorem 4.2.4. [65] Let \((\rho, \lambda_2, \lambda_3, \ldots, \lambda_m)\) be realisable, where \(\rho\) is the Perron eigenvalue, and let \((\mu_1, \mu_2, \ldots, \mu_n)\) be the spectrum of a nonnegative matrix with a diagonal element greater than or equal to \(\rho\). Then

\[
(\mu_1, \mu_2, \ldots, \mu_n, \lambda_2, \lambda_3, \ldots, \lambda_m)
\]

is realisable.

In [66], Šmigoc gives a construction to replace both the Perron eigenvalue and another real eigenvalue:

Theorem 4.2.5. [66] Let \((\rho, \lambda_2, \lambda_3, \ldots, \lambda_m)\) be realisable, where \(\rho\) is the Perron eigenvalue and \(\lambda_2\) is real. Let \(a\) and \(t_1\) be any nonnegative numbers and let \(t_2\) be any real number such that \(|t_2| \leq t_1\). Then

\[
(\mu_1, \mu_2, \mu_3, \lambda_3, \lambda_4, \ldots, \lambda_m)
\]

is realisable, where \(\mu_1, \mu_2, \mu_3\) are the roots of the polynomial

\[
w(x) = (x - \rho)(x - \lambda_2)(x - a) - (t_1 + t_2)x + t_1\lambda_2 + t_2\rho.
\]

In Section 4.3, we expand on the work done in [66] by presenting some new lists which may replace the eigenvalues \(\rho\) and \(\lambda_2\). In Section 4.4, we give a construction which allows us to replace the Perron eigenvalue and a complex conjugate pair of eigenvalues, i.e. given a realisable list

\[
(\rho, \alpha + i\beta, \alpha - i\beta, \lambda_4, \lambda_5, \ldots, \lambda_m),
\]

where \(\rho\) is the Perron eigenvalue and \(\alpha\) and \(\beta\) are real, we find some conditions on the list \((\mu_1, \mu_2, \mu_3, \mu_4)\) which imply that

\[
(\mu_1, \mu_2, \mu_3, \mu_4, \lambda_4, \lambda_5, \ldots, \lambda_m)
\]

is realisable.

To this end, we give a Lemma from [66], which is the foundation of this chapter.

Lemma 4.2.6. [66] Let the following assumptions hold:

(i) \(Y\) is an invertible matrix with a partition \(Y = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}\), where \(Y_1\) is an \(m \times p\) matrix and \(Y_2\) is an \(m \times m_1\) matrix with \(p + m_1 = m\);
(ii) \( B \) is an \( m \times m \) matrix such that
\[
Y^{-1}BY = \begin{bmatrix} C & E \\ 0 & F \end{bmatrix}
\]
for a \( p \times p \) matrix \( C \) and an \( m_1 \times m_1 \) matrix \( F \);

(iii) \( M \) is an \( n \times n \) matrix with a principal submatrix \( C \), partitioned in the
following way:
\[
M = \begin{bmatrix} A & K \\ L & C \end{bmatrix},
\]
where \( A \) is an \( n_1 \times n_1 \) matrix and \( p + n_1 = n \);

(iv) \( K = HY_1 \) for an \( n_1 \times m \) matrix \( H \).

Then for matrices
\[
N = \begin{bmatrix} A & H \\ Y_1 & L & B \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} I_{n_1} & 0 \\ 0 & Y \end{bmatrix},
\]
we have
\[
Z^{-1}NZ = \begin{bmatrix} A & K & HY_2 \\ L & C & E \\ 0 & 0 & F \end{bmatrix}.
\]

In particular, Lemma 4.2.6 produces a matrix \( N \) with spectrum
\( \sigma(N) = (\sigma(M), \sigma(F)) \). In order to apply this construction to the \( \text{NIEP} \), it
is necessary to determine when the matrix \( N \) produced in this way is
nonnegative. In [66], Šmigoc gives the following answer to this ques-
tion:

For an \( m \times p \) matrix \( Y_1 \), we define the sets:
\[
\mathcal{L}(Y_1) := \{ l \in \mathbb{R}^p : Y_1l \geq 0 \}
\]
and
\[
\mathcal{K}(Y_1) := \{ k \in \mathbb{R}^p : k^T = h^TY_1 \text{ for some nonnegative } h \in \mathbb{R}^m \}.
\]

For a \( p \times p \) matrix \( C \) and an \( m \times p \) matrix \( Y_1 \), we define \( M_n(Y_1, C) \) to be the set of all \( n \times n \) matrices
\[
M = \begin{bmatrix} A & K \\ L & C \end{bmatrix},
\]
such that \( A \) is an \( n_1 \times n_1 \) nonnegative matrix, \( n = n_1 + p \), every
column of \( L \) lies in \( \mathcal{L}(Y_1) \) and the transpose of every row of \( K \) lies in \( \mathcal{K}(Y_1) \).
Theorem 4.2.7. [66] Let the assumptions (i)–(iv) in Lemma 4.2.6 hold. Assume also that $B$ is nonnegative, that the Perron eigenvalue of $B$ lies in $\sigma(C)$ and that $M \in \mathcal{M}_n(Y_1, C)$. Then the matrix $N$ of the lemma is nonnegative, i.e. the list $(\sigma(M), \sigma(F))$ is realisable by a nonnegative matrix with principal submatrices $A$ and $B$.

Theorem 4.2.7 provides a method of producing new realisable lists from old. With $p = 1$, it allows us to replace the Perron eigenvalue of a known realisable list, for example as in Theorem 4.2.4. The $p = 1$ case has been dealt with in detail in [65]. With $p = 2$, it allows us to replace the Perron eigenvalue and another real eigenvalue, for example as in Theorem 4.2.5. The $p = 2$ case is dealt with in [66] and we give further results in Section 4.3. With $p = 3$, Theorem 4.2.7 allows us to replace the Perron eigenvalue and a complex conjugate pair of eigenvalues (see Section 4.4).

4.3 A $p = 2$ construction

In this section, given a realisable list $(\rho, \lambda_2, \lambda_3, \ldots, \lambda_m)$, where $\rho$ is the Perron eigenvalue and $\lambda_2$ is real, we present some lists $\lambda_1, \mu_2, \ldots, \mu_n, \lambda_3, \lambda_4, \ldots, \lambda_m$ is also realisable. This corresponds to letting $p = 2$ in Lemma 4.2.6.

In [66], Šmigoc characterises $\mathcal{L}(Y_1)$ and $\mathcal{K}(Y_1)$ for the $p = 2$ case. Using Lemma 4.2.1, we may assume without loss of generality that the eigenvector corresponding to $\rho$ is $e$. Let $z$ be a real eigenvector corresponding to $\lambda_2$ and let $z_{\text{max}}$ and $z_{\text{min}}$ denote the maximal and minimal entries of $z$, respectively. In [66, Section 4], Šmigoc shows that we may assume $z_{\text{max}} > 0$ and $z_{\text{min}} \leq 0$. She then gives the following characterisations of $\mathcal{L}(Y_1)$ and $\mathcal{K}(Y_1)$.

**Proposition 4.3.1.** [66] If $z_{\text{max}} > 0$ and $z_{\text{min}} < 0$, then

$$
\mathcal{L}(Y_1) = \left\{ \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} : -\frac{l_1}{z_{\text{max}}} \leq l_2 \leq -\frac{l_1}{z_{\text{min}}} \right\}.
$$

If $z_{\text{max}} > 0$ and $z_{\text{min}} = 0$, then

$$
\mathcal{L}(Y_1) = \left\{ \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} : -\frac{l_1}{z_{\text{max}}} \leq l_2 \quad \text{and} \quad l_1 \geq 0 \right\}.
$$

**Proposition 4.3.2.** [66]

$$
\mathcal{K}(Y_1) = \left\{ \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} : z_{\text{min}}k_1 \leq k_2 \leq z_{\text{max}}k_1 \right\}.
$$

We now give our $p = 2$ construction.

**Lemma 4.3.3.** Let the following assumptions hold:
(i) the list \( \sigma_0 := (\rho, \lambda_2, \lambda_3, \ldots, \lambda_m) \) is realisable, where \( \rho \) is the Perron eigenvalue, \( \lambda_2 \) is real and \( \rho \neq \lambda_2 \);

(ii) \( C' \) is a \( 2 \times 2 \) matrix of the form

\[
C' := \begin{bmatrix} \gamma & 1 \\ b_2 & b_1 + \gamma \end{bmatrix},
\]

where \( b_1 \) is real, \( \gamma = (\rho + \lambda_2 - b_1)/2 \geq 0 \) and \( b_2 = ((\rho - \lambda_2)^2 - b_1^2)/4 \);

(iii) \( K' := \begin{bmatrix} f & g \end{bmatrix} \), where \( f, g \in \mathbb{R}^{n-2} \), \( g \geq 0 \) and \( f \geq (\gamma - \lambda_2)g \);

(iv) \( L' := \begin{bmatrix} c^T \\ d^T \end{bmatrix} \), where \( c, d \in \mathbb{R}^{n-2} \), \( c \geq 0 \) and \( d \geq (\rho - \gamma)c \);

(v) \( A \) is an \( (n-2) \times (n-2) \) nonnegative matrix;

(vi) \( M' \) is the \( n \times n \) matrix defined by

\[
M' := \begin{bmatrix} A & K' \\ L' & C' \end{bmatrix}.
\]

Then the list \( (\sigma(M'), \lambda_3, \lambda_4, \ldots, \lambda_m) \) is realisable.

**Proof.** Let \( B \) be a nonnegative matrix with spectrum \( \sigma_0 \). As in the construction of Lemma 4.2.6, let \( Y \) be an invertible matrix such that

\[
Y^{-1}BY = \begin{bmatrix} C & * \\ 0 & * \end{bmatrix},
\]

where

\[
C := \begin{bmatrix} \rho & 0 \\ 0 & \lambda_2 \end{bmatrix}.
\]

By Lemma 4.2.1, we may assume without loss of generality that the Perron eigenvector of \( B \) is \( e \). Let \( z \) be a real eigenvector of \( B \) corresponding to \( \lambda_2 \), appropriately scaled so that \( z_{\text{max}} = 1 \) and \( z_{\text{min}} \leq 0 \) and let us write \( Y = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} \), where \( Y_1 = [e \ z] \).

Note that the definitions of \( \gamma \) and \( b_2 \) assure \( \sigma(C') = (\rho, \lambda_2) \). Therefore, since \( \rho \) and \( \lambda_2 \) are distinct, we may diagonalise \( C' \). Indeed, \( X^{-1}C'X = C \), where

\[
X := \begin{bmatrix} 1 & 1 \\ \rho - \gamma & \lambda_2 - \gamma \end{bmatrix} \quad \text{and} \quad X^{-1} = \frac{1}{\rho - \lambda_2} \begin{bmatrix} -\lambda_2 + \gamma & 1 \\ \rho - \gamma & -1 \end{bmatrix}.
\]

Now define

\[
K := K'X = \begin{bmatrix} f + (\rho - \gamma)g & f + (\lambda_2 - \gamma)g \end{bmatrix},
\]

---

1 See the discussion preceding Proposition 4.3.1.
\[ L := X^{-1}L' = \frac{1}{\rho - \lambda_2} \begin{bmatrix} (-\lambda_2 + \gamma)c^T + d^T \\ (\rho - \gamma)c^T - d^T \end{bmatrix} \]

and

\[ M := \begin{bmatrix} A & K \\ L & C \end{bmatrix}. \]

We will show that \( M \in M_n(Y_1, C) \) and that \( M \) and \( M' \) are similar (and hence cospectral). The result will then follow by Theorem 4.2.7.

To see that \( M \in M_n(Y_1, C) \), we first note that, since \( g \geq 0 \) and \( f \geq (\gamma - \lambda_2)g \geq (\gamma - \rho)g \), we have

\[ z_{\min}(f + (\rho - \gamma)g) \leq 0 \leq f + (\lambda_2 - \gamma)g \leq f + (\rho - \gamma)g \]

and hence, by Proposition 4.3.2, the transpose of every row of \( K \) lies in \( K(Y_1) \). Similarly, since \( c \geq 0 \) and \( d \geq (\rho - \gamma)c \geq (\lambda_2 - \gamma)c \), we have that

\[ -((\lambda_2 + \gamma)c + d) \leq (\rho - \gamma)c - d \leq 0 \leq -\frac{1}{z_{\min}}((\lambda_2 + \gamma)c + d), \]

where the right-most inequality holds provided \( z_{\min} \neq 0 \). Then, by Proposition 4.3.1, every column of \( L \) lies in \( L(Y_1) \).

Therefore, we have shown that \( M \in M_n(Y_1, C) \). Finally, it is easy to see that \( M \) and \( M' \) are similar:

\[ M = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix}^{-1} M' \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix}. \]

In the proof of Lemma 4.3.3, we have shown that \( M' \) is similar to a matrix in \( M_n(Y_1, C) \). In the applications of this lemma, we will choose \( A, K' \) and \( L' \) in such a way that \( M' \) has a structure which makes its characteristic polynomial easy to compute. Several such structured matrices—such as companion matrices, doubly companion matrices and block companion matrices—have been studied in the context of the NIEP, for example by Friedland, Laffey, Šmigoc and Cronin [12, 17, 30, 36] and indeed, the form of the matrix \( C' \) in Lemma 4.3.3 has been chosen with such matrices in mind.

For example, letting

\[ A = \begin{bmatrix} \gamma & 1 \\ \gamma & \ddots \\ \vdots & \ddots & 1 \\ \gamma & \end{bmatrix}, \tag{48} \]

\( d \geq 0, f = [0 \ 0 \ \ldots \ 0 \ 1]^T \) and \( c = g = 0 \), the matrix \( M' \) becomes a companion matrix plus a scalar, and as such, the characteristic polynomial of \( M' \) is easy to write down. The case where \( M' \) is
a companion matrix plus a scalar is developed formally in Theorem 4.3.6.

Alternatively, keeping \(c, d, f\) and \(g\) as above, but setting

\[
A = \begin{bmatrix}
\gamma & 1 & & & & \\
& \ddots & \ddots & & & \\
& & \gamma & 1 & & \\
& & & * & \cdots & * \\
& & & & \gamma & \ddots \\
& & & & & \ddots & 1 \\
& & & & & & \gamma
\end{bmatrix},
\]

the matrix \(M'\) becomes a 2-block companion matrix plus a scalar.

If we take \(f, g\) and \(d\) as above, \(c = [\ast \ 0 \ 0 \ \cdots \ 0]^T\) and

\[
A = \begin{bmatrix}
\ast & 1 \\
& \ast & \gamma & \ddots \\
& & \ddots & \ddots & 1 \\
& & & \ast & \gamma
\end{bmatrix},
\]

then \(M'\) becomes a doubly companion matrix plus a scalar. We will illustrate the use of these structured matrices below.

**Example 4.3.4.** Let \(\sigma\) be any list such that \((8, 2, \sigma)\) is realisable. In Lemma 4.3.3, take \(\rho = 8, \lambda_2 = 2, b_1 = 10\) and \(n = 4\). It is easily verified that the matrices

\[
K' := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad L' := \begin{bmatrix} 42 & 0 \\ 336 & 28 \end{bmatrix}
\]

and

\[
A := \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}
\]

satisfy the hypotheses of the lemma and the matrix \(M'\) of the lemma then becomes

\[
M' = \begin{bmatrix}
0 & 1 & 0 & 0 \\
3 & 0 & 1 & 0 \\
42 & 0 & 0 & 1 \\
336 & 28 & -16 & 10
\end{bmatrix}.
\]

The matrix \(M'\) is a doubly companion matrix, with characteristic polynomial

\[
w(x) = x^4 - 10x^3 + 13x^2 - 40x + 36
\]

\[
= (x - 9)(x - 1)(x^2 + 4).
\]

Hence the list \((9, 1, 2i, -2i, \sigma)\) is realisable.
Looking at Example 4.3.4, one may wonder if it is possible to use Theorem 4.2.5 to show that \((9, 2i, -2i, \sigma)\) is realisable without the eigenvalue 1; however this is not the case. In fact, there exist \(\sigma\) such that \((8, 2, \sigma)\) is realisable, but \((9, 2i, -2i, \sigma)\) is not. For one such \(\sigma\), consider the classical example given in Example 1.3.12: if \(\tau := (3 + t, 3 - t, -2, -2, -2)\), Laffey and Meehan [33] showed that \(\tau\) is realisable if and only if \(t \geq \sqrt{16\sqrt{6} - 39}\). Hence (by scaling this list by a factor of 5/3), we have that

\[
(5 + t', 5 - t', -10/3, -10/3, -10/3)
\]

is realisable if and only if \(t' \geq 5\sqrt{16\sqrt{6} - 39/3} \approx 0.73\). In particular,

\[
\sigma_0 := (8, 2, -10/3, -10/3, -10/3)
\]

is realisable, but

\[
\sigma_1 := (9, 2i, -2i, -10/3, -10/3, -10/3)
\]

is not, since \(s_1(\sigma_1) = -1\).

**Example 4.3.5.** Let \(\sigma\) be any list such that \((8, -2, \sigma)\) is realisable. In Lemma 4.3.3, take \(\rho = 8, \lambda_2 = -2, b_1 = 6\) and \(n = 7\). Then the matrix

\[
M' = \begin{bmatrix}
A & K' \\
L' & C'
\end{bmatrix}
\]

satisfies the hypotheses of the lemma. The matrix \(M'\) is an example of a 2-block companion matrix. Its characteristic polynomial is

\[
w(x) = \frac{1}{29} (29x^7 + 174x^6 + 469x^5 + 266x^4 - 1856x^3 + 11136x^2
\]

\[
+30016x + 17024)
\]

\[
= \frac{1}{29} (29x^2 - 203x - 266) (x^4 + 64) (x + 1)
\]

and hence the list

\[
(8.128 \ldots, -1.128 \ldots, 2 + 2i, 2 - 2i, -2 + 2i, -2 - 2i, -1, \sigma)
\]

is realisable.

**Theorem 4.3.6.** Let the list \(\sigma_0 := (\rho, \lambda_2, \lambda_3, \ldots, \lambda_m)\) be realisable, where \(\rho\) is the Perron eigenvalue, \(\lambda_2\) is real and \(\rho \neq \lambda_2\). Let \(b_1\) be any real number such that

\[
\gamma := \frac{\rho + \lambda_2 - b_1}{2} \geq 0,
\]

(49)
\[ b_2 := \frac{(\rho - \lambda_2)^2 - b_1^2}{4} \quad (50) \]

and let \( b_3, b_4, \ldots, b_n \) be any nonnegative numbers. Then the list
\[ (\mu_1, \mu_2, \ldots, \mu_n, \lambda_3, \lambda_4, \ldots, \lambda_m) \]
is realisable, where \( \mu_1, \ldots, \mu_n \) are the roots of the polynomial
\[ w(x) := (x - \gamma)^n - b_1(x - \gamma)^{n-1} - b_2(x - \gamma)^{n-2} - \cdots - b_{n-1}(x - \gamma) - b_n. \]

Proof. In Lemma 4.3.3, let \( A \) be as in (48) and let
\[
\begin{align*}
\mathbf{d} &= \begin{bmatrix} b_n & b_{n-1} & \cdots & b_3 \end{bmatrix}^T, \\
\mathbf{f} &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T
\end{align*}
\]
and \( c = g = 0. \) Then, \( M' - \gamma I_n \) becomes a companion matrix (where \( M' \) is defined in the statement of the lemma) and as such, it has characteristic polynomial \( w(x + \gamma). \) Hence \( M' \) has characteristic polynomial \( w(x). \)

Example 4.3.7. Let \( \sigma \) be any list such that \((4, 2, \sigma)\) is realisable. Taking \( \rho = 4 \) and \( \lambda_2 = 2 \) in Theorem 4.3.6, let us choose \( n = 4, b_1 = 6, b_3 = 10 \) and \( b_4 = 25. \) Then the polynomial \( w(x) \) of the theorem becomes
\[ w(x) = x^4 - 6x^3 + 8x^2 - 10x - 25 \]
\[ = (x - 5)(x^2 - 2x + 5)(x + 1) \]
and so the list \((5, 1 + 2i, 1 - 2i, -1, \sigma)\) is realisable.

At this point, we wish to use Theorem 1.3.5 in conjunction with Theorem 4.3.6 to produce a class of spectra which may replace the eigenvalues \( \rho \) and \( \lambda_2; \) however, Theorem 1.3.5 deals with realisation by matrices of the form \( G + \gamma I_n, \) where \( G \) has trace zero, and so applying this directly would limit us to taking \( b_1 = 0 \) in Theorem 4.3.6. With this in mind, we will present a slight modification of Theorem 1.3.5, in which we examine realisation by a matrix of the form \( G + \gamma I_n, \) where \( G \) may have nonzero trace.

Theorem 4.3.8. Let \( \sigma := (\rho, \lambda_2, \lambda_3, \ldots, \lambda_n) \) be realisable, where \( \rho \) is the Perron eigenvalue and \( \text{Re}\lambda_i \leq 0 \) for all \( i = 2, 3, \ldots, n. \) Then for any nonnegative number \( b_1 \) with \( b_1 \leq s_1(\sigma) \) and \( (n - 1)b_1^2 \leq ns_2(\sigma) - s_1(\sigma)^2, \) \( \sigma \) may be realised by a matrix of the form \( G + \gamma I_n, \) where \( G \) is a nonnegative companion matrix with trace \( b_1 \) and \( \gamma \) is a nonnegative scalar.
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Proof. Since σ is realisable, note that \( s_1(\sigma) \geq 0 \) and the JLL condition
\[ s_1(\sigma)^2 \leq n s_2(\sigma) \] holds. Choose any nonnegative \( b_1 \) such that \( b_1 \leq s_1(\sigma) \)
and \( (n-1)b_1^2 \leq n s_2(\sigma) - s_1(\sigma)^2 \). Let \( \gamma := (s_1(\sigma) - b_1)/n \),
\[ \sigma' := (\rho - \gamma, \lambda_2 - \gamma, \lambda_3 - \gamma, \ldots, \lambda_n - \gamma) \]
and
\[ g(x) := (x - \rho + \gamma) \prod_{i=2}^{n}(x - \lambda_i + \gamma). \]
It is clear from the definition of \( \gamma \) that \( s_1(\sigma') = b_1 \). Therefore, we may
write \( g(x) \) as
\[ g(x) = x^n - b_1 x^{n-1} - b_2 x^{n-2} - \cdots - b_n. \]
The elements of \( \sigma' \) are the roots of \( g \) and hence, using Newton’s
Identities, we have that
\[
\begin{align*}
b_2 &= \frac{1}{2} (s_2(\sigma') - b_1^2) \\
&= \frac{1}{2} (s_2(\sigma) - 2\gamma s_1(\sigma) + n\gamma^2 - b_1^2) \\
&= \frac{1}{2n} (ns_2(\sigma) - s_1(\sigma)^2 - (n-1)b_1^2) \\
&\geq 0.
\end{align*}
\]
The complex numbers \( \lambda_2 - \gamma, \lambda_3 - \gamma, \ldots, \lambda_n - \gamma \) have nonpositive real
parts and hence by Lemma 1.3.7, \( b_i \geq 0 \) for all \( i = 3, 4, \ldots, n \). Therefore,
the companion matrix of \( g \), \( G \) say, is nonnegative, has trace \( b_1 \)
and has spectrum \( \sigma' \). It follows that \( G + \gamma I_n \) has spectrum \( \sigma \). \qed

Remark. As in Observation 1.3.6, we note that, in the proof of
Theorem 4.3.8, it was only required that \( \lambda_2 - \gamma, \lambda_3 - \gamma, \ldots, \lambda_n - \gamma \) have nonpositive real
parts. Therefore, the condition that \( \text{Re} \lambda_i \leq 0 \) for
all \( i = 2, 3, \ldots, n \) in the statement of the theorem can be relaxed to
\( \text{Re} \lambda_i \leq (s_1(\sigma) - b_1)/n \).

In Chapter 5, we will explore realisation by a companion matrix
plus a diagonal matrix in considerably more depth.

Theorem 4.3.9. Let \( \sigma_0 := (\rho, \lambda_2, \lambda_3, \ldots, \lambda_m) \) be realisable, where \( \rho \) is the
Perron eigenvalue, \( \lambda_2 \) is real and \( \rho \neq \lambda_2 \). Let
\[ (n-2) \max(0,\lambda_2) \leq \delta \leq \frac{1}{2}(n-2)(\rho + \lambda_2) \] (51)
and let \( \mu := (\mu_1, \mu_2, \ldots, \mu_n) \) be a list of self-conjugate complex numbers
with \( \mu_1 \geq 0 \) and \( \text{Re} \mu_i \leq \delta/(n-2) \) for all \( i = 2, 3, \ldots, n \). Assume also that
\[ s_1(\mu) = \rho + \lambda_2 + \delta \] (52)
and
\[ s_2(\mu) = \rho^2 + \lambda_2^2 + \frac{\delta^2}{n-2}. \] (53)
Then the list \( (\mu_1, \mu_2, \ldots, \mu_n, \lambda_3, \lambda_4, \ldots, \lambda_m) \) is realisable.
Proof. We will show that $\mu$ is the spectrum of a nonnegative matrix of the form

$$
\begin{bmatrix}
\gamma & 1 & 0 & 0 \\
\gamma & \ddots & \ddots & \vdots \\
\ddots & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & \gamma & 1 \\
b_n & b_{n-1} & \cdots & b_3 & b_2 & b_1 + \gamma
\end{bmatrix},
$$

(54)

where $\gamma$ and $b_2$ satisfy (49) and (50), respectively. The result will then follow by Theorem 4.3.6.

To see that $\mu$ is realisable, from Theorem 1.3.5 and Observation 1.3.6, it suffices to check that $s_1(\mu)^2 \leq ns_2(\mu)$ and that $\text{Re} \mu_i \leq s_1(\mu)/n$ for all $i = 2, 3, \ldots, n$. For the first of these two conditions, consider $ns_2(\mu) - s_1(\mu)^2$ as a quadratic in $\delta$:

$$
ns_2(\mu) - s_1(\mu)^2 = \frac{2}{n-2} \delta^2 - 2(\rho + \lambda_2)\delta + (n-1)(\rho^2 + \lambda_2^2) - 2\rho\lambda_2.
$$

The coefficient of $\delta^2$ in this quadratic is positive and its discriminant is

$$
-\frac{4n(\rho - \lambda_2)^2}{n-2} < 0.
$$

Therefore, as required, $ns_2(\mu) - s_1(\mu)^2 > 0$ for all real $\delta$. For the second condition, let

$$
b_1 := \rho + \lambda_2 - \frac{2\delta}{n-2}.
$$

(55)

For all $\delta$ satisfying (51), we have $0 \leq b_1 \leq s_1(\mu)$ and equations (52) and (55) then give

$$
\text{Re} \mu_i \leq \frac{\delta}{n-2} = \frac{s_1(\mu) - b_1}{n} \leq \frac{s_1(\mu)}{n},
$$

as required. Therefore $\mu$ is realisable.

Furthermore, since

$$(n-2)\lambda_2 \leq \delta \leq \frac{1}{2}(n-2)(\rho + \lambda_2) \leq (n-2)\rho,$$

we have that

$$
ns_2(\mu) - s_1(\mu)^2 - (n-1)b_1^2 = \frac{2n(\delta - (n-2)\lambda_2)((n-2)\rho - \delta)}{(n-2)^2} \geq 0,
$$

so $b_1$ satisfies the conditions imposed on it by Theorem 4.3.8. Hence, by Theorem 4.3.8 and the remark that follows it, $\mu$ may be realised by a nonnegative matrix of the form (54), and so $\mu_1, \mu_2, \ldots, \mu_n$ are the roots of a polynomial of the form
\[ w(x) = (x - \gamma)^n - b_1(x - \gamma)^{n-1} - b_2(x - \gamma)^{n-2} - \ldots - b_{n-1}(x - \gamma) - b_n, \]

where
\[ \gamma = \frac{s_1(\mu) - b_1}{n}. \]  

(56)

So it remains to show that \( \gamma \) and \( b_2 \) satisfy (49) and (50). To see this, consider the list
\[ \mu' := (\mu_1 - \gamma, \mu_2 - \gamma, \ldots, \mu_n - \gamma) \]

and the polynomial
\[ w'(x) := x^n - b_1x^{n-1} - b_2x^{n-2} - \ldots - b_{n-1}x - b_n. \]

The elements of \( \mu' \) are the roots of \( w' \) and so, using Newton’s Identities, we have that
\[ b_2 = \frac{1}{2} \left( s_2(\mu') - b_1^2 \right) \]
\[ = \frac{1}{2} \left( s_2(\mu) - 2\gamma s_1(\mu) + n\gamma^2 - b_1^2 \right). \]  

(57)

By eliminating \( \delta \) from (52) and (55), we see that
\[ s_1(\mu) = \frac{n(\rho + \lambda_2) - (n-2)b_1}{2} \]  

(58)

and by eliminating \( \delta \) from (53) and (55), we have
\[ s_2(\mu) = \rho^2 + \lambda_2^2 + \frac{1}{4}(n-2)(\rho + \lambda_2 - b_1)^2. \]  

(59)

Substituting (58) in (56), we obtain (49)\(^2 \) and then, substituting (58), (59) and (49) into (57) gives (50).

Finally, from Theorem 4.3.6, we conclude that
\[(\mu_1, \mu_2, \ldots, \mu_n, \lambda_3, \lambda_4, \ldots, \lambda_m)\]
is realisable. \(\square\)

**Example 4.3.10.** Let \( \sigma \) be any list such that \( (1, 0, \sigma) \) is realisable. Letting \( \rho = 1, \lambda_2 = 0, n = 4 \) and \( \delta = 0 \) in Theorem 4.3.9, we see that the list \( (\mu_1, \mu_2, \mu_3, \mu_4, \sigma) \) is also realisable, provided \( \mu_1 \geq 0, (\mu_2, \mu_3, \mu_4) \) is self-conjugate, \( \text{Re} \mu_2, \text{Re} \mu_3, \text{Re} \mu_4 \leq 0 \) and \( \sum_{i=1}^{4} \mu_i = \sum_{i=1}^{4} \mu_i^2 = 1. \) For example,
\[
\left( \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}, i, -i, \sigma \right)
\]
is realisable.

\(^2\) The fact that \( \gamma \) is nonnegative is easily seen from (55).
Example 4.3.11. Let \( \sigma \) be any list such that \((1, -1, \sigma)\) is realisable. Letting \( \rho = 1, \lambda_2 = -1 \) and \( \delta = 0 \) in Theorem 4.3.9, we have that for any \( n \geq 3 \), the list

\[
(\rho, -\lambda, -\lambda, \ldots, -\lambda, \sigma)
\]

is realisable, where

\[
\rho := \sqrt{\frac{2(n-1)}{n}} \quad \text{and} \quad \lambda := \sqrt{\frac{2}{n(n-1)}}.
\]

Alternatively (again taking \( \delta = 0 \)), for any \( m \in \mathbb{N} \), Theorem 4.3.9 also implies that the list

\[
\left(\sqrt{2}, -\frac{1}{\sqrt{2m}} \pm \frac{1}{\sqrt{2m}} i, \ldots, -\frac{1}{\sqrt{2m}} \pm \frac{1}{\sqrt{2m}} i, \sigma\right)
\]

is realisable.

In Examples 4.3.10 and 4.3.11, it was possible to construct a new realisable list with the same trace as the original list. This was made possible by the fact that \( \lambda_2 \leq 0 \) in both cases and thus we could choose \( \delta = 0 \) in Theorem 4.3.9; however, even when \( \lambda_2 > 0 \), it may be possible to preserve the trace of the original spectrum using Theorem 4.3.6 (see Example 4.3.7).

4.4 A \( p = 3 \) construction

In this section, we let \( p = 3 \) in Lemma 4.2.6. For ease of calculation of the characteristic polynomial of \( M \), we will confine our attention to the case where \( n_1 = 1 \) and so \( M \) is a \( 4 \times 4 \) matrix. In this case, we seek to replace the eigenvalues \( \rho, \alpha + i\beta, \alpha - i\beta \) of a realisable list with eigenvalues \( \mu_1, \mu_2, \mu_3, \mu_4 \), where \( \sigma(M) = (\mu_1, \mu_2, \mu_3, \mu_4) \).

**Theorem 4.4.1.** Let the list \( \sigma_0 := (\rho, \alpha + i\beta, \alpha - i\beta, \lambda_4, \lambda_5, \ldots, \lambda_m) \) be realisable, where \( \rho \) is the Perron eigenvalue, \( \alpha \) is real and \( \beta > 0 \). Let \( a, t, \) and \( \eta \) be any real numbers satisfying \( a, t \geq 0 \) and \( 0 < \eta \leq 1 \). Then the list

\[
\sigma_1 := (\mu_1, \mu_2, \mu_3, \mu_4, \lambda_4, \lambda_5, \ldots, \lambda_m)
\]

is realisable, where \( \mu_1, \mu_2, \mu_3, \mu_4 \) are the roots of the polynomial

\[
q(x) := (x - \rho) (\left((x - \alpha)^2 + \beta^2\right) (x - a))
- t \left((x - \alpha)((1 + \eta)x - \alpha - \eta \rho) + \beta^2\right). \quad (60)
\]
Proof. Let the assumptions (i) and (ii) in Lemma 4.2.6 hold, where $B$ is a nonnegative matrix with spectrum $\sigma_0$ and

$$C = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{bmatrix}.$$ 

By Lemma 4.2.1, we may assume without loss of generality that the eigenvector corresponding to $\rho$ is $e$ and so we may write

$$Y_1 = \begin{bmatrix} e & u & v \end{bmatrix},$$

where $u$ and $v$ are real vectors and $u \pm iv$ are eigenvectors corresponding to the eigenvalues $\alpha \pm i\beta$, respectively. We may also assume that

$$\eta = u_1^2 + v_1^2 = \max_i (u_i^2 + v_i^2).$$

To see this, suppose instead that $\tau = u_k^2 + v_k^2 = \max_i (u_i^2 + v_i^2)$. Then we may replace $B$ by $PBP^T$ and $Y$ by $PYD$, where $P$ is the permutation matrix obtained by swapping rows 1 and $k$ of $I_m$ and $D$ is the diagonal matrix

$$D := \begin{bmatrix} 1 \\ \sqrt{\eta/\tau} \\ \sqrt{\eta/\tau} \\ 1 \\ 1 \\ \ddots \end{bmatrix}.$$ 

Now consider the matrix

$$M := \begin{bmatrix} a & tu_1 & tv_1 \\ 1 \\ u_1 \\ v_1 \\ C \end{bmatrix}.$$ 

For all $i = 1, 2, \ldots, m$, the Cauchy-Schwarz inequality gives

$$|u_i u_1 + v_i v_1| \leq \sqrt{(u_i^2 + v_i^2)(u_1^2 + v_1^2)} \leq \eta \leq 1$$

and therefore $-(u_i u_1 + v_i v_1) \leq 1$. Since $1 + u_i u_1 + v_i v_1$ is precisely the $i$th component of the vector

$$Y_1 \begin{bmatrix} 1 & u_1 & v_1 \end{bmatrix},$$
we see that
\[
\begin{bmatrix}
1 \\
u_1 \\
v_1
\end{bmatrix} \in \mathcal{L}(Y_1).
\]
Furthermore, since
\[
\begin{bmatrix}
t & tu_1 & tv_1
\end{bmatrix} = \begin{bmatrix}
t & 0 & 0 & \cdots & 0
\end{bmatrix} Y_1,
\]
we have that
\[
\begin{bmatrix}
t & tu_1 & tv_1
\end{bmatrix} \in \mathcal{K}(Y_1).
\]
Therefore \(M \in M_n(Y_1, C)\) and so by Theorem 4.2.7, the list
\((\sigma(M), \lambda_4, \ldots, \lambda_m)\)
is realisable.

Finally, the characteristic polynomial of \(M\) is
\[
q(x) = (x - \rho) \left( (x - \alpha)^2 + \beta^2 \right) (x - \alpha)
\]
\[- t ((x - \alpha) \left( (1 + u_1^2 + v_1^2) x - \alpha - (u_1^2 + v_1^2) \rho \right) + \beta^2),
\]
which, after the substitution \(u_1^2 + v_1^2 = \eta\), becomes the polynomial mentioned in the statement of the theorem.

**Example 4.4.2.** Consider the list
\[
\sigma_0 := (26, -12 + 2i, -12 - 2i, -1 + 14i, -1 - 14i).
\]
Since \(s_1(\sigma_0) = 0\) and \(s_2(\sigma_0) = 566\), \(\sigma_0\) is realisable by Theorem 1.3.5.

Applying Theorem 4.4.1 with \(\rho = 26\), \(\alpha = -12\), \(\beta = 2\), \(a = 0\), \(\eta = 1\) and \(t = 550\), we obtain a new realisable list
\((42.7876 \ldots, 5.17729 \ldots, -11.9818 \ldots, -33.9831 \ldots, -1 + 14i, -1 - 14i)\).

If desired, we may use three applications of Theorem 4.2.2 to round off these numbers and produce
\((43, 5, -12, -34, -1 + 14i, -1 - 14i)\).

Like \(\sigma_0\), this list is extreme in the sense that it is not realisable for any smaller Perron eigenvalue (it has trace zero).

In general, the roots of the polynomial \(q(x)\) given in (60) are complicated functions of \(\rho, \alpha, \beta, a, t\) and \(\eta\), but it is sometimes possible to choose the parameters \(a, t\) and \(\eta\) in such a way that we can explicitly write down formulae for these roots. This is illustrated by the following corollary.
Corollary 4.4.3. Let the list \((\rho, \alpha + i\beta, \alpha - i\beta, \lambda_4, \lambda_5, \ldots, \lambda_m)\) be realisable, where \(\rho\) is the Perron eigenvalue, \(\alpha < 0\) and \(\beta > 0\). If \(\rho \geq -((\alpha^2 + \beta^2)/\alpha)\), then for all \(s \geq 0\), the list
\[
(\rho + s, \mu_+, \mu_-, 0, \lambda_4, \lambda_5, \ldots, \lambda_m)
\]
is realisable, where
\[
\mu_{\pm} := \alpha - \frac{s}{2} \pm \sqrt{-\beta^2 \left( \frac{s}{\alpha} + 1 \right) + s^2 \left( \frac{1}{4} - \frac{\beta^2 (\alpha^2 + \beta^2)}{\alpha^2 (\beta^2 + (\alpha - \rho)^2) - s\alpha (\alpha^2 + \beta^2 - \alpha \rho)} \right)}.
\]

Proof. Taking \(a = 0\),
\[
\eta = \frac{\alpha^2 + \beta^2}{-\alpha \rho}
\]
and
\[
t = \frac{-s\alpha \rho (\beta^2 + (s - \alpha + \rho)^2)}{s (\alpha^2 + \beta^2 - \alpha \rho) - \alpha (\beta^2 + (\alpha - \rho)^2)}
\]
in Theorem 4.4.1, the roots of \(q(x)\) become \(\rho + s, \mu_+, \mu_-\) and \(0\).

Furthermore, our hypotheses that \(\alpha < 0\) and \(\rho \geq -((\alpha^2 + \beta^2)/\alpha)\) imply that the values of \(\eta\) and \(t\) given in (62) and (63) obey \(0 < \eta \leq 1\) and \(t \geq 0\).

Remark. We have seen that the hypotheses \(\alpha < 0\) and \(\rho \geq -((\alpha^2 + \beta^2)/\alpha)\) are necessary for Corollary 4.4.3 to hold. This region in the \((\alpha, \beta)\) plane is illustrated in Figure 3.

Example 4.4.4. Let \(\sigma\) be any list such that
\[
\sigma_0 := (6, -2 + 2\sqrt{2}i, -2 - 2\sqrt{2}i, \sigma)
\]
is realisable. By Corollary 4.4.3, for any \( s \geq 0 \), the list \( \sigma_1 := (\rho + s, \mu_+, \mu_-, 0, \sigma) \) is realisable, where

\[
\mu_\pm := \frac{1}{2} \left( -4 - s \pm \sqrt{(4 + s)^2 - \frac{288}{6 + s}} \right). \tag{64}
\]

In particular, taking \( s = 2 \), we have that \((8, -3, -3, 0, \sigma)\) is realisable.

This example is reminiscent of the kind of perturbation given in Theorem 4.2.3, except that we have also perturbed the imaginary part of the original complex conjugate pair \(-2 \pm 2\sqrt{2}i\). In fact, using a combination of Theorem 4.4.1 and Theorem 4.2.3, it is possible to show that

\[
(8, -3 + ib, -3 - ib, 0, \sigma) \tag{65}
\]

is realisable for all \( 0 \leq b \leq 2\sqrt{2} \). To see this, let us label the expression under the square root in (64) as

\[
h(s) := (4 + s)^2 - \frac{288}{6 + s}.
\]

Since \( h(0) = -32 \leq -4b^2 \leq 0 = h(2) \) and \( h \) is continuous on \([0, 2]\), there exists \( s_0 \in [0, 2] \) such that \( h(s_0) = -4b^2 \). Then, taking \( s = s_0 \) gives the realisable list

\[
\left( 6 + s_0, -2 - \frac{s_0}{2} + ib, -2 - \frac{s_0}{2} - ib, 0, \sigma \right).
\]

Finally, applying Theorem 4.2.3 by letting \( \epsilon = 1 - s_0/2 \) in (46), we see that lists of the form (65) are realisable.

One may wonder if the zero in (61) is necessary, i.e. given a realisable list \((\rho, \alpha + i\beta, \alpha - i\beta, \lambda_4, \lambda_5, \ldots, \lambda_m)\) satisfying the hypotheses of Corollary 4.4.3, perhaps \((\rho + s, \mu_+, \mu_-, \lambda_4, \lambda_5, \ldots, \lambda_m)\) is realisable without the additional zero eigenvalue. This question still remains open. To attempt to prove that the zero is necessary, one may seek to find some \( \rho, \alpha, \beta \) and \( \sigma \) satisfying the hypotheses of the corollary, such that \((\rho, \alpha + i\beta, \alpha - i\beta, \sigma)\) is realisable, but \((\rho + s, \mu_+, \mu_-, \sigma)\) violates one of the JLL conditions for some \( s > 0 \); however, consider any JLL condition of the form

\[
s_1^m \leq n^{m-1}s_m. \tag{66}
\]

For \((\rho + s, \mu_+, \mu_-, \sigma)\) to violate such an inequality, we would require a suitable \((\rho, \alpha, \beta, s)\) such that \( s_m(\rho + s, \mu_+, \mu_-) < s_m(\rho, \alpha + i\beta, \alpha - i\beta) \), but it would appear that such \((\rho, \alpha, \beta, s)\) do not exist. For example, with \( \rho = 1 \), the shaded region in Figure 4 illustrates those values of \( \alpha \) and \( \beta \) for which

\[
\frac{d}{ds}s_{40}(\rho + s, \mu_+, \mu_-) \bigg|_{s=0} < 0, \tag{67}
\]
but none of these points \((\alpha, \beta)\) lie within the region \(1 \geq -\frac{\alpha^2 + \beta^2}{-\alpha}\) in which the corollary holds. Attempts to find a suitable \(\rho, \alpha, \beta\) and \(\sigma\) which exploit JLL conditions other than those of the form (66) have also been unsuccessful.

To comment on the roots of the polynomial \(q(x)\) given in (60) when \(t\) is large, it is convenient to label

\[
\begin{align*}
  f(x) &:= (x - \rho) \left( (x - \alpha)^2 + \beta^2 \right) (x - a), \\
  g(x) &:= (x - \alpha)((1 + \eta)x - \alpha - \eta\rho) + \beta^2,
\end{align*}
\]

so that \(q(x) = f(x) - tg(x)\). As \(t\) approaches infinity, the quadratic, linear and constant terms of \(q(x)\) become increasingly dominated by those of \(-tg(x)\) and therefore two of the roots of \(q(x)\) will approach those of \(g(x)\). These limiting roots may be real or complex, depending on the values of \(\rho, \alpha, \beta\) and \(\eta\). We finish this chapter with an example where they are complex:

**Example 4.4.5.** Let \(\sigma\) be any list for which \(\sigma_0 = (2, i, -i, \sigma)\) is realisable. Applying Theorem 4.4.1 with \(\alpha = 0, \eta = 1\) and \(t = 1\) produces the realisable spectrum

\[
\sigma_1 = (2.4710 \ldots, 0.1868 \ldots + (0.6666 \ldots)i, 0.1868 \ldots - (0.6666 \ldots)i, -0.8445 \ldots, \sigma).
\]

t = 5 gives

\[
\sigma_1 = (3.8755 \ldots, 0.4100 \ldots + (0.5573 \ldots)i, 0.4100 \ldots - (0.5573 \ldots)i, -2.6954 \ldots, \sigma).
\]

t = 500 gives
\[ \sigma_1 = (32.1356\ldots, 0.499\ldots + (0.5007\ldots)i, 0.499\ldots - (0.5007\ldots)i, \\
-31.1336\ldots, \sigma), \]

illustrating the convergence of two of the eigenvalues of \( \sigma_1 \) to \( 1/2 \pm (1/2)i \).
5.1 INTRODUCTION

Let \( \sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n) \) be a self-conjugate list of complex numbers. Recall that if \( \sigma \) has at most three elements\(^1\), or if every entry in \( \sigma \) apart from the Perron eigenvalue has nonpositive real part\(^2\), then \( \sigma \) is realisable if and only if
\[
\sum_{i=1}^n \sigma_i \geq 0
\]
and \( \sigma \) obeys the JLL condition
\[
\sum_{i=1}^n \sigma_i^2 \leq n \sum_{i=1}^n \sigma_i. 
\] (68)

If a realising matrix is known to exist, one may consider characterising the possible diagonal elements of said matrix. This is the goal of this chapter. In the above cases, we essentially show that \( \sigma \) is the spectrum of a nonnegative matrix with diagonal elements \( \Delta := (a_1, a_2, \ldots, a_n) \) if and only if
\[
\sum_{i=1}^n \Delta_i = \sum_{i=1}^n \sigma_i 
\] (69)
and
\[
\sum_{i=1}^n \Delta_i^2 \leq \sum_{i=1}^n \sigma_i. 
\] (70)

This work is motivated by constructive methods in the NIEP which utilise diagonal elements. See, for example, Theorem 4.2.4 and Chapters 6 and 7.

The results are constructive, with realising matrices being of the form
\[
A := \begin{bmatrix}
1 & a_1 \\
a_2 & 1 \\
& \ddots & \ddots \\
a_n-1 & & 1 \\
b_n & \ldots & b_2 & a_n
\end{bmatrix}. 
\] (71)

Note that the above matrix is the sum of a companion matrix and a diagonal matrix.

---

\(^1\) See Theorem 1.3.2.

\(^2\) See Theorem 1.3.5.
5.2 The Sum of a Companion Matrix and a Diagonal Matrix

The observations in this section will apply to arbitrary (not necessarily nonnegative) matrices $A$ of the form (71).

Let us begin by computing the characteristic polynomial of $A$.

Lemma 5.2.1. Let $A$ be defined as in (71). Then the characteristic polynomial of $A$ is given by

$$f(x) := \prod_{i=1}^{n}(x - a_i) - b_2 \prod_{i=1}^{n-2}(x - a_i) - b_3 \prod_{i=1}^{n-3}(x - a_i) - \cdots - b_n. \quad (72)$$

Proof. The proof is by induction on $n$. If $n = 2$, then

$$A = \begin{bmatrix} a_1 & 1 \\ b_2 & a_2 \end{bmatrix}$$

has characteristic polynomial $(x - a_1)(x - a_2) - b_2$, as required. If $n > 2$, then by Laplace expansion along the first column of $xI_n - A$, we see that the characteristic polynomial of $A$ is given by

$$\text{Det}(xI_n - A) = (x - a_1)\text{Det}(B(x)) + (-1)^n b_n \text{Det}(C(x)),$$

where

$$B(x) := \begin{bmatrix} x - a_2 & -1 \\ x - a_3 & -1 \\ \vdots & \ddots \\ x - a_{n-1} & -1 \\ -b_{n-1} & -b_{n-2} & \cdots & -b_2 & x - a_n \end{bmatrix}$$

and

$$C(x) := \begin{bmatrix} -1 \\ x - a_2 & -1 \\ \vdots & \ddots \\ \vdots \\ x - a_{n-1} & -1 \end{bmatrix}.$$

By the inductive hypothesis,

$$\text{Det}(B(x)) = \prod_{i=2}^{n}(x - a_i) - b_2 \prod_{i=2}^{n-2}(x - a_i) - \cdots - b_{n-1}.$$ 

In addition, since $C(x)$ is lower triangular,

$$\text{Det}(C(x)) = (-1)^{n-1}.$$ 

Hence $\text{Det}(xI_n - A)$ is given by (72).
Given \(a_1, a_2, \ldots, a_n\) and an arbitrary polynomial

\[ p(x) := x^n - s_1(\Delta)x^{n-1} + q_2x^{n-2} + q_3x^{n-3} + \cdots + q_n, \]  

(73)

whose roots sum to \(s_1(\Delta)\), there exists a unique solution \((b_2, b_3, \ldots, b_n)\) such that the polynomials (72) and (73) coincide. For the remainder of this section, we will concern ourselves with the task of explicitly computing this solution.

Recall the definition of the \(k\)-th elementary symmetric function of the variables \(x_1, x_2, \ldots, x_n\) (12). In this section, we will also need to consider the \(k\)-th complete homogeneous symmetric function

\[
\begin{align*}
  h_0(x_1, x_2, \ldots, x_n) &:= 1, \\
  h_k(x_1, x_2, \ldots, x_n) &:= \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1}x_{i_2}\cdots x_{i_k} : k = 1, 2, \ldots 
\end{align*}
\]

As with the elementary symmetric functions, it is convenient to define \(h_k(x_1, x_2, \ldots, x_n) = 0\) if \(k < 0\). The following relation between the elementary symmetric functions and the complete homogeneous ones is well known. A proof is included for completeness.

**Lemma 5.2.2.** For all \(m > 0\),

\[
\sum_{i=0}^{m} (-1)^{i}e_i(x_1, x_2, \ldots, x_n)h_{m-i}(x_1, x_2, \ldots, x_n) = 0. 
\]

(74)

**Proof.** Consider the generating function for the complete homogeneous symmetric functions:

\[
\prod_{t=1}^{n} \frac{1}{1-x_t t} = \prod_{t=1}^{n} \sum_{i=0}^{\infty} (x_t t)^i = \sum_{r=0}^{\infty} h_r(x_1, x_2, \ldots, x_n) t^r.
\]

Since

\[
\prod_{t=1}^{n} (1-x_t t) = \sum_{r=0}^{\infty} (-1)^r e_r(x_1, x_2, \ldots, x_n) t^r,
\]

it follows that

\[
1 = \left( \sum_{r=0}^{n} (-1)^r e_r(x_1, x_2, \ldots, x_n) t^r \right) \left( \sum_{r'=0}^{\infty} h_{r'}(x_1, x_2, \ldots, x_n) t^{r'} \right). 
\]

(75)

Equating the coefficients of \(t^m\) in (75) gives (74).

We will now require some additional notation. Given \(a_1, a_2, \ldots, a_n\), define the list \(\Delta^{(j)}\) by

\[
\Delta^{(j)} := \begin{cases} 
  (a_1, a_2, \ldots, a_j) & : j = 1, 2, \ldots, n, \\
  (a_1, a_2, \ldots, a_n, 0, 0, \ldots, 0) & : j = n + 1, n + 2, \ldots \\
  \downarrow & \text{for } j-n \text{ zeros}
\end{cases}
\]
To avoid unnecessarily long expressions, for \( j = 1, 2, \ldots \), let

\[
e_{i}^{(j)} := e_{i} \left( \Delta^{(j)} \right),
\]

\[
\eta_{i}^{(j)} := h_{i} \left( \Delta^{(j)} \right)
\]

and let

\[
e_{i}^{(0)} = \eta_{i}^{(0)} = \begin{cases} 1 & : i = 0, \\ 0 & : i \neq 0. \end{cases}
\]

It will also be convenient to define

\[
e_{i}^{(j)} := e_{i} \left( a_{j+1}, a_{j+2}, \ldots, a_{n} \right) : i = 0, \ldots, n - j; \ j = 0, \ldots, n - 1.
\]

We will require a generalisation of Lemma 5.2.2:

**LEMMA 5.2.3.** *If* \( k \geq 0 \) *and* \( m > 0 \), *then*

\[
\sum_{i=0}^{m} (-1)^{i} e_{i}^{(k+i)} \eta_{m-i}^{(k+i+1)} = 0. \quad (76)
\]

**Proof.** We first note that (76) is independent of \( a_{k+m+2}, a_{k+m+3}, \ldots \), and hence, we need only consider \( n \leq k + m + 1 \). We note also that if \( n \leq j \), then \( e_{i}^{(j)} = e_{i}^{(n)} \) and \( \eta_{i}^{(j)} = \eta_{i}^{(n)} \). The proof is by induction on \( n \).

As a base of induction, we note that if \( n \leq k + 1 \), then the left-hand side of (76) reduces to

\[
\sum_{i=0}^{m} (-1)^{i} e_{i}^{(n)} \eta_{i}^{(n)},
\]

which, by Lemma 5.2.2, equals zero.

Now suppose \( n = k + l \), where \( 2 \leq l \leq m \) and assume the statement holds for lists of length \( n - 1 \). Note that by the inductive hypothesis,

\[
\sum_{i=0}^{1-2} (-1)^{i} e_{i}^{(n-1+i)} \eta_{m-i}^{(n-1+i+1)} + \sum_{i=1-1}^{m} (-1)^{i} e_{i}^{(n-1)} \eta_{m-i}^{(n-1)} = 0. \quad (77)
\]

The left-hand side of (76) may be written as

\[
\sum_{i=0}^{1-2} (-1)^{i} e_{i}^{(n-1+i)} \eta_{m-i}^{(n-1+i+1)}
\]

\[
+ (-1)^{1-1} e_{1}^{(n-1)} \eta_{m-1}^{(n)} + \sum_{i=1}^{m} (-1)^{i} e_{i}^{(n)} \eta_{m-i}. \quad (78)
\]

Furthermore, we note that for all \( j = 0, 1, \ldots \),

\[
e_{j}^{(n)} = e_{j}^{(n-1)} + a_{n} e_{j-1}^{(n-1)}
\]
and
\[ \eta_{j}^{(n)} = \sum_{r=0}^{j} a_{n} r \eta_{j-r}^{(n-1)}. \]

Hence (78) may be written in the form
\[
\sum_{i=0}^{1+2} (-1)^{i} \epsilon_{i}^{(n-1)} \eta_{m-i}^{(n-1)} + (-1)^{1} \epsilon_{1}^{(n-1)} \sum_{r=0}^{m-1+1} a_{n} r \eta_{m-1-r}^{(n-1)} \\
+ \sum_{i=1}^{m} (-1)^{i} \left( \epsilon_{i}^{(n-1)} + a_{n} \epsilon_{i-1}^{(n-1)} \right) \sum_{r=0}^{m-i} a_{n} r \eta_{m-i-r}^{(n-1)}. \quad (79)
\]

Let us now consider (79) as a polynomial in \( a_{n} \). The constant term is equal to (77), which, as previously noted, equals zero, and for \( s = 1, 2, \ldots \), the coefficient of \( a_{n}^{s} \) is given by
\[
\sum_{i=1}^{m} (-1)^{i} \epsilon_{i}^{(n-1)} \eta_{m-i-s}^{(n-1)} + \sum_{i=1}^{m} (-1)^{i} \epsilon_{i-1}^{(n-1)} \eta_{m-i-s+1}^{(n-1)} \\
= \sum_{i=1}^{m} (-1)^{i} \epsilon_{i}^{(n-1)} \eta_{m-i-s}^{(n-1)} - \sum_{i=1}^{m} (-1)^{i} \epsilon_{i}^{(n-1)} \eta_{m-i-s}^{(n-1)} \\
= (-1)^{m} \epsilon_{m}^{(n-1)} \eta_{-s}^{(n-1)} = 0.
\]

Hence (79) equals zero, as required.

Finally, if \( n = k + m + 1 \), then the left-hand side of (76) may be written as
\[
\sum_{i=0}^{m-1} (-1)^{i} \epsilon_{i}^{(n-m-i-1)} \eta_{m-i}^{(n-m+1)} + (-1)^{m} \epsilon_{m}^{(n-1)}. \quad (80)
\]

Since (80) is independent of \( a_{n} \), it must vanish by the inductive hypothesis.

Remark. Clearly, Lemma 5.2.3 holds for arbitrary (not necessarily real) variables \( a_{1}, a_{2}, \ldots, a_{n} \).

Lemma 5.2.4. Let
\[ g(x) := x^{n-1} + c_{1} x^{n-2} + c_{2} x^{n-3} + \cdots + c_{n-1} \]
be an arbitrary polynomial and let
\[ p := c_{1} + \epsilon_{1}^{(n)}. \]

Then the matrix \( A \) given in (71) has characteristic polynomial \((x - p)g(x)\) if and only if
\[ b_{j} = \sum_{i=0}^{j} \mathcal{X}_{i,j} : \quad j = 2, 3, \ldots, n, \quad (82)\]

where
\[ \mathcal{X}_{i,j} := \eta_{i}^{(n-j+1)} (c_{1} c_{j-i-1} - c_{j-i}) + \epsilon_{i}^{(n)} \eta_{i-1}^{(n-j+1)} c_{j-i}. \quad (83)\]
Proof. Equating the coefficients of \( x^{n-j} \) in \((x - \rho)g(x)\) and (72) gives

\[
c_j - \rho c_{j-1} = - \sum_{r=0}^{j} (-1)^r c_r^{(n-j+r)} b_{j-r}. \tag{84}
\]

Hence, in order for the polynomial in (72) to be equal to \((x - \rho)g(x)\), \((b_2, b_3, \ldots, b_n)\) must be the unique solution to the recurrence relation (84) with initial condition \(b_0 = -1\). Therefore, we must show that the solution (82) satisfies (84). For \(b_j\) given as in (82),

\[
\sum_{r=0}^{j} (-1)^r c_r^{(n-j+r)} b_{j-r} = \sum_{r=0}^{j} (-1)^r c_r^{(n-j+r)} \sum_{i=0}^{j-r} \mathcal{K}_{i,j-r} \tag{85}
\]

and hence, after substituting \(i = j - s - r\) in (85), we see that

\[
\sum_{r=0}^{j} (-1)^r c_r^{(n-j+r)} b_{j-r} = \sum_{r=0}^{j} \sum_{s=0}^{j-r} (-1)^r c_r^{(n-j+r)} \mathcal{K}_{j-s-r,j-r} = \sum_{s=0}^{j} \sum_{r=0}^{j-s} (-1)^r c_r^{(n-j+r)} \mathcal{K}_{j-s-r,j-r}.
\]

From (81) and the definition (83) of \(\mathcal{K}_{i,j}\), we have

\[
c_j - \rho c_{j-1} + \sum_{r=0}^{j} (-1)^r c_r^{(n-j+r)} b_{j-r} = - (c_1 c_{j-1} - c_j) - c_1^{[n]} c_{j-1}
\]

\[
+ \sum_{s=0}^{j} (c_1 c_{s-1} - c_s) \sum_{r=0}^{j-s} (-1)^r c_r^{(n-j+r)} \eta_{j-s-r}^{(n-j+r+1)}
\]

\[
+ c_1^{[n]} \sum_{s=0}^{j} c_s \sum_{r=0}^{j-s} (-1)^r c_r^{(n-j+r)} \eta_{j-s-r}^{(n-j+r+1)}
\]

however, we note that

\[
\sum_{s=0}^{j} \sum_{r=0}^{j-s} (-1)^r c_r^{(n-j+r)} \eta_{j-s-r}^{(n-j+r+1)} = \sum_{s=0}^{j-1} \sum_{r=0}^{j-s-1} (-1)^r c_r^{(n-j+r)} \eta_{j-s-r}^{(n-j+r+1)}, \tag{86}
\]

since the additional terms on the left-hand side of (86) vanish. Therefore

\[
c_j - \rho c_{j-1} + \sum_{r=0}^{j} (-1)^r c_r^{(n-j+r)} b_{j-r} = - (c_1 c_{j-1} - c_j) - c_1^{[n]} c_{j-1}
\]
where the final equality follows from Lemma 5.2.3.

Remark. Note that, if the solution (82) for \( b_j \) is considered as a multi-variable polynomial in \( a_1, a_2, \ldots, a_n \), then \( \mathcal{K}_{i,j} \) is the sum of all terms of degree \( i \). Therefore, by considering the sum of all terms of degree \( m \) on each side of (84), we see that

\[
\sum_{r=0}^{m} (-1)^r \epsilon_r^{(n-j+r)} \eta_{j-s-r}^{(n-j+r+1)} \mathcal{K}_{m-r,j-r} = 0
\]

for all \( j = 2, 3, \ldots, n \) and \( m = 2, 3, \ldots, j \).

The entry \( b_2 \) has a special significance: if \( A \) has spectrum \( \sigma \), then, applying Newton’s identities to the coefficient of \( x^{n-2} \) in (72), we see that

\[
b_2 = \frac{1}{2} (s_1(\Delta)^2 - s_2(\Delta)) - \frac{1}{2} (s_1(\sigma)^2 - s_2(\sigma)) = \frac{1}{2} (s_2(\sigma) - s_2(\Delta)).
\]

Hence condition (70) is directly related to the nonnegativity of \( b_2 \).

5.3 Necessity

It is easy to see that (69) and (70) are necessary for \( \sigma \) to be the spectrum of a nonnegative matrix with diagonal elements \( \Delta \):

**Observation 5.3.1.** Let \( \sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n) \) be a list of complex numbers and let \( \Delta := (a_1, a_2, \ldots, a_n) \) be a list of nonnegative numbers. If \( \sigma \) is the spectrum of a nonnegative matrix with diagonal elements \( \Delta \), then

\[
s_1(\Delta) = s_1(\sigma)
\]

and

\[
s_m(\Delta) \leq s_m(\sigma) : m = 2, 3, \ldots
\]
Proof. Let $A$ be a nonnegative matrix with spectrum $\sigma$ and diagonal elements $\Delta$. Obviously, $s_1(\Delta) = \text{Tr}(A) = s_1(\sigma)$. In addition, since $A$ is nonnegative, $(A^m)_{ii} \geq a_i^m$ for all positive integers $i$ and $m$. Hence

$$s_m(\sigma) = \text{Tr}(A^m) \geq \sum_{i=1}^{n} a_i^m = s_m(\Delta).$$

Notice that if $s_1(\Delta) = s_1(\sigma)$, then

$$\sum_{i=1}^{n} \left( a_i - \frac{s_1(\sigma)}{n} \right)^2 = s_2(\Delta) - \frac{s_1(\sigma)^2}{n}.$$  

In this case, (70) is equivalent to

$$\sum_{i=1}^{n} \left( a_i - \frac{s_1(\sigma)}{n} \right)^2 \leq s_2(\sigma) - \frac{s_1(\sigma)^2}{n},$$

i.e. condition (70) says that $(a_1, a_2, \ldots, a_n)$ must be sufficiently close to $(s_1(\sigma)/n, s_1(\sigma)/n, \ldots, s_1(\sigma)/n)$ with respect to the $\ell^2$ norm.

5.4 Solutions for small $n$

Suppose $\lambda_1 \geq \lambda_2$ and $a_1 \geq a_2$. The general $2 \times 2$ matrix

$$\begin{bmatrix}
    a_1 & a_{12} \\
    a_{21} & a_2
\end{bmatrix}$$

has spectrum $(a_1 + t, a_2 - t)$ if and only if

$$a_{12} a_{21} = t(a_1 - a_2 + t).$$

Since we require $a_{12}$ and $a_{21}$ to be nonnegative, it is not difficult to see that $(\lambda_1, \lambda_2)$ is the spectrum of a nonnegative matrix with diagonal elements $(a_1, a_2)$ if and only if $a_1 \leq \lambda_1$ and $a_1 + a_2 = \lambda_1 + \lambda_2$. If these conditions are satisfied, then choosing $a_{12} = 1$ yields a matrix of the form (71).

When $n = 3$, it is useful to distinguish the real and complex cases. Let us first consider the case when $\lambda_1$, $\lambda_2$ and $\lambda_3$ are real.

Proposition 5.4.1. If $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and $a_1 \geq a_2 \geq a_3 \geq 0$, then the list $(\lambda_1, \lambda_2, \lambda_3)$ is the spectrum of a nonnegative matrix with diagonal elements $(a_1, a_2, a_3)$ if and only if the following conditions hold:

(i) $\lambda_2 \leq a_1 \leq \lambda_1$;

(ii) $a_1 + a_2 + a_3 = \lambda_1 + \lambda_2 + \lambda_3$;

(iii) $a_1^2 + a_2^2 + a_3^2 \leq \lambda_1^2 + \lambda_2^2 + \lambda_3^2$.

Furthermore, if (i)–(iii) are satisfied, then $(\lambda_1, \lambda_2, \lambda_3)$ may be realised by a nonnegative matrix of the form (71).
Proof. The necessity of (ii) and (iii) was shown in Observation 5.3.1. Now suppose the matrix

\[ A := \begin{bmatrix} a_1 & a_{12} & a_{13} \\ a_{21} & a_2 & a_{23} \\ a_{31} & a_{32} & a_3 \end{bmatrix} \]

has spectrum \((\lambda_1, \lambda_2, \lambda_3)\). The characteristic polynomial of \(A\) is

\[
(x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = (x - a_1)(x - a_2)(x - a_3)
- a_{23}a_{32}(x - a_1) - a_{13}a_{31}(x - a_2)
- a_{12}a_{21}(x - a_3) - a_{12}a_{23}a_{31} - a_{21}a_{32}a_{13}.
\]

Letting \(x = a_1\) in (90) gives

\[
(a_1 - \lambda_1)(a_1 - \lambda_2)(a_1 - \lambda_3) = -a_{13}a_{31}(a_1 - a_2) - a_{12}a_{21}(a_1 - a_3)
- a_{12}a_{23}a_{31} - a_{21}a_{32}a_{13}
\leq 0.
\]

Therefore, either \(a_1 \leq \lambda_3\) or \(\lambda_2 \leq a_1 \leq \lambda_1\). Since \(a_1 + a_2 + a_3 = \lambda_1 + \lambda_2 + \lambda_3\), the former case would imply \(a_1 = a_2 = a_3 = \lambda_1 = \lambda_2 = \lambda_3\). Hence (i) holds.

Now suppose (i)–(iii) hold and the matrix

\[ A := \begin{bmatrix} a_1 & 1 & 0 \\ 0 & a_2 & 1 \\ b_3 & b_2 & a_3 \end{bmatrix} \]

has spectrum \((\lambda_1, \lambda_2, \lambda_3)\). By (88), \(b_2 \geq 0\). According to Lemma 5.2.1, the characteristic polynomial of \(A\) is

\[
(x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = (x - a_1)(x - a_2)(x - a_3) - b_2(x - a_1) - b_3.
\]

Hence, letting \(x = a_1\) in (92), we see that

\[
b_3 = -(a_1 - \lambda_1)(a_1 - \lambda_2)(a_1 - \lambda_3),
\]

which, by (i), is nonnegative. \(\square\)

Note that if we require the realising matrix to be symmetric, then the conditions on \((a_1, a_2, a_3)\) are more restrictive (see Section 6.2).

Example 5.4.2. The matrix

\[
\begin{bmatrix} 6 & 1 & 0 \\ 0 & 4 & 1 \\ 20 & 1 & 0 \end{bmatrix}
\]

has spectrum \((\lambda_1, \lambda_2, \lambda_3) = (7, 2, 1)\) and diagonal elements \((a_1, a_2, a_3) = (6, 4, 0)\); however, since \(a_3 < \lambda_3\), no symmetric nonnegative matrix exists with this spectrum and these diagonal elements (see (107)).
Let us now consider the case when \( \lambda_2 \) and \( \lambda_3 \) are complex.

**Proposition 5.4.3.** Let \( \sigma := (\rho, \alpha + i\beta, \alpha - i\beta) \), where \( \rho, \beta \geq 0 \) and \( \alpha \) is real, and let \( \Delta := (a_1, a_2, a_3) \), where \( a_1 \geq a_2 \geq a_3 \geq 0 \). Then \( \sigma \) is the spectrum of a nonnegative matrix with diagonal elements \( \Delta \) if and only if the following conditions hold:

(i) \( a_1 \leq \rho \);

(ii) \( s_1(\Delta) = s_1(\sigma) \);

(iii) \( s_2(\Delta) \leq s_2(\sigma) \).

Furthermore, if (i)–(iii) are satisfied, then \( \sigma \) may be realised by a nonnegative matrix of the form (71).

**Proof.** Suppose there exists a nonnegative matrix \( A \) with spectrum \( \sigma \) and diagonal elements \( \Delta \). The fact that (i) holds follows from property (iii) of Theorem 1.2.4. (In fact, it is well-known that the Perron eigenvalue of a nonnegative matrix must be at least as large as each of its entries.) The necessity of (ii) and (iii) was shown in Observation 5.3.1.

Now suppose (i)–(iii) hold and the matrix \( A \) given in (91) has spectrum \( \sigma \). As in the proof of Proposition 5.4.1, we note that \( b_2 \geq 0 \) by (88) and
\[
b_3 = -(a_1 - \rho) \left( (a_1 - \alpha)^2 + \beta^2 \right) \geq 0.
\]
\( \square \)

## 5.5 Lists with Nonpositive Real Parts

In this section, we consider lists of the form \( \sigma := (\rho, \lambda_2, \lambda_3, \ldots, \lambda_n) \), where \( \rho \geq 0 \) and \( \text{Re}(\lambda_i) \leq 0 : i = 2, 3, \ldots, n \). We will characterise the possible diagonal elements of a nonnegative matrix which realises \( \sigma \). We will employ several results from Chapters 2 and 3.

We now give the main result of this Chapter:

**Theorem 5.5.1.** Let \( \Delta := (a_1, a_2, \ldots, a_n) \), where \( a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \), let \( \rho \geq 0 \) and let \( (\lambda_2, \lambda_3, \ldots, \lambda_n) \) be a self-conjugate list of complex numbers with nonpositive real parts. Then the list \( \sigma := (\rho, \lambda_2, \lambda_3, \ldots, \lambda_n) \) is the spectrum of a nonnegative matrix with diagonal elements \( \Delta \) if and only if
\[
s_1(\Delta) = s_1(\sigma) \quad (93)
\]
and
\[
s_2(\Delta) \leq s_2(\sigma). \quad (94)
\]

Furthermore, if (93) and (94) are satisfied, then \( \sigma \) may be realised by a nonnegative matrix of the form (71).

**Proof.** Necessity was established in Observation 5.3.1, so now suppose (93) and (94) hold. We will show that \( \sigma \) is the spectrum of a nonnegative matrix \( A \) of the form (71).
Let 
\[ c_i := e_i(-\lambda_2, -\lambda_3, \ldots, -\lambda_n) : \quad i \in \mathbb{Z} \]
and let 
\[ g(x) := \prod_{i=2}^{n}(x - \lambda_i) = x^{n-1} + c_1x^{n-2} + c_2x^{n-3} + \cdots + c_{n-1}. \]

Since \( \text{Re}(\lambda_i) \leq 0; \ i = 2, 3, \ldots, n \), it follows that \( c_i \geq 0 \) for all \( i \). In addition, since \( s_1(\Delta) = s_1(\sigma) = \rho - c_1 \), we must have \( \rho = c_1 + s_1(\Delta) \), or, in other notation, \( \rho = c_1 + e_i^{(n)} \).

Therefore, by Lemma 5.2.4, it suffices to show that the quantities \( b_2, b_3, \ldots, b_n \) given in (82) are nonnegative. Firstly, we note that (83) can be rearranged to give 
\[ \mathcal{X}_{i,j} := \eta_i^{(n-j+1)}c_1c_{j-i-1} + \left( e_1^{(n)}\eta_i^{(n-j+1)} - \eta_i^{(n-j+1)} \right) c_{j-i}. \]

If \( i > 0 \), then 
\[ \eta_i^{(n-j+1)} = \sum_{r=1}^{n-j+1} a_r\eta_{i-1}^{(r)} \leq \sum_{r=1}^{n} a_r\eta_{i-1}^{(r)} = c_1^{(n)}\eta_i^{(n-j+1)}. \]

Hence 
\[ \mathcal{X}_{i,j} \geq 0 : \quad i = 1, 2, \ldots, j; \quad j = 2, 3, \ldots, n. \]

Note also that 
\[ \mathcal{X}_{0,j} = c_1c_{j-1} - c_j : \quad j = 2, 3, \ldots, n. \]

By Theorem 3.3.3, \( \mathcal{X}_{0,j} > 0 \) whenever \( j \) is odd. Hence \( b_1 > 0 \) for each odd index \( j \).

Let us now consider \( b_1 \) for even indices \( j \). By (88), \( b_2 \geq 0 \). Now fix \( k \in \{2, 3, \ldots, \lfloor n/2 \rfloor\} \). We will show that \( b_{2k} \geq 0 \). If \( \mathcal{X}_{0,2k} > 0 \), we are done, so from now on, assume \( \mathcal{X}_{0,2k} < 0 \). Now consider the polynomial 
\[ q(x) := (x - c_1)g(x) = x^n - \mathcal{X}_{0,2}x^{n-2} - \mathcal{X}_{0,3}x^{n-3} - \cdots - \mathcal{X}_{0,n}. \]

By considering the even terms in \( q(x) \), we see that Lemma 2.4.1 implies \( \mathcal{X}_{0,j} < 0 : \quad j = 1, 2, \ldots, k \).

At this point, it is helpful to consider the quantities 
\[ W_1 := \mathcal{X}_{0,2k}\mathcal{X}_{1,2} - \mathcal{X}_{0,2}\mathcal{X}_{1,2k}, \]
\[ W_2 := \mathcal{X}_{0,2k}\mathcal{X}_{2,2} - \mathcal{X}_{0,2}\mathcal{X}_{2,2k}. \]

We will show that \( W_1, W_2 \geq 0 \).

Let us first examine \( W_1 \). Note that the expression for \( \mathcal{X}_{1,2} \) simplifies to 
\[ \mathcal{X}_{1,2} = e_1^{(n)}c_1. \]
In addition, notice that $\mathcal{K}_{1,2k}$ may be written in the form
\[
\mathcal{K}_{1,2k} = e_1^{(n-2k+1)} c_1 c_{2k-2} + e_1^{(n-2k+1)} c_{2k-1}.
\] (95)

Since $c_1 c_{2k-2} \geq c_{2k-1}$ by Theorem 3.3.3, it follows that
\[
\mathcal{K}_{1,2k} \geq e_1^{(n)} c_{2k-1}.
\]

Therefore
\[
W_1 \geq e_1^{(n)} (\mathcal{K}_{0,2k} - e_1^{(n)} c_{2k-1})
\]
\[
= e_1^{(n)} (c_2 c_{2k-1} - c_1 c_{2k})
\]
\[
\geq 0,
\]
where the final inequality is due to Theorem 3.3.3.

Next we examine $W_2$. Similarly to (95), we note that
\[
\mathcal{K}_{1,2k-1} = e_1^{(n-2k+2)} c_1 c_{2k-3} + e_1^{(n-2k+2)} c_{2k-2}.
\]

Hence
\[
\mathcal{K}_{1,2k-1} + e_1^{(n-2k+2)} \mathcal{K}_{0,2k-2} = e_1^{(n)} c_1 c_{2k-3} \geq 0,
\]
i.e.
\[
\mathcal{K}_{1,2k-1} \geq -e_1^{(n-2k+2)} \mathcal{K}_{0,2k-2}.
\] (96)

Letting $m = 2$ and $j = 2k$ in (87), we see that
\[
\mathcal{K}_{2,2k} = e_1^{(n-2k+1)} \mathcal{K}_{1,2k-1} - e_2^{(n-2k+2)} \mathcal{K}_{0,2k-2}
\] (97)

and combining (97) with (96) gives
\[
\mathcal{K}_{2,2k} \geq -\left( e_1^{(n-2k+1)} e_1^{(n-2k+2)} + e_2^{(n-2k+2)} \right) \mathcal{K}_{0,2k-2}.
\] (98)

Next, we note that
\[
e_2^{(n)} = e_2^{(n-2k+2)} + e_2^{(n-2k+1)} + e_1^{(n-2k+1)}
\]
\[
= e_2^{(n-2k+2)} - a_{n-2k+2} e_1^{(n-2k+1)} + e_2^{(n-2k+1)}
\]
\[
+ e_1^{(n-2k+1)} a_{n-2k+2} + e_1^{(n-2k+2)}
\]
\[
= e_2^{(n-2k+2)} + e_2^{(n-2k+1)} + e_1^{(n-2k+1)} e_1^{(n-2k+2)},
\]

that is to say,
\[
e_1^{(n-2k+1)} e_1^{(n-2k+2)} + e_2^{(n-2k+2)} = e_2^{(n)} - e_2^{(n-2k+1)}.
\]

Hence (98) is equivalent to
\[
\mathcal{K}_{2,2k} \geq -\left( e_2^{(n)} - e_2^{(n-2k+1)} \right) \mathcal{K}_{0,2k-2}.
\]
We also observe that, since \( a_1 \geq a_2 \geq \cdots \geq a_n \),

\[
\left( \frac{n}{2} \right)^{-1} e_2^{(n)} \geq \left( \frac{2k-1}{2} \right)^{-1} e_2^{(n-2k+1)}.
\]

Therefore

\[
e_2^{(n-2k+1)} \leq \left( \frac{2k-1}{2} \right) e_2^{(n)} \leq \left( \frac{2k-1}{2} \right) e_2^{(n)} = \left( 1 - \frac{1}{k} \right) e_2^{(n)}
\]

and hence

\[
\mathcal{X}_{2,2k} \geq \frac{1}{k} e_2^{(n)} \mathcal{X}_{0,2k-2}.
\]

Now note that

\[
\mathcal{X}_{2,2} = e_1^{(n)} e_1^{(n-1)} - \eta_2^{(n-1)} = \sum_{r=1}^{n-1} a_r e_1^{(n)} - \sum_{r=1}^{n-1} a_r e_1^{(r)}
\]

\[
= \sum_{r=1}^{n-1} a_r e_1^{(r)} = e_2^{(n)}.
\]

By (99) and (100), it follows that

\[
W_2 \geq \left( \mathcal{X}_{0,2k} + \frac{1}{k} \mathcal{X}_{0,2} \mathcal{X}_{0,2k-2} \right) e_2^{(n)};
\]

however, in the notation of Section 3.5, we have

\[-\mathcal{X}_{0,2i} = \left( \frac{\lfloor n/2 \rfloor}{i} \right) Q_{2i}(e_1; -\lambda_2, -\lambda_3, \ldots, -\lambda_n) : \ i = 1, 2, \ldots, \lfloor n/2 \rfloor\]

and hence, Lemma 3.5.1 implies

\[
\frac{\mathcal{X}_{0,2} \mathcal{X}_{0,2k-2}}{-\mathcal{X}_{0,2k}} \geq \frac{\left( \frac{n/2}{i} \right) \left( \frac{n/2}{k-1} \right)}{\left( \frac{n/2}{k} \right)} = k \frac{n/2}{k} - k + 1 > k.
\]

Hence \( W_2 \geq 0 \), as claimed.

All that remains is to notice that

\[
\mathcal{X}_{0,2k} b_2 - \mathcal{X}_{0,2} (\mathcal{X}_{0,2k} + \mathcal{X}_{1,2k} + \mathcal{X}_{2,2k}) = W_1 + W_2 \geq 0
\]

and hence

\[
b_2 \geq \mathcal{X}_{0,2k} + \mathcal{X}_{1,2k} + \mathcal{X}_{2,2k} \geq \mathcal{X}_{0,2k} b_2 \geq 0.
\]

**Example 5.5.2.** Let \( \sigma := (4, i, -i, i, -i) \) and \( \Delta := (2, 2, 0, 0, 0) \). We have

\[
s_1(\Delta) = s_1(\sigma) = 4 \quad \text{and} \quad s_2(\Delta) = 8 < 12 = s_2(\sigma).
\]

Therefore, by Theorem 5.5.1, \( \sigma \) is the spectrum of a nonnegative matrix of the form (71),
with the diagonal elements appearing in descending order. Computing the matrix entries $b_2, b_3, b_4, b_5$ from (82) (or otherwise), we see that $\sigma$ is realised by the matrix

$$A = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
50 & 55 & 16 & 2 & 0
\end{bmatrix}.$$  

On the other hand, if we wish the diagonal elements to appear in ascending order, we are forced to choose

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
4 & -1 & 8 & 2 & 2
\end{bmatrix},$$

which is not nonnegative. Hence the requirement that $a_1 \geq a_2 \geq \cdots \geq a_n$ in Theorem 5.5.1 cannot be omitted.

Note that if $a_1 = a_2 = \cdots = a_n = s_1(\sigma)/n$ in Theorem 5.5.1, then (94) becomes the JLL condition (68) and the matrix $A$ takes the form $A = C + \frac{s_1(\sigma)}{n}I_n$, where $C$ is a companion matrix with trace zero. Hence Theorem 5.5.1 may be seen as a generalisation of Theorem 1.3.5.

If $\sigma := (\rho, \lambda_2, \lambda_3, \ldots, \lambda_n)$, where $\rho \geq 0$ and $\text{Re}(\lambda_i) \leq 0$: $i = 2, 3, \ldots, n$, then we can use Theorem 5.5.1 to calculate the maximum possible diagonal element and the minimal possible diagonal element of a nonnegative matrix with spectrum $\sigma$:

**Corollary 5.5.3.** Let $\sigma := (\rho, \lambda_2, \lambda_3, \ldots, \lambda_n)$ be realisable, where $\rho \geq 0$ and $\text{Re}(\lambda_i) \leq 0$: $i = 2, 3, \ldots, n$. Then $\sigma$ is the spectrum of a nonnegative matrix with a diagonal element $a$ if and only if

$$0 \leq a \leq s_1(\sigma)$$  \hspace{1cm} (101)

and

$$\left(a - \frac{s_1(\sigma)}{n}\right)^2 \leq \frac{(n-1)(ns_2(\sigma) - s_1(\sigma)^2)}{n^2}. \hspace{1cm} (102)$$

**Proof.** Let us write

$$\delta := \sqrt{(n-1)(ns_2(\sigma) - s_1(\sigma)^2)}.$$  

We note that since $\sigma$ is realisable, it must satisfy (68) and hence $\delta$ is real.
The necessity of (101) is obvious. To see that (102) is necessary, suppose \( \sigma \) is the spectrum of a nonnegative matrix with diagonal elements \( (a_1, a_2, \ldots, a_n) \) and define
\[
t_i := a_i - \frac{s_1(\sigma)}{n} : \quad i = 1, 2, \ldots, n.
\]
(103)

In order to show that (102) is necessary, we need to show that
\[
t_i^2 \leq \frac{\delta^2}{n^2} : \quad i = 1, 2, \ldots, n.
\]

In fact, since the \( a_i \) are labelled arbitrarily, it suffices to show that
\[
t_1^2 \leq \frac{\delta^2}{n^2}.
\]

Since \( \sum_{i=1}^n a_i = s_1(\sigma) \), it follows that \( \sum_{i=1}^n t_i = 0 \). Hence, by the Cauchy-Schwarz inequality,
\[
\sum_{i=1}^n t_i^2 = \left( \sum_{i=2}^n t_i \right)^2 + \sum_{i=2}^n t_i^2 \geq \frac{n}{n-1} \left( \sum_{i=2}^n t_i \right)^2 = \frac{n t_1^2}{n-1}.
\]

Combining (104) with (89) then gives
\[
\frac{n t_1^2}{n-1} \leq s_2(\sigma) - \frac{s_1(\sigma)^2}{n},
\]
or, equivalently, \( t_1^2 \leq \frac{\delta^2}{n^2} \), as required.

Now suppose (101) and (102) hold and consider the list \( \Delta = (a_1, a_2, \ldots, a_n) \), where
\[
a_1 = a,
\[
a_i = \frac{s_1(\sigma) - a}{n-1} : \quad i = 2, 3, \ldots, n.
\]

Define \( t_i \) as before.

By (101), \( a_i \geq 0 \) for all \( i = 1, 2, \ldots, n \) and it is clear that \( s_1(\Delta) = s_1(\sigma) \). Furthermore, we note that, since \( a_2 = a_3 = \cdots = a_n \), we must have equality in (104). Combine this with the fact that (102) implies \( t_1^2 \leq \frac{\delta^2}{n^2} \) and we see that
\[
\sum_{i=1}^n \left( a_i - \frac{s_1(\sigma)}{n} \right)^2 = \sum_{i=1}^n t_i^2 = \frac{n t_1^2}{n-1} \leq \frac{\delta^2}{n(n-1)} = s_2(\sigma) - \frac{s_1(\sigma)^2}{n},
\]
which, as noted in Section 5.2, is equivalent to \( s_2(\Delta) \leq s_2(\sigma) \). Therefore, by Theorem 5.5.1, \( \sigma \) is the spectrum of a nonnegative matrix with diagonal elements \( \Delta \).

**Remark.** If there is equality in the JLL condition (68), then it follows from Corollary 5.5.3 that \( \sigma \) can only be realised by a matrix with constant diagonal.
Part III

THE SYMMETRIC NONNEGATIVE INVERSE EIGENVALUE PROBLEM
6.1 INTRODUCTION

Let \( \sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n) \) be a list of \( n \) real numbers. Recall that, if there exists a nonnegative symmetric matrix \( A \) with spectrum \( \sigma \), then we say \( \sigma \) is \emph{symmetrically realisable} and that \( A \) realises \( \sigma \). The problem of characterising all symmetrically realisable lists is referred to as the “Symmetric Nonnegative Inverse Eigenvalue Problem”, or \textbf{SNIEP}.

Since the spectrum of a symmetric matrix is necessarily real, the restriction that \( \sigma \) consist only of real numbers is a natural one; however, if we allow \( A \) to be not-necessarily-symmetric, but consider only lists of real numbers, then the resulting problem is known as the “Real Nonnegative Inverse Eigenvalue Problem”, or \textbf{RNIEP}.

A large body of work on the \textbf{SNIEP}/\textbf{RNIEP} can be found in the literature, giving several different sufficient conditions. We unify this work by introducing a family of symmetrically realisable lists that contain essentially all previously known sufficient conditions. The recursive method we employ allows us to uncover characterising properties of these lists, which not only give us a solid understanding of this realisable family, but can also be checked algorithmically.

One of the most general methods for constructing symmetrically realisable lists is due to Soules [58] (later generalised by Elsner, Nabben and Neumann [13]). In this chapter, we describe an alternative recursive method of constructing symmetrically realisable lists, using a construction of Šmigoc [65]. We show that this recursive method is equivalent to the method of Soules in that the symmetrically realisable lists obtained are identical and the realising matrices have the same form. We also consider a sufficient condition for the \textbf{RNIEP} due to Borobia, Moro and Soto [4] called “C-realisability” and a family of sufficient conditions for the \textbf{SNIEP} due to Soto [57]. We show that C-realisability is also sufficient for the \textbf{SNIEP} and that \( \sigma \) is C-realisable if and only if it satisfies one of Soto’s conditions (C-realisability was not previously known to apply to the \textbf{SNIEP}). Furthermore, such \( \sigma \) are precisely those which may be obtained by our method or by the method of Soules. The equivalence of all four methods is proved in Section 6.4.

In Section 6.2, we outline the background to and terminology used in this chapter. In Section 6.3, we describe our recursive approach and prove several properties of the realisable lists which may be obtained in this manner. In Section 6.5, we mention several sufficient condi-
tions for the SNIEP given in the literature, including Suleimanova [60], Perfect [48], Ciarlet [11], Kellog [29], Salzmann [52], Fiedler [16], Borobia [3], Soto [55] and Holtz [25]. We show that if $\sigma$ obeys any of these sufficient conditions, then $\sigma$ may also be obtained by our method.

6.2 Preliminaries and Notation

To denote that $\sigma$ is symmetrically realisable, we may sometimes write $\sigma \in \mathcal{R}_n$. In this chapter and in Chapter 7, the diagonal elements of the realising matrix will also be important; hence, if there exists a non-negative symmetric matrix $A$ with diagonal elements $(a_1, a_2, \ldots, a_n)$ and spectrum $\sigma$, then we write

$$\sigma \in \mathcal{R}_n(a_1, a_2, \ldots, a_n).$$

If we wish to specify that $\lambda_1$ is the Perron eigenvalue of the realising matrix, we will separate $\lambda_1$ from the remaining entries in the list by a semicolon, e.g. we may write

$$(\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{R}_n$$

or

$$(\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{R}_n(a_1, a_2, \ldots, a_n).$$

The remaining eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_n$ will generally be considered unordered. The diagonal elements $a_1, a_2, \ldots, a_n$ will also generally be considered unordered and they may appear in any order on the diagonal of $A$, i.e. we do not assume that $a_i$ is the $(i, i)$ entry of $A$. Sometimes we will assume that the $\lambda_i$ or $a_i$ are arranged in non-increasing order and if this is the case, we will say so explicitly. In this chapter, $\mathcal{R}$ will always be replaced by either $\mathcal{S}$ or $\mathcal{H}$, depending on whether we are considering realisability via Soules or by our recursive method. We will make use of the $\mathcal{R}$ notation in Chapter 7 when considering realisability by more general means.

We begin by stating some necessary conditions due to Fiedler [16] for $\sigma$ to be the spectrum of a nonnegative symmetric matrix with specified diagonal elements.

**Theorem 6.2.1.** [16] If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the spectrum of a nonnegative symmetric matrix with diagonal elements $(a_1, a_2, \ldots, a_n)$, then

$$\lambda_1 \geq a_1,$$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_i$$

and

$$\sum_{i=1}^{s} \lambda_i + \lambda_k \geq \sum_{i=1}^{s-1} a_i + a_{k-1} + a_k$$

for all $1 \leq s < k \leq n$ (with the convention that $\sum_{i=1}^{0} a_i = 0$).
Fiedler also gave the following sufficient conditions.

**THEOREM 6.2.2.** [16] Let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) and \( a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \) satisfy the following conditions:

\[
\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} a_i : \ k = 1, 2, \ldots, n - 1,
\]

\[
\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_i,
\]

\[
\lambda_k \leq a_{k-1} : \ k = 2, 3, \ldots, n - 1.
\]  

(105)

Then \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \) is the spectrum of a nonnegative symmetric matrix with diagonal elements \( (a_1, a_2, \ldots, a_n) \).

For \( n \leq 3 \), the question of whether \( \sigma \in \mathbb{R}_n(a_1, a_2, \ldots, a_n) \) is completely solved by Theorems 6.2.1 and 6.2.2. If \( n = 2 \), the matrix

\[
\begin{bmatrix}
a_1 & \sqrt{(\lambda_1 - a_1)(\lambda_1 - a_2)} \\
\sqrt{(\lambda_1 - a_1)(\lambda_1 - a_2)} & a_2
\end{bmatrix}
\]

has spectrum \( (\lambda_1, a_1 + a_2 - \lambda_1) \) and hence if \( \lambda_1 \geq \lambda_2 \) and \( a_1 \geq a_2 \geq 0 \), then \( (\lambda_1, \lambda_2) \) is the spectrum of a nonnegative symmetric matrix with diagonal elements \( (a_1, a_2) \) if and only if the following conditions are satisfied:

\[
\lambda_1 \geq a_1,
\]

\[
\lambda_1 + \lambda_2 = a_1 + a_2.
\]  

(106)

If \( n = 3 \), then the conditions of Theorems 6.2.1 and 6.2.2 are identical and hence if \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \) and \( a_1 \geq a_2 \geq a_3 \geq 0 \), then \( (\lambda_1, \lambda_2, \lambda_3) \) is the spectrum of a nonnegative symmetric matrix with diagonal elements \( (a_1, a_2, a_3) \) if and only if the following conditions hold:

\[
\lambda_2 \leq a_1 \leq \lambda_1,
\]

\[
\lambda_3 \leq a_3,
\]

\[
\lambda_1 + \lambda_2 + \lambda_3 = a_1 + a_2 + a_3.
\]  

(107)

Note that if we do not require the realising matrix to be symmetric, then the conditions on \( (a_1, a_2, a_3) \) are less restrictive (see Proposition 5.4.1 and Example 5.4.2).

### 6.2.1 The Soules approach to the SNIEP

Soules’ approach to the **SNIEP** focuses on constructing the eigenvectors of the realising matrix \( A \). Starting from a positive vector \( x \in \mathbb{R}^n \), Soules [58] showed how to construct a real orthogonal \( n \times n \) matrix \( R \) with first column \( x \) such that for all \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \), the matrix \( R \Lambda R^T \)—where \( \Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \)—is nonnegative. This motivated Elsner, Nabben and Neumann [13] to make the following definition.
Definition 6.2.3. Let $R$ be an $n \times n$ real orthogonal matrix with columns $r_1, r_2, \ldots, r_n$. $R$ is called a Soules matrix if $r_1$ is positive and for every diagonal matrix $\Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, the matrix $R\Lambda R^T$ is nonnegative.

With regard to the SNIEP, a key property of Soules matrices is the following:

Theorem 6.2.4. [13] Let $R$ be a Soules matrix and let $\Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the off-diagonal entries of the matrix $R\Lambda R^T$ are nonnegative.

Therefore, if $R = (r_{ij})$ is an $n \times n$ Soules matrix and $\Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then $\sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the spectrum of a nonnegative symmetric matrix if the diagonal elements of $R\Lambda R^T$ are nonnegative. This motivates the following definition.

Definition 6.2.5. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and let $a_1, a_2, \ldots, a_n \geq 0$. We write

$$ (\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{S}_n(a_1, a_2, \ldots, a_n) \quad (108) $$

if there exists an $n \times n$ Soules matrix $R$ such that the matrix $R\Lambda R^T$—where $\Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$—has diagonal elements $(a_1, a_2, \ldots, a_n)$. We write

$$ (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{S}_n $$

if there exist $a_1, a_2, \ldots, a_n \geq 0$ such that (108) holds and we call $\mathcal{S}_n$ the Soules set.

Elsner, Nabben and Neumann generalised the work of Soules by characterising all Soules matrices. In order to state their characterisation, we require two definitions.

Definition 6.2.6. Let $N = (N_1, N_2, \ldots, N_n)$ be a sequence of partitions of $\{1, 2, \ldots, n\}$. We say that $N$ is Soules-type if $N$ has the following properties:

(i) for each $i \in \{1, 2, \ldots, n\}$, the partition $N_i$ consists of precisely $i$ subsets, say $N_i = \{N_{i,1}, N_{i,2}, \ldots, N_{i,i}\}$;

(ii) for each $i \in \{2, 3, \ldots, n\}$, there exist indices $j, k, l$ with $1 \leq j \leq i - 1$ and $1 \leq k < l \leq i$, such that $N_{i-1} \setminus N_{i-1,j} = N_i \setminus \{N_{i,k}, N_{i,l}\}$ and $N_{i-1,j} = N_{i,k} \cup N_{i,l}$, i.e. $N_i$ is constructed from $N_{i-1}$ by splitting one of the sets $N_{i-1,1}, N_{i-1,2}, \ldots, N_{i-1,i-1}$ into two subsets.

If $N = (N_1, N_2, \ldots, N_n)$ is a Soules-type sequence of partitions of $\{1, 2, \ldots, n\}$, then we label the sets $N_{i,k}$ and $N_{i,l}$ in (ii) as $N_{i}^k$ and $N_{i}^{l}$, i.e. for $i \in \{2, 3, \ldots, n\}$, we define $N_{i}^k$ and $N_{i}^{l}$ to be those sets in $N_i$ which do not coincide with any of the sets in $N_{i-1}$. 
**Definition 6.2.7.** Let \( x \in \mathbb{R}^n \) be a positive vector and let \( N = (N_1, N_2, \ldots, N_n) \) be a Soules-type sequence of partitions of \( \{1, 2, \ldots, n\} \). For each \( i \in \{2, 3, \ldots, n\} \), we define \( x_N^{(i)} \) to be the vector in \( \mathbb{R}^n \) whose \( i \)-th component is:

\[
\begin{align*}
&x_{N_i^{(i)}} : i \in N_i^{*} \\
&0 : i \not\in N_i^{*}
\end{align*}
\]

and we define \( x_N^{(i)} \) to be the vector in \( \mathbb{R}^n \) whose \( i \)-th component is:

\[
\begin{align*}
x_{N_i^{* *}} : i \in N_i^{* *}, \\
0 : i \not\in N_i^{* *}.
\end{align*}
\]

We are now ready to state the characterisation of Soules matrices due to Elsner, Nabben and Neumann:

**Theorem 6.2.8.** [13] Let \( x \in \mathbb{R}^n \) be a positive vector and let \( R \) be a Soules matrix with columns \( r_1, r_2, \ldots, r_n \), where \( r_1 = x \). Then there exists a Soules-type sequence \( N \) of partitions of \( \{1, 2, \ldots, n\} \) such that \( r_1 \) is given (up to a factor of \( \pm 1 \)) by

\[
r_i = \frac{1}{\sqrt{||x_N^{(i)}||^2 + ||\hat{x}_N^{(i)}||^2}} \left( \frac{||\hat{x}_N^{(i)}||_2}{||x_N^{(i)}||_2} x_N^{(i)} - \frac{||\hat{x}_N^{(i)}||_2}{||x_N^{(i)}||_2} \hat{x}_N^{(i)} \right), \tag{109}
\]

\( i = 2, 3, \ldots, n \).

Conversely, if \( x \in \mathbb{R}^n \) is a positive vector with \( ||x||_2 = 1 \) and \( N \) is a Soules-type sequence of partitions of \( \{1, 2, \ldots, n\} \), then the matrix \( R = [r_1 \ r_2 \ \cdots \ r_n] \) — with \( r_1 = x \) and \( r_2, r_3, \ldots, r_n \) given by (109) — is a Soules matrix.

**Remark.** Note that, by (109), the \( j \)-th entry of \( r_1 \) is nonzero if and only if \( j \in N_i^{*} \cup N_i^{* *} \).

**Example 6.2.9.** Let us show that \( (7, 5, -2, -4, -6) \in S_5(0, 0, 0, 0, 0) \). To see this, consider the vector

\[
x = \begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} \end{bmatrix}^T
\]

and the partition sequence \( N = (N_1, N_2, N_3, N_4, N_5) \), illustrated in Figure 5, where

\[
N_1 = \{\{1, 2, 3, 4, 5\}\}, \\
N_2 = \{\{1, 2\}, \{3, 4, 5\}\}, \\
N_3 = \{\{1, 2\}, \{3\}, \{4, 5\}\}, \\
N_4 = \{\{1, 2\}, \{3\}, \{4\}, \{5\}\}, \\
N_5 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{9\}\}.
\]

Using (109), we construct the Soules matrix.
Figure 5: Partition sequence $\mathcal{N}$

$$R = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} \\
\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & 0 & 0 \\
\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0
\end{bmatrix}$$

and the realising matrix

$$A = RR^T = \begin{bmatrix}
0 & 6 & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \\
6 & 0 & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \\
\frac{1}{2\sqrt{2}} & 0 & \sqrt{6} & \sqrt{6} & 0 \\
\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \sqrt{6} & 0 & 4 \\
\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \sqrt{6} & 4 & 0
\end{bmatrix}, \quad (110)$$

where $\Lambda := \text{diag}(7, 5, -2, -4, -6)$.

Observe that the restriction that Soules matrices must have positive first column excludes certain boundary cases from the Soules set. In order to complete the equivalence we prove in Section 6.4, we would like to include these spectra. Hence we define a property which we call piecewise Soules:

**Definition 6.2.10.** Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and let $a_1, a_2, \ldots, a_n \geq 0$. We write

$$(\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{S}_n(a_1, a_2, \ldots, a_n) \quad (111)$$

if there exist two partitions

$$\{1, \ldots, n\} = (\alpha_1^{(1)}, \ldots, \alpha_{n_1}^{(1)}) \cup (\alpha_1^{(2)}, \ldots, \alpha_{n_2}^{(2)}) \cup \cdots \cup (\alpha_1^{(k)}, \ldots, \alpha_{n_k}^{(k)}),$$

$$\{1, \ldots, n\} = (\beta_1^{(1)}, \ldots, \beta_{n_1}^{(1)}) \cup (\beta_1^{(2)}, \ldots, \beta_{n_2}^{(2)}) \cup \cdots \cup (\beta_1^{(k)}, \ldots, \beta_{n_k}^{(k)})$$

such that

$$(\lambda_{\alpha_1^{(i)}}, \lambda_{\alpha_2^{(i)}}, \ldots, \lambda_{\alpha_{n_1}^{(i)}}) \in \mathcal{S}_{n_1} (a_{\beta_1^{(i)}}, a_{\beta_2^{(i)}}, \ldots, a_{\beta_{n_1}^{(i)}}): \quad i = 1, 2, \ldots, k.$$
We write
\[(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{S}_n\]
if there exist \(a_1, a_2, \ldots, a_n \geq 0\) such that (111) holds.

Remark. \(\overline{\mathcal{S}}_n\) is the topological closure of \(\mathcal{S}_n\).

The Soules set and its role in the SNIEP have been extensively studied, for example by McDonald and Neumann [41] and Loewy and McDonald [38]. Soules matrices and the associated orthonormal bases have also been considered elsewhere in the literature, for example in [8–10, 14, 46]. In addition, Soules matrices have been applied to other areas of linear algebra, including nonnegative matrix factorisation [7], the cp-rank problem [53] and describing the relationships between various classes of matrices [13, 46].

6.2.2 A constructive lemma

In [65, Lemma 5], given a nonnegative matrix \(B\) with Perron eigenvalue \(c\) and spectrum \((c, \nu_2, \nu_3, \ldots, \nu_l)\) and a nonnegative matrix \(A\) with spectrum \((\mu_1, \mu_2, \ldots, \mu_k)\) and a diagonal element \(c\), Šmigoc shows how to construct a nonnegative matrix \(C\) with spectrum \((\mu_1, \mu_2, \ldots, \mu_k, \nu_2, \nu_3, \ldots, \nu_l)\). This construction is the foundation of the proof of Theorem 4.2.4. Furthermore, if \(A\) and \(B\) are symmetric, then \(C\) will be symmetric also. We state this special case below.

**Lemma 6.2.11.** [65] Let \(B\) be an \(l \times l\) nonnegative symmetric matrix with Perron eigenvalue \(c\) and spectrum \((c, \nu_2, \nu_3, \ldots, \nu_l)\) and let \(Y \in \mathbb{R}^{l \times l}\) be an orthogonal matrix such that

\[Y^T B Y = \text{diag}(c, \nu_2, \nu_3, \ldots, \nu_l)\]

Let \(Y\) be partitioned as

\[Y = \begin{bmatrix} v & V \end{bmatrix}\]

where \(v \in \mathbb{R}^l\) and \(V \in \mathbb{R}^{l \times (l-1)}\).

Let

\[A := \begin{bmatrix} A_1 & a \\ a^T & c \end{bmatrix},\]

where \(A_1\) is an \((k-1) \times (k-1)\) nonnegative symmetric matrix and \(a \in \mathbb{R}^{k-1}\) is nonnegative and let \(X \in \mathbb{R}^{k \times k}\) be an orthogonal matrix such that

\[X^T A X = \text{diag}(\mu_1, \mu_2, \ldots, \mu_k)\]

Let \(X\) be partitioned as

\[X = \begin{bmatrix} U & u^T \end{bmatrix},\]

where \(u \in \mathbb{R}^k\) and \(U \in \mathbb{R}^{(k-1) \times k}\).
Then for matrices
\[ C := \begin{bmatrix} A_1 & av^T \\ va^T & B \end{bmatrix} \]
and
\[ Z := \begin{bmatrix} U & 0 \\ vu^T & V \end{bmatrix}, \]
we have
\[ Z^TCZ = \text{diag}(\mu_1, \mu_2, \ldots, \mu_k, \nu_2, \nu_3, \ldots, \nu_l). \]

6.2.3 C-realisability and the RNIEP

In [4], Borobia, Moro and Soto construct realisable lists in the RNIEP, starting from trivially realisable lists, using three well known results.

Specifically, recall Theorem 4.2.2 which states that we may perturb a real eigenvalue of a realisable list by ±ε, provided we also increase the Perron eigenvalue by ε. For convenience, we restate the theorem here:

**Theorem 6.2.12.** [21] If \((\rho, \lambda_2, \lambda_3, \ldots, \lambda_n)\) is realisable, where \(\rho\) is the Perron eigenvalue and \(\lambda_2\) is real, then
\[(\rho + \epsilon, \lambda_2 \pm \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_n)\]
is realisable for all \(\epsilon \geq 0\).

Recall also Theorem 1.3.8, which, in particular, says:

**Theorem 6.2.13.** [21] If \((\rho, \lambda_2, \lambda_3, \ldots, \lambda_n)\) is the spectrum of a nonnegative matrix with Perron eigenvalue \(\rho\), then for all \(\epsilon \geq 0\), \((\rho + \epsilon, \lambda_2, \lambda_3, \ldots, \lambda_n)\) is the spectrum of a nonnegative matrix also.

Finally, recall that the spectrum of a block diagonal matrix is the union of the spectra of the diagonal blocks, in other words:

**Observation 6.2.14.** If \((\lambda_1, \lambda_2, \ldots, \lambda_m)\) and \((\mu_1, \mu_2, \ldots, \mu_n)\) are realisable, then \((\lambda_1, \lambda_2, \ldots, \lambda_m, \mu_1, \mu_2, \ldots, \mu_n)\) is realisable.

Borobia, Moro and Soto make the following definition.

**Definition 6.2.15.** A list of real numbers \((\lambda_1, \lambda_2, \ldots, \lambda_n)\) is called C-realisable if it may be obtained by starting with the \(n\) trivially realisable lists \((0), (0), \ldots, (0)\) and then using results 6.2.12, 6.2.13 and 6.2.14 any number of times in any order.

**Example 6.2.16.** In Example 6.2.9, we showed that \((7, 5, -2, -4, -6) \in \mathcal{S}_5\). To see that \((7, 5, -2, -4, -6)\) is C-realisable, consider the following series of steps:

1. \((0), (0), (0), (0), (0)\)
2. (0, 0), (0, 0), (0)
3. (6, -6), (4, -4), (0)
4. (6, -6), (4, 0, -4)
5. (6, -6), (6, -2, -4)
6. (6, 6, -2, -4, -6)
7. (7, 5, -2, -4, -6)

We used Observation 6.2.14 at steps 1 → 2, 3 → 4 and 5 → 6. We used Theorem 6.2.12 at steps 2 → 3, 4 → 5 and 6 → 7.

Of course, if \( \sigma \) is C-realisable, then \( \sigma \) is realisable. Note that while the symmetric analogues of Theorem 6.2.13 and Observation 6.2.14 hold, it is an open question whether the symmetric version of Theorem 6.2.12 is true. We prove in Section 6.4 that if \( \sigma \) is C-realisable, then \( \sigma \) is symmetrically realisable.

### 6.2.4 A family of realizability criteria in the SNIEP

Based on a theorem of Brauer, Soto [57] gives a family of realizability criteria denoted \( S_1, S_2, \ldots \) (not to be confused with \( S_n \)), such that, if a list of real numbers \( \sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n) \) satisfies the criterion \( S_p \) for some \( p = 1, 2, \ldots \), then \( \sigma \) is realisable. Soto also shows that the \( S_p \) criteria are sufficient for symmetric realizability. In order to state \( S_p \), we will require some terminology and notation from [57].

Let \( \sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n) \), where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \), and let \( K \) be a realizability criterion. Then we write

\[
\sigma \in \mathcal{Q}_K
\]

if \( \sigma \) satisfies the criterion \( K \). The **Brauer K-negativity** of \( \sigma \) is defined to be the nonnegative number

\[
\mathcal{N}_K(\sigma) := \min\{\epsilon \geq 0 : (\lambda_1 + \epsilon, \lambda_2, \lambda_3, \ldots, \lambda_n) \in \mathcal{Q}_K\} \tag{112}
\]

and if \( \sigma \in \mathcal{Q}_K \), then the **Brauer K-realisability margin** of \( \sigma \) is defined to be

\[
\mathcal{M}_K(\sigma) := \max\{\epsilon \in [0, \lambda_1 - \lambda_2] : (\lambda_1 - \epsilon, \lambda_2, \lambda_3, \ldots, \lambda_n) \in \mathcal{Q}_K\} \tag{113}
\]

The \( S_p \) criteria are now defined recursively: We say \( \sigma \) satisfies the \( S_1 \) criterion if

\[
\lambda_1 \geq -\lambda_n - \sum_{T_i < 0} T_i,
\]

where

\[
T_i := \lambda_i + \lambda_{n-i+1} : i = 2, 3, \ldots, \left\lfloor \frac{n}{2} \right\rfloor
\]
and for odd \( n \geq 3 \), \( T_{n+1} := \min \{ \lambda_{n+1}, 0 \} \).

For \( p = 2, 3, \ldots \), we say that \( \sigma \) satisfies the \( S_p \) criterion if there exists a partition of \( \sigma \) into sublists \( \sigma_1, \sigma_2, \ldots, \sigma_r \), where

\[
\begin{align*}
\sigma_i &= \left( \lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_{n_i}^{(i)} \right) : i = 1, 2, \ldots, r, \\
\lambda_1^{(i)} &= \lambda_1, \\
\lambda_1^{(i)} &\geq 0 : i = 1, 2, \ldots, r, \\
\lambda_1^{(i)} &\geq \lambda_2^{(i)} \geq \ldots \geq \lambda_{n_i}^{(i)} : i = 1, 2, \ldots, r,
\end{align*}
\]

such that \( \sigma_1 \in \mathcal{D}_{p-1} \) and

\[
\lambda_1 \geq \gamma + \sum_{\sigma_i \not\in \mathcal{D}_{p-1}} \mathcal{M}_{p-1}(\sigma_1),
\]

where

\[
\gamma := \max\{\lambda_1 - \mathcal{M}_{p-1}(\sigma_1), \lambda_1, \lambda_1^{(2)}, \lambda_1^{(3)}, \ldots, \lambda_1^{(r)}\}.
\]

Note that if we allow \( r = 1 \) above, then we have:

**Observation 6.2.17.** If \( \sigma \) satisfies \( S_p \), then \( \sigma \) satisfies \( S_{p+1} \).

**Theorem 6.2.18.** [57] If \( \sigma \) satisfies \( S_p \) for any \( p \), then \( \sigma \) is symmetrically realisable.

**Example 6.2.19.** In Example 6.2.9, we showed that \( \sigma := (7, 5, -2, -4, -6) \in \mathcal{S}_5 \) and in Example 6.2.16, we showed that \( \sigma \) is C-realisable. It is easy to check that \( \sigma \) does not satisfy \( S_1 \); however, for the partition \( \sigma = (\sigma_1, \sigma_2) \), where \( \sigma_1 = (7, -6) \) and \( \sigma_2 = (5, -2, -4) \), we have \( \mathcal{M}_{S_1}(\sigma_1) = \mathcal{M}_{S_1}(\sigma_2) = 1 \) and hence \( \sigma \) satisfies \( S_2 \).

### 6.3 A Recursive Approach to the Sniep

Here, we describe a method of recursively constructing symmetrically realisable lists, starting with lists of length 2 and repeatedly applying Lemma 6.2.11. Formally, we define the set \( \mathcal{H}_n \) in the following way:

**Definition 6.3.1.** For \( a \geq 0 \), we write \( (\lambda) \in \mathcal{H}_1(a) \) if \( \lambda = a \). For \( a_1, a_2 \geq 0 \), we write \( (\lambda_1; \lambda_2) \in \mathcal{H}_2(a_1, a_2) \) if \( \lambda_1 \geq \max\{a_1, a_2\} \) and \( \lambda_1 + \lambda_2 = a_1 + a_2 \). For \( a_1, a_2, \ldots, a_m \geq 0 \), we write

\[
(\lambda_1; \lambda_2, \ldots, \lambda_m) \in \mathcal{H}_m(a_1, a_2, \ldots, a_m)
\]

if there exist two partitions

\[
\{2, 3, \ldots, m\} = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \cup \{\beta_1, \beta_2, \ldots, \beta_{m-k-1}\}
\]

\[
\{1, 2, \ldots, m\} = \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \cup \{\delta_1, \delta_2, \ldots, \delta_{m-k}\}
\]

and a nonnegative number \( c \) such that

\[
(\lambda_1; \lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_k}) \in \mathcal{H}_{k+1}(a_{\gamma_1}, a_{\gamma_2}, \ldots, a_{\gamma_k}, c)
\]
and
\[(c; \lambda_β_1, \lambda_β_2, \ldots, \lambda_β_{m-k-1}) \in \mathcal{H}_{m-k}(a_δ_1, a_δ_2, \ldots, a_δ_{m-k}).\]

We write
\[ (λ_1, λ_2, \ldots, λ_n) \in \mathcal{H}_n \]
if there exist \(a_1, a_2, \ldots, a_n \geq 0\) such that (117) holds.

**Theorem 6.3.2.** If \((λ_1; λ_2, \ldots, λ_n) \in \mathcal{H}_n(a_1, a_2, \ldots, a_n)\), then \((λ_1, λ_2, \ldots, λ_n)\) is the spectrum of a nonnegative symmetric matrix with Perron eigenvalue \(λ_1\) and diagonal elements \((a_1, a_2, \ldots, a_n)\).

**Proof.** We proceed by induction on \(n\). If \(n = 1\), there is nothing to prove. If \(n = 2\), then the necessary and sufficient conditions given in (106) coincide with the definition of \(\mathcal{H}_2(a_1, a_2)\) and so \((λ_1, λ_2)\) is the spectrum of a nonnegative symmetric matrix with diagonal elements \((a_1, a_2)\) if and only if \((λ_1; λ_2) \in \mathcal{H}_2(a_1, a_2)\). Note also that the definition of \(\mathcal{H}_2(a_1, a_2)\) implies \(λ_1 \geq λ_2\) and hence \(λ_1\) is the Perron eigenvalue of the realising matrix.

Now assume the statement holds for all \(n \in \{1, 2, \ldots, m-1\}\) and consider the case when \(n = m\). Let \((λ_1; λ_2, \ldots, λ_m) \in \mathcal{H}_m(a_1, a_2, \ldots, a_m)\). Then there exist a partition of \([2, 3, \ldots, m]\) into two subsets \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) and \((β_1, β_2, \ldots, β_{m-k-1})\), a partition of \([1, 2, \ldots, m]\) into two subsets \((γ_1, γ_2, \ldots, γ_k)\) and \((δ_1, δ_2, \ldots, δ_{m-k})\), and a nonnegative number \(c\) such that
\[ (λ_1; λ_α_1, λ_α_2, \ldots, λ_α_k) \in \mathcal{H}_{k+1}(a_γ_1, a_γ_2, \ldots, a_γ_k, c) \]
and
\[ (c; λ_β_1, λ_β_2, \ldots, λ_β_{m-k-1}) \in \mathcal{H}_{m-k}(a_δ_1, a_δ_2, \ldots, a_δ_{m-k}). \]

By the inductive hypothesis, there exists a \((k+1) \times (k+1)\) nonnegative symmetric matrix \(A\) with Perron eigenvalue \(λ_1\), spectrum \((λ_1, λ_α_1, λ_α_2, \ldots, λ_α_k)\) and diagonal elements \((a_γ_1, a_γ_2, \ldots, a_γ_k, c)\). Without loss of generality, we may assume that \(c\) is the \((k+1, k+1)\) entry of \(A\), as otherwise we may replace \(A\) with \(PAP^T\), where \(P\) is a suitable permutation matrix. Similarly, the inductive hypothesis guarantees the existence of a \((m-k) \times (m-k)\) nonnegative symmetric matrix \(B\) with Perron eigenvalue \(c\), spectrum \((c, λ_β_1, λ_β_2, \ldots, λ_β_{m-k-1})\) and diagonal elements \((a_δ_1, a_δ_2, \ldots, a_δ_{m-k})\). Hence, by Lemma 6.2.11, there exists an \(m \times m\) nonnegative symmetric matrix \(C\) with spectrum \((λ_1, λ_2, \ldots, λ_m)\) and diagonal elements \((a_1, a_2, \ldots, a_m)\).

Finally, since \(λ_1\) is the Perron eigenvalue of \(A\), we have \(λ_1 \geq λ_α_i; i = 1, 2, \ldots, k\) and \(λ_1 \geq c \geq λ_β_j; j = 1, 2, \ldots, m-k-1\). Hence \(λ_1\) is the Perron eigenvalue of \(C\). \(\square\)

**Example 6.3.3.** In Example 6.2.9, we showed that \(σ := (7, 5, -2, -4, -6) \in S_5\), in Example 6.2.16, we showed that \(σ\) is C-realisable and
in Example 6.2.19, we showed that \( \sigma \) satisfies \( S_2 \). Let us now show that \( \sigma \in \mathcal{K}_5 \). To do this, we need only show how to progressively decompose \( \sigma \) according to Definition 6.3.1. One such decomposition is given in Figure 6.

\[
\begin{align*}
(7; 5, -2, -4, -6) &\in \mathcal{K}_5(0, 0, 0, 0, 0) \\
(7; 5, -6) &\in \mathcal{K}_3(0, 0, 6) \\
(6; -2, -4) &\in \mathcal{K}_3(0, 0, 0) \\
(7; 5) &\in \mathcal{K}_2(6, 6) \\
(6; -2) &\in \mathcal{K}_2(0, 4) \\
(6; -6) &\in \mathcal{K}_2(0, 0) \\
(4; -4) &\in \mathcal{K}_2(0, 0)
\end{align*}
\]

Figure 6: Decomposition of \( \sigma \) into lists of length 2

We noted in the proof that the converse of Theorem 6.3.2 holds for \( n = 2 \). In the following lemma, we show that the converse holds for \( n = 3 \) also. The converse does not hold for \( n = 4 \). (See Example 7.1.1 and Chapter 7 in general.)

**Lemma 6.3.4.** Let \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \) and \( a_1 \geq a_2 \geq a_3 \geq 0 \). If \( (\lambda_1, \lambda_2, \lambda_3) \) is the spectrum of a nonnegative symmetric matrix with diagonal elements \( (a_1, a_2, a_3) \), then

\[
(\lambda_1; \lambda_2) \in \mathcal{K}_2(a_1, c) \quad \text{and} \quad (c; \lambda_3) \in \mathcal{K}_2(a_2, a_3),
\]

where \( c := \lambda_1 + \lambda_2 - a_1 \). In particular, \( (\lambda_1; \lambda_2, \lambda_3) \in \mathcal{K}_3(a_1, a_2, a_3) \).

**Proof.** The result follows easily from (107) and the definition of \( \mathcal{K}_2 \).

Suppose that for all \( n \in \{2, 3, \ldots, m - 1\} \) and \( a_1, a_2, \ldots, a_{m-1} \geq 0 \), the sets \( \mathcal{K}_n(a_1, a_2, \ldots, a_n) \) are known and we wish to determine whether \( (\lambda_1; \lambda_2, \ldots, \lambda_m) \in \mathcal{K}_m(b_1, b_2, \ldots, b_m) \). From Definition 6.3.1, we would need to consider all \( k \in \{1, 2, \ldots, m - 2\} \), all \( k \)-subsets of \( \{2, 3, \ldots, m\} \) and all \( k \)-subsets of \( \{1, 2, \ldots, m\} \), a total of

\[
\sum_{k=1}^{m-2} \binom{m-1}{k} \binom{m}{k} = \binom{2m-1}{m-1} - m - 1
\]

possibilities. Our next result reduces the number of possibilities to \( \binom{m}{2} \) and shows that \( \mathcal{K}_m \) depends only on \( \mathcal{K}_{m-1} \) and \( \mathcal{K}_2 \).
Theorem 6.3.5. Let \( n \geq 3 \), let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) and let \( a_1, a_2, \ldots, a_n \geq 0 \). Then \( (\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n(a_1, a_2, \ldots, a_n) \) if and only if there exist \( s, t \in \{1, 2, \ldots, n\}, s < t \) such that

\[
(\lambda_1; \lambda_2, \ldots, \lambda_{n-1}) \in \mathcal{H}_{n-1}(a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{t-1}, a_{t+1}, \ldots, a_n, c) \quad (118)
\]

and

\[
(c; \lambda_n) \in \mathcal{H}_2(a_s, a_t), \quad (119)
\]

where \( c := a_s + a_t - \lambda_n \).

Proof. That (118) and (119) imply \( (\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n(a_1, a_2, \ldots, a_n) \) follows from Definition 6.3.1. Conversely, assume \( (\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n(a_1, a_2, \ldots, a_n) \). We claim that there exist \( s \) and \( t \) such that (118) and (119) hold.

We prove our claim by induction on \( n \). If \( n = 3 \), then the claim follows from Lemma 6.3.4. Now assume the claim holds for all \( n \in \{3, 4, \ldots, m - 1\}, m \geq 4 \), and suppose \( (\lambda_1; \lambda_2, \ldots, \lambda_m) \in \mathcal{H}_m(a_1, a_2, \ldots, a_m) \). Then there exist a partition of \( \{2, 3, \ldots, m\} \) into two subsets \( \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \) and \( \{\beta_1, \beta_2, \ldots, \beta_{m-k-1}\} \), a partition of \( \{1, 2, \ldots, m\} \) into two subsets \( \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \) and \( \{\delta_1, \delta_2, \ldots, \delta_{m-k}\} \) and a nonnegative number \( \hat{c} \) such that

\[
(\lambda_1; \lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_k}) \in \mathcal{H}_{k+1}(a_{\gamma_1}, a_{\gamma_2}, \ldots, a_{\gamma_k}, \hat{c}) \quad (120)
\]

and

\[
(\hat{c}; \lambda_{\beta_1}, \lambda_{\beta_2}, \ldots, \lambda_{\beta_{m-k-1}}) \in \mathcal{H}_{m-k}(a_{\delta_1}, a_{\delta_2}, \ldots, a_{\delta_{m-k}}). \quad (121)
\]

We will show that this implies the existence of some \( s \) and \( t \) such that

\[
(\lambda_1; \lambda_2, \ldots, \lambda_{m-1}) \in \mathcal{H}_{m-1}(a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{t-1}, a_{t+1}, \ldots, a_m, c) \quad (122)
\]

and

\[
(c; \lambda_m) \in \mathcal{H}_2(a_s, a_t), \quad (123)
\]

where \( c := a_s + a_t - \lambda_m \).

Without loss of generality, assume that the \( \alpha_i, \beta_i, \gamma_i \) and \( \delta_i \) are labelled so that \( \alpha_1 < \alpha_2 < \cdots < \alpha_k, \beta_1 < \beta_2 < \cdots < \beta_{m-k-1} \), \( \gamma_1 < \gamma_2 < \cdots < \gamma_k \) and \( \delta_1 < \delta_2 < \cdots < \delta_{m-k} \). Since the \( \lambda_i \) are ordered also, we must have either \( \alpha_k = m \) or \( \beta_{m-k-1} = m \). If \( \alpha_k = m \), then we must distinguish the cases \( k = 1 \) and \( k > 1 \). If \( \beta_{m-k-1} = m \), then we must distinguish the cases \( k = m - 2 \) and \( k < m - 2 \). In summary, we need to consider four possible cases:

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( k = m - 2, \beta_1 = m )</td>
</tr>
<tr>
<td>2</td>
<td>( k &lt; m - 2, \beta_{m-k-1} = m )</td>
</tr>
<tr>
<td>3</td>
<td>( k &gt; 1, \alpha_k = m )</td>
</tr>
<tr>
<td>4</td>
<td>( k = 1, \alpha_1 = m )</td>
</tr>
</tbody>
</table>
Case 1: There is nothing to prove.

Case 2: Applying the inductive hypothesis to (121), there exist $s, t \in \{1, 2, \ldots, m - k\}$, $s < t$, such that

\[(\hat{c}; \lambda_1, \lambda_2, \ldots, \lambda_{m-k-2}) \in \mathcal{H}_{m-k-1}(a_{\delta_1}, \ldots, a_{\delta_{s-1}}, a_{\delta_{s+1}}, \ldots, a_{\delta_{t-1}}, a_{\delta_{t+1}}, \ldots, a_{\delta_{m-k}}, c) \quad (124)\]

and

\[(c; \lambda_m) \in \mathcal{H}_2(a_{\delta_t}, a_{\delta_t}), \quad (125)\]

where $c := a_{\delta_s} + a_{\delta_t} - \lambda_m$. Hence, if we let $s = \delta_s$ and $t = \delta_t$, then by Definition 6.3.1, (120) and (124) imply (122), and (125) becomes (123). Hence, in Case 2, we have completed the inductive step and established our claim. In the remainder of the proof, we will make frequent use of Definition 6.3.1.

Case 3: Applying the inductive hypothesis to (120), we see that one of the following two sub-cases must hold:

Case 3 (a): There exist $s, t \in \{1, 2, \ldots, k\}$, $s < t$, such that

\[(\lambda_1; \lambda_1, \lambda_2, \ldots, \lambda_{k-1}) \in \mathcal{H}_k(a_{\gamma_1}, \ldots, a_{\gamma_{s-1}}, a_{\gamma_{s+1}}, \ldots, a_{\gamma_{t-1}}, a_{\gamma_{t+1}}, \ldots, a_{\gamma_k}, c) \quad (126)\]

and

\[(c; \lambda_m) \in \mathcal{H}_2(a_{\gamma_t}, a_{\gamma_t}), \quad (127)\]

where $c := a_{\gamma_s} + a_{\gamma_t} - \lambda_m$. In this case, if we let $s = \gamma_s$ and $t = \gamma_t$, then (126) and (121) give (122), and (127) becomes (123), which establishes the claim in Case 3 (a).

Case 3 (b): There exists $h \in \{1, 2, \ldots, k\}$ such that

\[(\lambda_1; \lambda_1, \lambda_2, \ldots, \lambda_{k-1}) \in \mathcal{H}_k(a_{\gamma_1}, \ldots, a_{\gamma_{h-1}}, a_{\gamma_{h+1}}, \ldots, a_{\gamma_k}, c') \quad (128)\]

and

\[(c'; \lambda_m) \in \mathcal{H}_2(a_{\gamma_h}, c'), \quad (129)\]

where $c' := a_{\gamma_h} + c - \lambda_m$. In this case, (129) and (121) imply

\[(c'; \lambda_1, \lambda_{m-k+1}, \lambda_{m-k}) \in \mathcal{N}_{m-k+1}(a_{\delta_1}, a_{\delta_2}, \ldots, a_{\delta_{m-k}}, a_{\gamma_h}). \quad (130)\]

By examining equations (128) and (130), we see that we have reduced the problem to Case 2. More formally, let us label

\[
\hat{k} := k - 1,
\]
\[
\hat{\alpha}_i := \alpha_i \quad : \quad i = 1, 2, \ldots, k - 1,
\]

\[
\hat{\beta}_i := \begin{cases} 
\beta_i & : \quad i = 1, 2, \ldots, m - k - 1 \\
m & : \quad i = m - k 
\end{cases}
\]

\[
\hat{\gamma}_i := \begin{cases} 
\gamma_i & : \quad i = 1, 2, \ldots, s - 1 \\
\gamma_{i+1} & : \quad i = s, s + 1, \ldots, k - 1
\end{cases}
\]
\[\delta_i := \begin{cases} 
\delta_i : & i = 1, 2, \ldots, m - k \\
\gamma_h : & i = m - k + 1.
\end{cases}\]

Then the partitions
\[\{2, 3, \ldots, m\} = (\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_k) \cup (\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_{m-k-1}),\]
\[\{1, 2, \ldots, m\} = (\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_k) \cup (\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_{m-k})\]
are such that
\[(\lambda_1; \lambda_{\hat{\alpha}_1}, \lambda_{\hat{\alpha}_2}, \ldots, \lambda_{\hat{\alpha}_k}) \in \mathcal{H}_{k+1} \left( a_{\hat{\varphi}_1}, a_{\hat{\varphi}_2}, \ldots, a_{\hat{\varphi}_k}, c', \right),\]
\[(c'; \lambda_{\hat{\beta}_1}, \lambda_{\hat{\beta}_2}, \ldots, \lambda_{\hat{\beta}_{m-k-1}}) \in \mathcal{H}_{m-k} \left( a_{\hat{\delta}_1}, a_{\hat{\delta}_2}, \ldots, a_{\hat{\delta}_{m-k}} \right),\]
\[\hat{k} < m - 2\] and \[m = \hat{\beta}_{m-k-1}.\] Since we have already established the claim in Case 2, we have now established it in Case 3 (b) also.

**Case 4:** In Case 4, if we set \(h := \gamma_1\), then (120) and (121) become
\[(\lambda_1; \lambda_{\hat{\alpha}_m}) \in \mathcal{H}_2 (a_h, c)\] (131)
and
\[(\hat{c}; \lambda_2, \lambda_3, \ldots, \lambda_{m-1}) \in \mathcal{H}_{m-1} (a_1, \ldots, a_{h-1}, a_{h+1}, \ldots, a_m),\] (132)
respectively. Applying the inductive hypothesis to (132), there exist \(p, q \in \{1, 2, \ldots, m\} \setminus \{h\}\), \(p < q\), such that
\[(\hat{c}; \lambda_2, \lambda_3, \ldots, \lambda_{m-2}) \in \mathcal{H}_{m-2} (a_{r_1}, a_{r_2}, \ldots, a_{r_{m-3}}, c')\] (133)
and
\[(c'; \lambda_{m-1}) \in \mathcal{H}_2 (a_p, a_q),\] (134)
where \(\{r_1, r_2, \ldots, r_{m-3}\} = \{1, 2, \ldots, m\} \setminus \{p, q, h\}\) and \(c' := a_p + a_q - \lambda_{m-1}\). Then, by (131) and (133),
\[(\lambda_1; \lambda_2, \lambda_3, \ldots, \lambda_{m-2}, \lambda_m) \in \mathcal{H}_{m-1} (a_{r_1}, a_{r_2}, \ldots, a_{r_{m-3}}, a_h, c'),\] (135)
and now, examining (135) and (134), we see that we have reduced the problem to Case 3. This establishes our claim in Case 4 and completes the proof. \(\square\)

**Example 6.3.6.** Note that the decomposition given in Figure 6 does not conform to Theorem 6.3.5. For a decomposition which does conform to Theorem 6.3.5, see Figure 7. This shows that the decomposition into lists of length 2 need not be unique.

Recall Observation 6.2.14 and Theorems 6.2.13 and 6.2.12, which are the foundation of the definition of C-realisability. In order to prove that the lists in \(\mathcal{H}_n\) are precisely the C-realisable lists, we need to show that the analogues of these three theorems for lists in \(\mathcal{H}_n\) hold also. This is the focus of our next three results.

Firstly, observe that, trivially, \((\lambda_1; \mu_1) \in \mathcal{H}_2 (\lambda_1, \mu_1)\). Therefore
\[(\lambda_1; \lambda_2, \ldots, \lambda_m, \mu_1) \in \mathcal{H}_{m+1} (a_1, a_2, \ldots, a_m, \mu_1)\]
and hence we have:
We write $\mathcal{O}$ for lists in $\mathcal{O}$.

If there exist $L \in \mathcal{O}_n$ such that $\mathcal{O}$, then for all $\mathcal{O}$.

By Observation 6.3.7, $\mathcal{O}$ is closed under union. Mirroring our definitions of $\mathcal{O}$ and $\mathcal{O}$, we would like to have a notion of “irreducibility” for lists in $\mathcal{O}_n$. Therefore, we make the following definition.

**Definition 6.3.8.** We write

$$ (\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{O}_n^+ (a_1, a_2, \ldots, a_n) \quad (136) $$

if $(\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{O}_n(a_1, a_2, \ldots, a_n)$ and there do not exist partitions

$$ (1, 2, \ldots, n) = \{p_1, p_2, \ldots, p_l\} \cup \{q_1, q_2, \ldots, q_{n-1}\}, $$

$$ (1, 2, \ldots, n) = \{r_1, r_2, \ldots, r_l\} \cup \{s_1, s_2, \ldots, s_{n-1}\} $$

such that $$(\lambda_{p_1}; \lambda_{p_2}, \ldots, \lambda_{p_l}) \in \mathcal{O}_l(a_{r_1}, a_{r_2}, \ldots, a_{r_l})$$

and $$(\lambda_{q_1}; \lambda_{q_2}, \ldots, \lambda_{q_{n-1}}) \in \mathcal{O}_{n-1}(a_{s_1}, a_{s_2}, \ldots, a_{s_{n-1}}).$$

We write $$(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{O}_n^*$$

if there exist $a_1, a_2, \ldots, a_n \geq 0$ such that (136) holds.

We now give the analogue of Theorem 6.2.13 for lists in $\mathcal{O}_n$.

**Lemma 6.3.9.** If $(\rho; \lambda_2, \lambda_3, \ldots, \lambda_n) \in \mathcal{O}_n(a_1, a_2, \ldots, a_n)$, then for all $\epsilon \geq 0$, $$(\rho + \epsilon; \lambda_2, \lambda_3, \ldots, \lambda_n) \in \mathcal{O}_n(a_1 + \epsilon, a_2, a_3, \ldots, a_n).$$
Remark. Note that, since the $a_i$ are unordered in Lemma 6.3.9, $a_1$ can take the place of any of the diagonal elements $a_1, a_2, \ldots, a_n$.

Proof of Lemma 6.3.9. We proceed by induction on $n$. If $n = 2$ and $(\rho; \lambda_2) \in \mathcal{I}_2(a_1, a_2)$, then $\rho \geq \max(a_1, a_2)$ and $\rho + \lambda_2 = a_1 + a_2$. Hence $\rho + \epsilon \geq \max(a_1 + \epsilon, a_2)$ and $(\rho + \epsilon) + \lambda_2 = (a_1 + \epsilon) + a_2$. Therefore $(\rho + \epsilon; \lambda_2) \in \mathcal{I}_2(a_1 + \epsilon, a_2)$.

Now assume that the assertion holds for all $n \in \{2, 3, \ldots, m - 1\}$ and consider the case when $n = m$. If $(\rho; \lambda_2, \lambda_3, \ldots, \lambda_m) \in \mathcal{I}_m(a_1, a_2, \ldots, a_m)$, then there exist a partition of $\{2, 3, \ldots, m\}$ into two subsets $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ and $\{\beta_1, \beta_2, \ldots, \beta_{m - k - 1}\}$, a partition of $\{1, 2, \ldots, m\}$ into two subsets $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ and $\{\delta_1, \delta_2, \ldots, \delta_{m - k}\}$, and a nonnegative number $c$ such that

$$(\rho; \lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_k}) \in \mathcal{I}_{k + 1}(a_{\gamma_1}, a_{\gamma_2}, \ldots, a_{\gamma_k}, c) \quad (137)$$

and

$$(c; \lambda_{\beta_1}, \lambda_{\beta_2}, \ldots, \lambda_{\beta_{m - k - 1}}) \in \mathcal{I}_{m - k}(a_{\delta_1}, a_{\delta_2}, \ldots, a_{\delta_{m - k}}). \quad (138)$$

We now distinguish two possible cases: the case when $1 \in \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ and the case when $1 \in \{\delta_1, \delta_2, \ldots, \delta_{m - k}\}$.

Suppose $1 \in \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$. Without loss of generality, we may assume that $1 = \gamma_1$. By (137) and the inductive hypothesis, we have that

$$(\rho + \epsilon; \lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_k}) \in \mathcal{I}_{k + 1}(a_1 + \epsilon, a_{\gamma_2}, a_{\gamma_3}, \ldots, a_{\gamma_k}, c) \quad (139)$$

and hence, by (139) and (138),

$$(\rho + \epsilon; \lambda_2, \lambda_3, \ldots, \lambda_m) \in \mathcal{I}_m(a_1 + \epsilon, a_2, a_3, \ldots, a_m). \quad (140)$$

Now suppose $1 \in \{\delta_1, \delta_2, \ldots, \delta_{m - k}\}$. Without loss of generality, we may assume that $1 = \delta_1$. Applying the inductive hypothesis to (137) and (138) gives

$$(\rho + \epsilon; \lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_k}) \in \mathcal{I}_{k + 1}(a_{\gamma_1}, a_{\gamma_2}, \ldots, a_{\gamma_k}, c + \epsilon)$$

and

$$(c + \epsilon; \lambda_{\beta_1}, \lambda_{\beta_2}, \ldots, \lambda_{\beta_{m - k - 1}}) \in \mathcal{I}_{m - k}(a_1 + \epsilon, a_{\delta_2}, a_{\delta_3}, \ldots, a_{\delta_{m - k}}),$$

respectively. Hence (140) holds, as before. \qed

Note that Lemma 6.3.9 is true in general: in [35], Laffey and Šmigoc show that if $(\rho, \lambda_2, \lambda_3, \ldots, \lambda_n)$ is the spectrum of an irreducible nonnegative symmetric matrix with diagonal elements $(a_1, a_2, \ldots, a_n)$, then for all $\epsilon \geq 0$, $(\rho + \epsilon, \lambda_2, \lambda_3, \ldots, \lambda_n)$ is the spectrum of a nonnegative symmetric matrix with diagonal elements $(a_1 + \epsilon, a_2, a_3, \ldots, a_n)$.

We now give the analogue of Theorem 6.2.12 for lists in $\mathcal{I}_n$.
Theorem 6.3.10. Suppose \((\rho; \lambda_2, \lambda_3, \ldots, \lambda_n) \in \mathcal{H}_n(a_1, a_2, \ldots, a_n)\) and \(\epsilon \geq 0\). Then

(i) \((\rho + \epsilon; \lambda_2 - \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_n) \in \mathcal{H}_n(a_1, a_2, \ldots, a_n)\);

(ii) there exist \(s, t \in \{1, 2, \ldots, n\}\), \(s < t\), such that

\[
(\rho + \epsilon, \lambda_2 + \epsilon, \lambda_3, \ldots, \lambda_n) \in \mathcal{H}_n(a_1, \ldots, a_{s-1}, a_s + \epsilon, a_{s+1}, \ldots, a_{t-1}, a_t + \epsilon, a_{t+1}, \ldots, a_n).
\]

In particular, if \((\rho, \lambda_2, \lambda_3, \ldots, \lambda_n) \in \mathcal{H}_n\), then \((\rho + \epsilon, \lambda_2 \pm \epsilon, \lambda_3, \ldots, \lambda_n) \in \mathcal{H}_n\).

Remark. Since the \(\lambda_i\) are unordered, \(\lambda_2\) may take the place of any of the eigenvalues \(\lambda_2, \lambda_3, \ldots, \lambda_n\).

Proof of Theorem 6.3.10. We first consider the case when \(n = 2\). Suppose \((\rho; \lambda_2) \in \mathcal{H}_2(a_1, a_2)\). Then \(\rho \geq \max\{a_1, a_2\}\) and \(\rho + \lambda_2 = a_1 + a_2\). Therefore \(\rho + \epsilon \geq \max\{a_1, a_2\}\) and \((\rho + \epsilon) + (\lambda_2 - \epsilon) = a_1 + a_2\). Hence \((\rho + \epsilon; \lambda_2 - \epsilon) \in \mathcal{H}_2(a_1, a_2)\). Similarly, \(\rho + \epsilon \geq \max\{a_1 + \epsilon, a_2 + \epsilon\}\) and \((\rho + \epsilon) + (\lambda_2 + \epsilon) = (a_1 + \epsilon) + (a_2 + \epsilon)\). Hence \((\rho + \epsilon; \lambda_2 + \epsilon) \in \mathcal{H}_2(a_1 + \epsilon, a_2 + \epsilon)\).

Now assume the statement holds for all \(n \in \{2, 3, \ldots, m - 1\}\) and suppose \((\rho; \lambda_2, \lambda_3, \ldots, \lambda_m) \in \mathcal{H}_m(a_1, a_2, \ldots, a_m)\). Then there exist a partition of \(\{2, 3, \ldots, m\}\) into two subsets \(\{\alpha_1, \alpha_2, \ldots, \alpha_k\}\) and \(\{\beta_1, \beta_2, \ldots, \beta_{m-k-1}\}\), a partition of \(\{1, 2, \ldots, m\}\) into two subsets \(\{\gamma_1, \gamma_2, \ldots, \gamma_k\}\) and \(\{\delta_1, \delta_2, \ldots, \delta_{m-k}\}\), and a nonnegative number \(c\) such that

\[
(\rho; \lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_k}) \in \mathcal{H}_{k+1}(a_{\gamma_1}, a_{\gamma_2}, \ldots, a_{\gamma_k}, c) \tag{141}
\]

and

\[
(\epsilon; \lambda_{\beta_1}, \lambda_{\beta_2}, \ldots, \lambda_{\beta_{m-k-1}}) \in \mathcal{H}_{m-k}(a_{\delta_1}, a_{\delta_2}, \ldots, a_{\delta_{m-k}}). \tag{142}
\]

Assume also that the \(\gamma_i\) and \(\delta_i\) are labelled so that \(\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_k\) and \(\delta_1 \leq \delta_2 \leq \cdots \leq \delta_{m-k}\). We must distinguish between two possible cases: \(2 \in \{\alpha_1, \alpha_2, \ldots, \alpha_k\}\) and \(2 \in \{\beta_1, \beta_2, \ldots, \beta_{m-k-1}\}\).

Case 1: Suppose \(2 \in \{\alpha_1, \alpha_2, \ldots, \alpha_k\}\). Without loss of generality, assume \(2 = \alpha_1\). By (141) and the inductive hypothesis,

\[
(\rho + \epsilon; \lambda_2 - \epsilon, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_k}) \in \mathcal{H}_{k+1}(a_{\gamma_1}, a_{\gamma_2}, \ldots, a_{\gamma_k}, c), \tag{143}
\]

and hence, by (143) and (142),

\[
(\rho + \epsilon; \lambda_2 - \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_m) \in \mathcal{H}_m(a_1, a_2, \ldots, a_m). \tag{144}
\]

This proves (i) in Case 1. The inductive hypothesis also guarantees that one of the following sub-cases holds:

Case 1 (a): There exist \(\hat{s}, \hat{t} \in \{1, 2, \ldots, k\}\), \(\hat{s} < \hat{t}\), such that
\[(\rho + \epsilon, \lambda_2 + \epsilon, \lambda_3, \ldots, \lambda_\alpha_0) \in \mathcal{H}_{k+1}\]
\[(a_{\gamma_1}, \ldots, a_{\gamma_{s-1}}, a_{\gamma_s} + \epsilon, a_{\gamma_{s+1}}, \ldots, a_{\gamma_{t-1}}, a_{\gamma_t} + \epsilon, a_{\gamma_{t+1}}, \ldots, a_{\gamma_m}, c).\]  

(145)

In this case, by (145) and (142),
\[(\rho + \epsilon, \lambda_2 + \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_m) \in \mathcal{H}_m\]
\[(a_1, \ldots, a_{s-1}, a_s + \epsilon, a_{s+1}, \ldots, a_{t-1}, a_t + \epsilon, a_{t+1}, \ldots, a_m),\]  

(146)

where \(s = \gamma_3\) and \(t = \gamma_1\). This proves (ii) in Case 1 (a).

**Case 1 (b):** There exists \(s \in \{1, 2, \ldots, k\}\), such that
\[(\rho + \epsilon, \lambda_2 + \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_\alpha_k) \in \mathcal{H}_{k+1}\]
\[(a_{\gamma_1}, \ldots, a_{\gamma_{s-1}}, a_{\gamma_s} + \epsilon, a_{\gamma_{s+1}}, \ldots, a_{\gamma_m}, c + \epsilon).\]  

(147)

In this case, applying Lemma 6.3.9 to (142), we have
\[(c + \epsilon; \lambda_\beta_1, \lambda_\beta_2, \ldots, \lambda_\beta_{m-k-1}) \in \mathcal{H}_{m-k}\]
\[(a_{\delta_1} + \epsilon, a_{\delta_2}, a_{\delta_3}, \ldots, a_{\delta_{m-k}})\]  

(148)

and hence (146) follows from (147) and (148), where \(s = \min(\gamma_3, \delta_1)\) and \(t = \max(\gamma_3, \delta_1)\). This proves (ii) in Case 1 (b).

**Case 2:** Suppose \(2 \in (\beta_1, \beta_2, \ldots, \beta_{m-k-1})\) and without loss of generality, assume \(2 = \beta_1\). By (141) and Lemma 6.3.9,
\[(\rho + \epsilon; \lambda_\alpha_0, \lambda_\alpha_2, \ldots, \lambda_\alpha_k) \in \mathcal{H}_{k+1}\]
\[(a_{\gamma_1}, a_{\gamma_2}, \ldots, a_{\gamma_m}, c + \epsilon).\]  

(149)

Applying the inductive hypothesis to (142),
\[(c + \epsilon; \lambda_2 + \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_\beta_{m-k-1}) \in \mathcal{H}_{m-k}\]
\[(a_{\delta_1}, a_{\delta_2}, a_{\delta_3}, \ldots, a_{\delta_{m-k}})\]  

(150)

and there exist \(\hat{s}, \hat{t} \in \{1, 2, \ldots, m-k\}\), \(\hat{s} < \hat{t}\), such that
\[(c + \epsilon; \lambda_2 + \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_\beta_{m-k-1}) \in \mathcal{H}_{m-k}\]
\[(a_{\delta_1}, \ldots, a_{\delta_{\hat{s}-1}}, a_{\delta_{\hat{s}}} + \epsilon, a_{\delta_{\hat{s}+1}}, \ldots, a_{\delta_{\hat{t}-1}}, a_{\delta_{\hat{t}}} + \epsilon, a_{\delta_{\hat{t}+1}}, \ldots, a_{\delta_{m-k}}).\]  

(151)

Equation (144) then follows from (149) and (150), which proves (i) in Case 2. Equation (146) follows from (149) and (151), where \(s = \delta_3\) and \(t = \delta_1\). This proves (ii) in Case 2.

In [21], Guo conjectured that the symmetric analogue of Theorem 6.2.10 holds, i.e. that if \(\sigma := (\rho, \lambda_2, \lambda_3, \ldots, \lambda_n)\) is symmetrically realisable, then \((\rho + \epsilon, \lambda_2 \pm \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_n)\) is symmetrically realisable also. Whether this conjecture is true remains an open question; however, Theorem 6.3.10 shows the conjecture holds when \(\sigma \in \mathcal{H}_n\). In particular, it says that if \(\sigma \in \mathcal{H}_n\), then we may always increase the spectral gap whilst preserving symmetric realisability. Next, we show that if \(\sigma \in \mathcal{H}_n\), then it is also possible to decrease the spectral gap in the following sense: if \((\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n^*, \) where \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\), then there exists \(0 < \epsilon \leq \frac{1}{2}(\lambda_1 - \lambda_2)\) such that \((\lambda_1 - \epsilon, \lambda_2 + \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_n) \in \mathcal{H}_n \setminus \mathcal{H}_n^*.\)
Theorem 6.3.11. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $a_1, a_2, \ldots, a_n \geq 0$. Then $(\lambda_1; \lambda_2, \ldots, \lambda_n) \in H_n(a_1, a_2, \ldots, a_n)$ if and only if there exist

$$0 \leq \epsilon \leq \frac{1}{2}(\lambda_1 - \lambda_2)$$

and two partitions

$$\{3, 4, \ldots, n\} = \{p_1, p_2, \ldots, p_{l-1}\} \cup \{q_1, q_2, \ldots, q_{n-l-1}\},$$

$$\{1, 2, \ldots, n\} = \{r_1, r_2, \ldots, r_l\} \cup \{s_1, s_2, \ldots, s_{n-l}\}$$

such that

$$(\lambda_1 - \epsilon; \lambda_{p_1}, \lambda_{p_2}, \ldots, \lambda_{p_{l-1}}) \in H_1(a_{r_1}, a_{r_2}, \ldots, a_{r_l})$$

and

$$(\lambda_2 + \epsilon; \lambda_{q_1}, \lambda_{q_2}, \ldots, \lambda_{q_{n-l-1}}) \in H_{n-1}(a_{s_1}, a_{s_2}, \ldots, a_{s_{n-l}}).$$

We allow the possibilities $l = 1$, in which case $\{p_1, p_2, \ldots, p_{l-1}\}$ is the empty set, and $l = n - 1$, in which case $\{q_1, q_2, \ldots, q_{n-l-1}\}$ is the empty set.

Proof. First suppose there exist $\epsilon \in [0, \frac{1}{2}(\lambda_1 - \lambda_2)]$ and partitions of the form (152) such that (153) and (154) hold. Then by Observation 6.3.7,

$$(\lambda_1 - \epsilon; \lambda_2 + \epsilon, \lambda_3, \lambda_4, \ldots, \lambda_n) \in H_n(a_1, a_2, \ldots, a_n)$$

and hence by Theorem 6.3.10, $(\lambda_1; \lambda_2, \ldots, \lambda_n) \in H_n(a_1, a_2, \ldots, a_n)$. We claim the converse holds also.

If $n = 2$, then for $\epsilon = \lambda_1 - a_1$, we have $\lambda_1 - \epsilon = a_1$ and $\lambda_2 + \epsilon = a_2$. Hence $(\lambda_1 - \epsilon) \in H_1(a_1)$ and $(\lambda_2 + \epsilon) \in H_1(a_2)$ and so our claim holds in this case. Now assume the claim holds for $n = m - 1$ and consider the case when $n = m$.

Suppose $(\lambda_1; \lambda_2, \ldots, \lambda_m) \in H_m(a_1, a_2, \ldots, a_m)$. Then by Theorem 6.3.5, there exist $s, t \in \{1, 2, \ldots, m\}$, $s < t$, such that

$$(\lambda_1; \lambda_2, \ldots, \lambda_{m-1}) \in H_{m-1}(a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{t-1}, a_{t+1}, \ldots, a_m, c)$$

and

$$(c; \lambda_m) \in H_2(a_s, a_t),$$

where $c := a_s + a_t - \lambda_m$. Let us now apply the inductive hypothesis to (155); we will need to distinguish between the two possible cases $c \in \{r_1, r_2, \ldots, r_l\}$ and $c \in \{s_1, s_2, \ldots, s_{n-l-1}\}$.

Case 1: There exist $\epsilon \in [0, (\lambda_1 - \lambda_2)/2]$ and two partitions

$$\{3, 4, \ldots, m-1\} = \{\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_{l-1}\} \cup \{q_1, q_2, \ldots, q_{m-l-2}\},$$
There exist \( \{1, 2, \ldots, m\} \setminus \{s, t\} = \{\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_{\hat{l} - 1}\} \cup \{s_1, s_2, \ldots, s_{m - \hat{l} - 1}\} \)

such that

\[
(\lambda_1 - \varepsilon; \lambda_{p_1}, \lambda_{p_2}, \ldots, \lambda_{p_{\hat{l} - 1}}) \in \mathcal{H}_l(a_{\hat{r}_1}, a_{\hat{r}_2}, \ldots, a_{\hat{r}_{\hat{l} - 1}}, c)
\]  \hspace{1cm} (157)

and

\[
(\lambda_2 + \varepsilon; \lambda_{q_1}, \lambda_{q_2}, \ldots, \lambda_{q_{m - \hat{l} - 1}}) \in \mathcal{H}_{m - \hat{l} - 1}(a_{s_1}, a_{s_2}, \ldots, a_{s_{m - \hat{l} - 1}}).
\]  \hspace{1cm} (158)

In this case, by Definition 6.3.1, (157) and (156) imply

\[
(\lambda_1 - \varepsilon; \lambda_{p_1}, \lambda_{p_2}, \ldots, \lambda_{p_{\hat{l} - 1}}, \lambda_m) \in \mathcal{H}_{l - 1}(a_{\hat{r}_1}, a_{\hat{r}_2}, \ldots, a_{\hat{r}_{\hat{l} - 1}}, a_{s}, a_{t})
\]  \hspace{1cm} (159)

and after the relabelling

\[
\hat{l} = \hat{l} + 1,
\]

\[
p_l = \begin{cases} 
\hat{r}_i : & i = 1, 2, \ldots, \hat{l} - 1 \\
\hat{r}_i : & i = \hat{l}, \\
m : & i = \hat{l} + 1 \end{cases}
\]

\[
\hat{r}_l = \begin{cases} 
\hat{r}_i : & i = 1, 2, \ldots, \hat{l} - 1 \\
s : & i = \hat{l} \\
t : & i = \hat{l} + 1 \end{cases}
\]

(159) and (158) become

\[
(\lambda_1 - \varepsilon; \lambda_{p_1}, \lambda_{p_2}, \ldots, \lambda_{p_{\hat{l} - 1}}) \in \mathcal{H}_l(a_{r_1}, a_{r_2}, \ldots, a_{r_1})
\]

and

\[
(\lambda_2 + \varepsilon; \lambda_{q_1}, \lambda_{q_2}, \ldots, \lambda_{q_{m - \hat{l} - 1}}) \in \mathcal{H}_{m - \hat{l} - 1}(a_{s_1}, a_{s_2}, \ldots, a_{s_{m - \hat{l} - 1}}),
\]

respectively. This completes the inductive step and establishes the claim in Case 1.

**Case 2:** There exist \( \varepsilon \in [0, (\lambda_1 - \lambda_2)/2] \) and two partitions

\[
\{3, 4, \ldots, m - 1\} = \{p_1, p_2, \ldots, p_{l - 1}\} \cup \{\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{\hat{l} - 2}\},
\]

\[
\{1, 2, \ldots, m\} \setminus \{s, t\} = \{r_1, r_2, \ldots, r_1\} \cup \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{m - \hat{l} - 2}\}
\]

such that

\[
(\lambda_1 - \varepsilon; \lambda_{p_1}, \lambda_{p_2}, \ldots, \lambda_{p_{l - 1}}) \in \mathcal{H}_l(a_{r_1}, a_{r_2}, \ldots, a_{r_1})
\]

and

\[
(\lambda_2 + \varepsilon; \lambda_{q_1}, \lambda_{q_2}, \ldots, \lambda_{q_{m - \hat{l} - 2}}) \in \mathcal{H}_{m - \hat{l} - 1}(a_{s_1}, a_{s_2}, \ldots, a_{s_{m - \hat{l} - 2}}, c).
\]  \hspace{1cm} (160)

In this case, we may apply Definition 6.3.1 to (160) and (156) and the remainder of the proof is analogous to Case 1. \(\square\)
Example 6.3.12. Consider \( \sigma := (9, 6, 3, 3, -5, -5, -5) \), a list shown to be C-realisable in [4]. For a decomposition of \( \sigma \) in accordance with Theorem 6.3.11, see Figure 8. In particular, note the leaves in Figure 8; by Theorem 6.3.11, we can construct a realising matrix having these leaves as diagonal elements, i.e.

\[(9; 6, 3, 3, -5, -5, -5, -5) \in \mathcal{H}(1, 0, 0, 0, 0, 0).
\]

\[
\begin{array}{c}
\lambda_1, \lambda_5, \lambda_3, \lambda_6, \lambda_2, \lambda_7, \lambda_4, \lambda_8 \\
(9, -5, 3, -5, 6, -5, 3, -5) \\
(8, -5, 3, -5) \\
(6, -5) \\
(1) \\
(0) \\
\end{array}
\]

\[
\begin{array}{c}
\epsilon_1 = 1 \\
\epsilon_2 = 2 \\
\epsilon_3 = 5 \\
\epsilon_4 = 5 \\
\end{array}
\]

\[
\begin{array}{c}
\epsilon_5 = 2 \\
(5, -5) \\
(0) \\
\epsilon_6 = 5 \\
(0) \\
\end{array}
\]

\[
\begin{array}{c}
\epsilon_7 = 5 \\
(5, -5) \\
(0) \\
(0) \\
(0) \\
\end{array}
\]

Figure 8: Decomposition of \((9, 6, 3, 3, -5, -5, -5, -5)\)

In Appendix B, we show how to determine algorithmically whether a given list \( \sigma \) lies in \( \mathcal{H}_n \). Theorem 6.3.11 is the basis of this algorithm.

We finish this section by proving a property of \( \mathcal{H}_n \) which will not have an application in proving our main equivalence result, but which is of independent interest. The effect of adding zeros to a list \( \sigma \) has been extensively studied in the \textit{SNIEP} (see Section 1.3.3). The effect of adding zeros has also been studied in the \textit{SIEP} (see [28]). Our next result shows that adding zeros to \( \sigma \) does not affect whether \( \sigma \in \mathcal{H}_n \).

Theorem 6.3.13. If 

\[(\lambda_1; \lambda_2, \ldots, \lambda_n, 0) \in \mathcal{H}_{n+1}(a_1, a_2, \ldots, a_{n+1}),\]

then there exist \( s, t \in \{1, 2, \ldots, n+1\} \), \( s < t \), such that

\[(\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n(a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{t-1}, a_{t+1}, \ldots, a_{n+1}, a_s + a_t).\]

In particular, if \( (\lambda_1, \lambda_2, \ldots, \lambda_n, 0) \in \mathcal{H}_{n+1} \), then \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n \).

Proof. If \( n = 1 \), then the conclusion follows trivially. Now suppose \( n = 2 \) and let

\[(\lambda_1; \lambda_2, 0) \in \mathcal{H}_3(a_1, a_2, a_3),\]

where \( a_1 \geq a_2 \geq a_3 \geq 0 \). Then from the conditions given in (107),

\[\lambda_1 \geq a_1,\]

(161)
\[\lambda_2 \leq a_1, \quad (162)\]

and
\[\lambda_1 + \lambda_2 = a_1 + a_2 + a_3. \quad (163)\]

Combining (162) and (163),
\[\lambda_1 \geq a_2 + a_3 \quad (164)\]

and hence from (161), (164) and (163), we see that
\[(\lambda_1; \lambda_2) \in \mathcal{H}(a_1, a_2 + a_3). \]

Now assume the assertion holds for all \(n \in \{1, 2, \ldots, m-1\}\) and consider the case when \(n = m\). If
\[(\lambda_1; \lambda_2, \ldots, \lambda_m, 0) \in \mathcal{H}(a_1, a_2, \ldots, a_{m+1}), \]
then by Definition 6.3.1, one of the following cases must hold:

**Case 1:** There exist a partition of \(\{2, 3, \ldots, m\}\) into two subsets \(\{\alpha_1, \alpha_2, \ldots, \alpha_k\}\) and \(\{\beta_1, \beta_2, \ldots, \beta_{m-k-1}\}\), a partition of \(\{1, 2, \ldots, m+1\}\) into two subsets \(\{\gamma_1, \gamma_2, \ldots, \gamma_k\}\) and \(\{\delta_1, \delta_2, \ldots, \delta_{m-k-1}\}\), and a non-negative number \(c\) such that
\[(\lambda_1; \lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_k}) \in \mathcal{H}(\alpha_1, a_{\gamma_2}, \ldots, a_{\gamma_k}, c) \quad (165)\]

and
\[(c; \lambda_{\beta_1}, \lambda_{\beta_2}, \ldots, \lambda_{\beta_{m-k-1}}, 0) \in \mathcal{H}(a_{\delta_1}, a_{\delta_2}, \ldots, a_{\delta_{m-k-1}}). \quad (166)\]

In this case, applying the inductive hypothesis to (166), we see that there exist \(s, t \in \{1, 2, \ldots, m-k+1\}\), \(s < t\), such that
\[(c; \lambda_{\beta_1}, \lambda_{\beta_2}, \ldots, \lambda_{\beta_{m-k-1}}) \in \mathcal{H}(a_{\delta_1}, \ldots, a_{\delta_{s-1}}, a_{\delta_{s+1}}, \ldots, a_{\delta_{t-1}}, a_{\delta_{t+1}}, \ldots, a_{\delta_{m-k-1}}, a_{\delta_s} + a_{\delta_t}) \quad (167)\]

and hence, assuming the \(\delta_i\) are labelled so that \(\delta_1 < \delta_2 < \cdots < \delta_{m-k+1}\), (165), (167) and Definition 6.3.1 imply
\[(\lambda_1; \lambda_2, \ldots, \lambda_m) \in \mathcal{H}(a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{t-1}, a_{t+1}, \ldots, a_{n+1}, a_s + a_t) \quad (168)\]

where \(s = \delta_s\) and \(t = \delta_t\). This completes the inductive step in Case 1.

**Case 2:** There exist a partition of \(\{2, 3, \ldots, m\}\) into two subsets \(\{\alpha_1, \alpha_2, \ldots, \alpha_k\}\) (which may be empty) and \(\{\beta_1, \beta_2, \ldots, \beta_{m-k-1}\}\), a partition of \(\{1, 2, \ldots, m+1\}\) into two subsets \(\{\gamma_1, \gamma_2, \ldots, \gamma_{k+1}\}\) and \(\{\delta_1, \delta_2, \ldots, \delta_{m-k}\}\) and a non-negative number \(c\), such that
\[(\lambda_1; \lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_k}, 0) \in \mathcal{H}(\alpha_1, a_{\gamma_2}, \ldots, a_{\gamma_{k+1}}, c) \quad (169)\]
and
\[
(c; \lambda_1, \lambda_2, \ldots, \lambda_{m-k}) \in \mathcal{H}_{m-k}(a_{\delta_1}, a_{\delta_2}, \ldots, a_{\delta_{m-k}}). \tag{170}
\]

If \(\{\alpha_1, \alpha_2, \ldots, \alpha_k\}\) is empty, then (169) reduces to \(\lambda_1 = a_{\gamma_1} + c\) and so, applying Lemma 6.3.9 to (170) with \(\epsilon = a_{\gamma_1}\), gives
\[
(\lambda_1; \lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_k}) \in \mathcal{H}_{k+1}(a_{\gamma_1}, a_{\gamma_{k+1}}, a_{\gamma_{k+1}}, \ldots, a_{\gamma_{k+1}}, a_{\gamma_{k+1}} + a_{\gamma_1} + c)
\]
\[
(\lambda_1; \lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_k}) \in \mathcal{H}_{k+1}(a_{\gamma_1}, a_{\gamma_{k+1}}, a_{\gamma_{k+1}}, \ldots, a_{\gamma_{k+1}} + a_{\gamma_1} + c + a_{\gamma_{t}}). \tag{171}
\]

In this case, assuming the \(\gamma_l\) are labelled so that \(\gamma_1 < \gamma_2 < \cdots < \gamma_{k+1}\), (168) holds by (171) and (170), with \(s = \gamma_s\) and \(t = \gamma_t\).

- **Case 2 (a):** There exist \(\hat{s}, \hat{t} \in \{1, 2, \ldots, k+1\}\), \(\hat{s} < \hat{t}\), such that
\[
(\lambda_1; \lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_k}) \in \mathcal{H}_{k+1}(a_{\gamma_{\hat{s}+1}}, a_{\gamma_{\hat{s}+1}}, a_{\gamma_{\hat{s}+1}}, \ldots, a_{\gamma_{\hat{s}+1}} + a_{\gamma_{\hat{s}}} + c)
\]
\[
(\lambda_1; \lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_k}) \in \mathcal{H}_{k+1}(a_{\gamma_{\hat{s}+1}}, a_{\gamma_{\hat{s}+1}}, a_{\gamma_{\hat{s}+1}}, \ldots, a_{\gamma_{\hat{s}+1}} + a_{\gamma_{\hat{s}}} + c + a_{\gamma_{\hat{t}}}). \tag{172}
\]

In this case, applying Lemma 6.3.9 to (170) with \(\epsilon = a_{\gamma_t}\), we have that
\[
(e + a_{\gamma_t}; \lambda_{\beta_1}, \lambda_{\beta_2}, \ldots, \lambda_{\beta_{m-k+1}}) \in \mathcal{H}_{m-k}(a_{\delta_1} + a_{\gamma_t}, a_{\delta_2}, a_{\delta_{m-k}}, \ldots, a_{\delta_{m-k}})
\]
\[
(\lambda_1; \lambda_2, \ldots, \lambda_n) \text{ is C-realizable;}
\]
\[
(\lambda_1, \lambda_2, \ldots, \lambda_n) \text{ satisfies } S_p \text{ for some } p.
\]

Furthermore, if \(a_1, a_2, \ldots, a_n \geq 0\) are given, then \((\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_{n}(a_1, a_2, \ldots, a_n)\) if and only if \((\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_{n}(a_1, a_2, \ldots, a_n)\).
\textbf{Proof.} The proof is divided into six parts. In part 1, we show that \( S_n(a_1, a_2, \ldots, a_n) \subseteq \mathcal{H}_n(a_1, a_2, \ldots, a_n) \). In Part 2, we show that \( \mathcal{H}^*_n(a_1, a_2, \ldots, a_n) \subseteq S_n(a_1, a_2, \ldots, a_n) \). In Part 3, we use parts 1 and 2 to show that \( S_n(a_1, a_2, \ldots, a_n) \subseteq \mathcal{H}_n(a_1, a_2, \ldots, a_n) \). In Part 4, we show that (ii) implies (iv). In Part 5, we show that (iv) implies (ii) and finally, in Part 6, we show that (iii) implies (ii).

\textbf{Part 1:} First, we claim that \( S_n(a_1, a_2, \ldots, a_n) \subseteq \mathcal{H}_n(a_1, a_2, \ldots, a_n) \). If \( n = 1 \), there is nothing to prove. If \( (\lambda_1; \lambda_2) \in S_2(a_1, a_2) \), then in particular, \( (\lambda_1, \lambda_2) \) is the spectrum of a nonnegative symmetric matrix with diagonal elements \( (a_1, a_2) \). Therefore, the conditions given in (106) hold and hence \( (\lambda_1; \lambda_2) \in \mathcal{H}_2(a_1, a_2) \). Now assume the claim holds for all \( n \in \{1, 2, \ldots, m-1\} \), \( m \geq 3 \), and consider the case when \( n = m \).

Suppose \( (\lambda_1; \lambda_2, \ldots, \lambda_m) \in S_m(a_1, a_2, \ldots, a_m) \), where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \). Then there exists an \( m \times m \) Soules matrix \( R = (r_{ij}) \) such that the matrix \( RAR^T \) —where \( \Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \)—has diagonal elements \( (a_1, a_2, \ldots, a_m) \). Assume also that the \( a_i \) are labelled so that \( a_i \) is the \( (j, j) \) entry of \( RAR^T \) and let us label the columns of \( R \) as \( x = r_1, r_2, \ldots, r_m \). Our aim is to construct two smaller Soules matrices \( R_1 \) and \( R_2 \) from \( R \) and then apply the inductive hypothesis.

By Theorem 6.2.8, there exists a Soules-type sequence \( N = (N_1, N_2, \ldots, N_m) \) of partitions of \( \{1, 2, \ldots, m\} \) such that for each \( i \in \{2, 3, \ldots, m\} \), \( r_i \) is given (up to a factor of \( \pm 1 \)) by (109). As before, let us write \( N_i = (N_{i,1}, N_{i,2}, \ldots, N_{i,i}) \), \( i = 1, 2, \ldots, m \). Suppose \( N_{2,1} = \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \), where \( \gamma_1 < \gamma_2 < \cdots < \gamma_k \) and \( N_{2,2} = \{\delta_1, \delta_2, \ldots, \delta_{m-k}\} \), where \( \delta_1 < \delta_2 < \cdots < \delta_{m-k} \). Without loss of generality, we may assume that \( k < m-1 \), since if \( k = m-1 \), we may relabel the set \( N_{2,1} \) as \( N_{2,2} \) and vice versa. Let \( \alpha_1, \alpha_2, \ldots, \alpha_{k-1} \)—where \( \alpha_1 < \alpha_2 < \cdots < \alpha_{k-1} \)—be those indices in \( \{3, 4, \ldots, m\} \) such that \( N_3^*, N_4^* \subseteq N_{2,1} \). Similarly, \( N_3^*, N_4^* \subseteq N_{2,2} \).

Let \( S_1 \) be the \( k \times (k+1) \) submatrix of \( R \) obtained by selecting rows \( \gamma_1, \gamma_2, \ldots, \gamma_k \) and columns \( 1, 2, \alpha_1, \alpha_2, \ldots, \alpha_{k-1} \). Let \( u \) denote the first column of \( S_1 \). Similarly, let \( S_2 \) be the \( (m-k) \times (m-k+1) \) submatrix of \( R \) obtained by selecting rows \( \delta_1, \delta_2, \ldots, \delta_{m-k} \), and columns \( 1, 2, \beta_1, \beta_2, \ldots, \beta_{m-k-1} \). We denote the first column of \( S_2 \) by \( v \). By (109), either the second column of \( S_1 \) is \((|v|/|u|)u\) and the second column of \( S_2 \) is \(-(|v|/|u|)u\) or the second column of \( S_1 \) is \((|v|/|u|)u\) and the second column of \( S_2 \) is \((|v|/|u|)u\). Without loss of generality, we may assume the former, as otherwise we may replace \( r_2 \) with \(-r_2\).

Hence we may write

\[
S_1 = \begin{bmatrix} u & \frac{|v|}{|u|}u & T_1 \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} v & -\frac{|u|}{|v|}v & T_2 \end{bmatrix},
\]

where \( T_1 \) and \( T_2 \) are \( k \times (k-1) \) and \((m-k) \times (m-k-1)\) matrices, respectively.
We now define the \((k+1) \times (k+1)\) matrix
\[
R_1 := \begin{bmatrix} \cdots & S_1 & \cdots \\ \|v\| - \|u\| & 0 & 0 & \cdots & 0 \\ \|v\| & 0 & 0 & \cdots & 0 \\ \|v\| & 0 & 0 & \cdots & 0 \\ \|v\| & 0 & 0 & \cdots & 0 \\ \end{bmatrix} = \begin{bmatrix} 0 & u & v \\ \|v\| & \|u\| & u & T_1 \\ \|v\| & \|u\| & 0 & \cdots & 0 \\ \|v\| & \|u\| & 0 & \cdots & 0 \\ \|v\| & \|u\| & 0 & \cdots & 0 \\ \end{bmatrix}
\]
and the \((m-k) \times (m-k)\) matrix
\[
R_2 := \begin{bmatrix} \frac{v}{\|v\|} & T_2 \end{bmatrix}.
\]
Note that \(|\|u\|^2 + |v|^2| = |x|^2 = 1\) and so \(R_1\) and \(R_2\) have normalised, positive first columns. Moreover, \(R_1\) and \(R_2\) are Soules matrices. To see this, consider the sequence \(N = (N_1, N_2, \ldots, N_{k+1})\) of partitions of \((1,2,\ldots,k+1)\), where \(N_1 = \{1,2,\ldots,k+1\}\), \(N_2 = \{1,2,\ldots,k\}\) and for \(i = 3,4,\ldots,k+1, N_{l,j} := \{s : \gamma_s \in \mathcal{M}_{l,j}\}\), where \(\mathcal{M}_l\) is obtained from \(N_{\alpha_{i-2}}\) by removing those sets \(N_{\alpha_{i-2},1} \in N_{\alpha_{i-2}}\) which are subsets of \(N_{2,2}\). Then, labelling the columns of \(R_1\) as \(y = \tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_{k+1}\), we see that
\[
\tilde{r}_i = \pm \frac{1}{\sqrt{\|y_N^{(i)}\|^2 + \|y_N^{(i)}\|^2}} \left( \frac{\|y_N^{(i)}\| y_N^{(i)} - \|y_N^{(i)}\| g_N^{(i)}}{\|y_N^{(i)}\|^2} \right),
\]
for all \(i = 2,3,\ldots,k+1\) and hence \(R_1\) is a Soules matrix. Similarly, \(R_2\) may be seen to be a Soules matrix by considering the sequence \(N' = (N'_1, N'_2, \ldots, N'_{m-k})\) of partitions of \((1,2,\ldots,m-k)\), where \(N'_1 = \{1,2,\ldots,m-k\}\) and for \(i = 2,3,\ldots,m-k, N'_{l,j} := \{s : \delta_s \in \mathcal{M}'_{l,j}\}\), where \(\mathcal{M}'_l\) is obtained from \(N_{\beta_{i-1}}\) by removing those sets \(N_{\beta_{i-1},1} \in N_{\beta_{i-1}}\) which are subsets of \(N_{2,1}\).

Now set
\[
c := |v|^2 \lambda_1 + |u|^2 \lambda_2 \tag{174}
\]
and observe that if
\[
\Lambda_1 := \text{diag}(\lambda_1, \lambda_2, \lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_{k-1}}),
\]
then the diagonal elements of the matrix \(R_1 \Lambda_1 R_1^T\) are \((a_{\gamma_1}, a_{\gamma_2}, \ldots, a_{\gamma_k}, c)\). To see this, note that for \(i = 1,2,\ldots,k\), the definitions of \(\gamma_i\) and \(\alpha_i\) and the structure of \(R\) imply
\[
\sum_{t=3}^{m} r_{\gamma_1 t}^2 \lambda_t = \sum_{s=3}^{k+1} r_{\gamma_1 \alpha_{s-2}}^2 \lambda_{\alpha_{s-2}}
\]
and hence
\[
(R_1 \Lambda_1 R_1^T)_{ii} = \sum_{s=1}^{k+1} (R_1)_{is}^2 (\Lambda_1)_{ss}
\]
and
\[
= \sum_{s=1}^{k+1} (S_1)_{is}^2 (\Lambda_1)_{ss}
\]
Therefore, by the inductive hypothesis,

\[ r_{y_1}^2 \lambda_1 + r_{y_2}^2 \lambda_2 + \sum_{s=3}^{k+1} r_{y_1,\alpha_{s-2}}^2 \lambda_{\alpha_{s-2}} = r_{y_1}^2 \lambda_1 + r_{y_2}^2 \lambda_2 + \sum_{t=3}^{m} r_{y_1,\lambda_t}^2 \lambda_t = (RA^T)_{y_1, y_1} = a_{y_1}. \]

In addition,

\[ (R_1 \Lambda_1 R_1^T)_{k+1,k+1} = \sum_{s=1}^{k+1} (R_1)_{k+1,s}^2 (\Lambda_1)_{ss} = ||v||^2 \lambda_1 + ||u||^2 \lambda_2 = c. \]

Therefore, by the inductive hypothesis,

\[ (\lambda_1, \lambda_2, \lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_{k-1}}) \in \mathcal{K}_{k+1}(a_{y_1}, a_{y_2}, \ldots, a_{y_k}, c). \quad (175) \]

Similarly, if \( \Lambda_2 := \text{diag}(c, \lambda_{\beta_1}, \lambda_{\beta_2}, \ldots, \lambda_{\beta_{m-k-1}}) \), then the diagonal elements of the matrix \( R_2 \Lambda_2 R_2^T \) are \( (a_{\delta_1}, a_{\delta_2}, \ldots, a_{\delta_{m-k}}) \). To see this, note that for \( i = 1, 2, \ldots, m-k \), the definitions of \( \delta_i \) and \( \beta_i \) and the structure of \( R \) imply

\[ \sum_{t=3}^{m} r_{\delta_i, \lambda_t}^2 \lambda_t = \sum_{s=3}^{m-k+1} r_{\delta_i, \beta_{s-2}}^2 \lambda_{\beta_{s-2}} \]

and hence

\[ (R_2 \Lambda_2 R_2^T)_{ii} = \sum_{s=1}^{m-k} (R_2)_{i,s}^2 (\Lambda_2)_{ss} = \frac{v_i^2}{||v||^2} c + \sum_{s=2}^{m-k} (T_2)_{i,s-1}^2 \lambda_{\beta_{s-1}} = \frac{v_i^2}{||v||^2} c + \sum_{s=3}^{m-k+1} (T_2)_{i,s-2}^2 \lambda_{\beta_{s-2}} = v_i^2 \lambda_1 + \frac{||u||^2}{||v||^2} v_i^2 \lambda_2 + \sum_{s=3}^{m-k+1} (S_2)_{i,s}^2 \lambda_{\beta_{s-2}} = r_{\delta_i, \lambda_t}^2 \lambda_1 + r_{\delta_i, \beta_{s+2}}^2 \lambda_{\beta_{s+2}} = r_{\delta_i, \lambda_t}^2 \lambda_1 + r_{\delta_i, \beta_{s+2}}^2 \lambda_{\beta_{s+2}} = r_{\delta_i, \lambda_t}^2 \lambda_1 + r_{\delta_i, \beta_{s+2}}^2 \lambda_{\beta_{s+2}} = (R \Lambda R^T)_{\delta_i, \delta_i} = a_{\delta_i}. \]
Note also that $c \geq \lambda_2 \geq \lambda_{\beta_1}$ and hence the inductive hypothesis gives

\[ (c; \lambda_{\beta_1}, \lambda_{\beta_2}, \ldots, \lambda_{\beta_{m-k-1}}) \in \mathcal{H}_{m-k}(a_{s_1}, a_{s_2}, \ldots, a_{s_{m-k}}). \]  

(176)

Therefore, by Definition 6.3.1, (175) and (176) imply

\[ (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathcal{H}_m(a_1, a_2, \ldots, a_m). \]  

(177)

We have now shown that $\mathcal{S}_n(a_1, a_2, \ldots, a_n) \subseteq \mathcal{H}_n(a_1, a_2, \ldots, a_n)$.

**PART 2:** We now claim $\mathcal{H}^*_n(a_1, a_2, \ldots, a_n) \subseteq \mathcal{S}_n(a_1, a_2, \ldots, a_n)$. If $n = 1$, there is nothing to prove. If $(\lambda_1; \lambda_2) \in \mathcal{S}_2(a_1, a_2)$, then there exists $\epsilon > 0$ such that $\lambda_1 = a_1 + \epsilon$ and $\lambda_2 = a_2 - \epsilon$. If $(\lambda_1; \lambda_2) \in \mathcal{H}^*_2(a_1, a_2)$, then $\epsilon > 0$ and hence the matrix

\[ R_0 := \frac{1}{\sqrt{a_1 - a_2 + 2\epsilon}} \begin{bmatrix} a_1 - a_2 + \epsilon & \sqrt{\epsilon} \\ \sqrt{\epsilon} & -\sqrt{a_1 - a_2 + \epsilon} \end{bmatrix} \]

is a Soules matrix with

\[ R_0 \Lambda R_0^T = \begin{bmatrix} a_1 & \sqrt{\epsilon} (a_1 - a_2 + \epsilon) \\ \sqrt{\epsilon} (a_1 - a_2 + \epsilon) & a_2 \end{bmatrix}, \]

where $\Lambda := \text{diag}(\lambda_1, \lambda_2)$. Therefore $(\lambda_1; \lambda_2) \in \mathcal{S}_2(a_1, a_2)$.

Now assume the claim holds for $n = m - 1$, $m \geq 3$, and consider the case when $n = m$.

Suppose $(\lambda_1; \lambda_2, \ldots, \lambda_m) \in \mathcal{H}^*_m(a_1, a_2, \ldots, a_m)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. Then by Theorem 6.3.5, there exist $s, t \in \{1, 2, \ldots, m\}$, $s < t$, such that

\[ (\lambda_1; \lambda_2, \ldots, \lambda_{m-1}) \in \mathcal{H}_{m-1}(a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{t-1}, a_{t+1}, \ldots, a_{m}, c) \]  

(178)

and

\[ (c; \lambda_m) \in \mathcal{H}_2(a_s, a_t), \]  

(179)

where $c := a_s + a_t - \lambda_m$. It is not difficult to see that in (178) and (179), it is possible to replace $\mathcal{H}$ by $\mathcal{H}^*$. Let us assume the contrary. If

\[ (\lambda_1; \lambda_2, \ldots, \lambda_{m-1}) \notin \mathcal{H}_{m-1}(a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{t-1}, a_{t+1}, \ldots, a_{m}, c), \]

then there exist partitions

\[ \{1, 2, \ldots, m-1\} = \{p_1, p_2, \ldots, p_l\} \cup \{q_1, q_2, \ldots, q_{m-1-l}\}, \]

\[ \{1, 2, \ldots, m\} \setminus \{s, t\} = \{r_1, r_2, \ldots, r_{l-1}\} \cup \{s_1, s_2, \ldots, s_{m-1-l}\}, \]

where $\{r_1, r_2, \ldots, r_{l-1}\}$ may be empty, such that

\[ (\lambda_{p_1}; \lambda_{p_2}, \ldots, \lambda_{p_l}) \in \mathcal{H}_l(a_{r_1}, a_{r_2}, \ldots, a_{r_{l-1}}, c) \]  

(180)
and

\[(\lambda_{q_1}; \lambda_{q_2}, \ldots, \lambda_{q_{m-1}}) \in \mathcal{H}_{m-1}(a_{s_1}, a_{s_2}, \ldots, a_{s_{m-1}}). \tag{181}\]

In this case, (180) and (179) imply

\[(\lambda_{p_1}; \lambda_{p_2}, \ldots, \lambda_{p_{t}}, \lambda_m) \in \mathcal{H}_{t+1}(a_{r_1}, a_{r_2}, \ldots, a_{r_{t-1}}, a_s, a_t); \tag{182}\]

however, (181) and (182) contradict the fact that \((\lambda_1; \lambda_2, \ldots, \lambda_m) \in \mathcal{H}_m^+(a_1, a_2, \ldots, a_m).\) Hence

\[(\lambda_1; \lambda_2, \ldots, \lambda_{m-1}) \in \mathcal{H}_{m-1}^+(a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{t-1}, a_{t+1}, \ldots, a_m, c). \tag{183}\]

Similarly, suppose \((c; \lambda_m) \in \mathcal{H}_2(a_s, a_t) \setminus \mathcal{H}_2^+(a_s, a_t)\) and assume, without loss of generality, that \(a_s \geq a_t.\) Then \(c = a_s\) and \(\lambda_m = a_t,\) and thus, replacing \(c\) by \(a_s\) in (178), we have

\[(\lambda_1; \lambda_2, \ldots, \lambda_{m-1}) \in \mathcal{H}_{m-1}(a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{t-1}, a_{t+1}, \ldots, a_m),\]

\[(\lambda_m) \in \mathcal{H}_1(a_t),\]

again contradicting the fact that \((\lambda_1; \lambda_2, \ldots, \lambda_m) \in \mathcal{H}_m^+(a_1, a_2, \ldots, a_m).\) Therefore

\[(c; \lambda_m) \in \mathcal{H}_2^+(a_s, a_t). \tag{184}\]

Applying the inductive hypothesis to (183), there exists an \((m - 1) \times (m - 1)\) Soules matrix \(R_1,\) such that the matrix \(R_1\Lambda_1R_1^T—\)where \(\Lambda_1 := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{m-1})—\)has diagonal elements

\[(a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{t-1}, a_{t+1}, \ldots, a_m, c).\]

Note that if \(R\) is a \(k \times k\) Soules matrix and \(P\) is a \(k \times k\) permutation matrix, then \((PR)\Theta(PR)^T = P(R\Theta R^T)P^T\) is nonnegative for every nonnegative diagonal matrix \(\Theta\) with non-increasing diagonal entries. Hence \(PR\) is a Soules matrix also. Therefore, we may assume, without loss of generality, that \(c\) is the \((m - 1, m - 1)\) entry of \(R_1\Lambda_1R_1^T,\) since otherwise, we may replace \(R_1\) with \(PR_1,\) where \(P\) is a suitable permutation matrix. Similarly, by (184), there exists a \(2 \times 2\) Soules matrix \(R_2\) such that the matrix \(R_2\Lambda_2R_2^T—\)where \(\Lambda_2 := \text{diag}(c, \lambda_m)—\)has diagonal elements \((a_s, a_t).\) Let \(R_1\) be partitioned as

\[R_1 = \begin{bmatrix} U & \ast \\ u^T & \ast \end{bmatrix},\]

where \(u \in \mathbb{R}^{m-1}\) and \(U \in \mathbb{R}^{(m-2) \times (m-1)}\) and let \(R_2\) be partitioned as

\[R_2 = \begin{bmatrix} v & v' \\ v'' & \ast \end{bmatrix},\]

where \(v, v' \in \mathbb{R}^2.\)
By Lemma 6.2.11, for matrices

\[ R := \begin{bmatrix} \mathbb{U} & 0 \\ \mathbb{V}^T & \mathbb{V} \end{bmatrix} \]

and \( \Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \), the matrix \( \Lambda R \Lambda^T \) has diagonal elements \((a_1, a_2, \ldots, a_m)\). We will show that \( R \) is a Soules matrix. To see this, let \( \mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_{m-1}) \) be the Soules-type sequence of partitions of \( \{1, 2, \ldots, m-1\} \) associated with \( R_1 \), where \( \mathcal{N}_i = \{\mathcal{N}_{i+1}, \mathcal{N}_{i+2}, \ldots, \mathcal{N}_{i+j}\} \). The matrix \( R \) may then be seen to be a Soules matrix by considering the Soules-type sequence \( \mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_m), \mathcal{N}_i = \{\mathcal{N}_{i+1}, \mathcal{N}_{i+2}, \ldots, \mathcal{N}_{i+j}\} \) of partitions of \( \{1, 2, \ldots, m\} \), where for \( i \in \{1, 2, \ldots, m-1\} \), \( \mathcal{N}_{i,j} \) is given by

\[ \begin{align*}
\mathcal{N}_{i,j} & = \mathcal{N}_{i,j} : m-1 \notin \mathcal{N}_{i,j} \\
\mathcal{N}_{i,j} & = \mathcal{N}_{i,j} \cup \{m\} : m-1 \in \mathcal{N}_{i,j}
\end{align*} \]

and \( \mathcal{N}_m = \{\{1\}, \{2\}, \ldots, \{m\}\} \).

Therefore \((\lambda_1; \lambda_2, \ldots, \lambda_m) \in \mathbb{S}_m(a_1, a_2, \ldots, a_m)\). Hence \( \mathbb{H}_n(a_1, a_2, \ldots, a_n) \subseteq \mathbb{S}_n(a_1, a_2, \ldots, a_n) \).

\textbf{PART 3}: We will now use parts 1 and 2 to show that \( \mathcal{S}_n(a_1, a_2, \ldots, a_n) = \mathbb{H}_n(a_1, a_2, \ldots, a_n) \). By definition, if \((\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{S}_n(a_1, a_2, \ldots, a_n)\), then there exist partitions

\[ \{1, \ldots, n\} = \{a_{1}^{(1)}, \ldots, a_{n_1}^{(1)}\} \cup \{a_{1}^{(2)}, \ldots, a_{n_2}^{(2)}\} \cup \cdots \cup \{a_{1}^{(k)}, \ldots, a_{n_k}^{(k)}\}, \]

\[ \{1, \ldots, n\} = \{\beta_{1}^{(1)}, \ldots, \beta_{n_1}^{(1)}\} \cup \{\beta_{1}^{(2)}, \ldots, \beta_{n_2}^{(2)}\} \cup \cdots \cup \{\beta_{1}^{(k)}, \ldots, \beta_{n_k}^{(k)}\} \quad (185) \]

such that

\[ (\lambda_{a_{1}^{(i)}}; \lambda_{a_{2}^{(i)}}, \ldots, \lambda_{a_{n_{i}^{(i)}}}) \in \mathbb{S}_{n_{i}}(a_{\beta_{1}^{(i)}}, a_{\beta_{2}^{(i)}}, \ldots, a_{\beta_{n_{i}^{(i)}}}) : i = 1, 2, \ldots, k \]

(186)

By Part 1, in (186), we may replace \( \mathbb{S} \) by \( \mathbb{H} \) and hence, by Observation 6.3.7, \((\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathbb{H}_n(a_1, a_2, \ldots, a_n)\).

Conversely, if \((\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathbb{H}_n(a_1, a_2, \ldots, a_n)\), then there exist partitions of the form (185) such that

\[ (\lambda_{a_{1}^{(i)}}; \lambda_{a_{2}^{(i)}}, \ldots, \lambda_{a_{n_{i}^{(i)}}}) \in \mathbb{H}_{n_{i}}^*(a_{\beta_{1}^{(i)}}, a_{\beta_{2}^{(i)}}, \ldots, a_{\beta_{n_{i}^{(i)}}}) : i = 1, 2, \ldots, k \]

(187)

By Part 2, in (187), we may replace \( \mathbb{H}^* \) by \( \mathbb{S} \) and hence, by definition, \((\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{S}_n(a_1, a_2, \ldots, a_n)\).

\textbf{PART 4}: We now claim that (ii) implies (iv). If \((\lambda) \in \mathcal{H}_1\), then \( \lambda \geq 0 \) and \( (\lambda) \) trivially satisfies \( S_1 \). Now assume the claim holds for \( n \in \{1, 2, \ldots, m-1\} \) and suppose \( (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathcal{H}_m \), where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \). Then by Theorem 6.3.11, there exist \( \epsilon \in [0, (\lambda_1 - \lambda_2)/2] \) and a partition

\[ \{3, 4, \ldots, m\} = \{p_1, p_2, \ldots, p_{l-1}\} \cup \{q_1, q_2, \ldots, q_{m-l-1}\} \]
We now claim (iv) implies (iii). The cases where we may assume $k$ (Theorem 6 and $s$ satisfy $S$) have been dealt with in $\mathcal{H}_t$. By the inductive hypothesis, there exist $k$ and $t$ such that

$$
\sigma_1^t := (\lambda_1 - \epsilon, \lambda_{p_1}, \lambda_{p_2}, \ldots, \lambda_{p_{t-1}})
$$

and

$$
\sigma_2^t := (\lambda_2 + \epsilon, \lambda_{q_1}, \lambda_{q_2}, \ldots, \lambda_{q_{m-t-1}})
$$

satisfy $S_k$ and $S_t$, respectively. Furthermore, by Observation 6.2.17, we may assume $k = t$, i.e. (in the notation of Section 6.2.4) $\sigma_1^t, \sigma_2^t \in \mathcal{D}_{S_k}$.

Define also

$$
\sigma_1 := (\lambda_1, \lambda_{p_1}, \lambda_{p_2}, \ldots, \lambda_{p_{t-1}}),
\sigma_2 := (\lambda_2, \lambda_{q_1}, \lambda_{q_2}, \ldots, \lambda_{q_{m-t-1}}).
$$

Recalling definitions (113) and (112), we have $\mathcal{M}_k(\sigma_1) = \mathcal{M}_k(\sigma_1^t) + \epsilon \geq \epsilon$, $\mathcal{M}_k^*(\sigma_2) \leq \epsilon$ and

$$
\gamma := \max(\lambda_1 - \epsilon, \lambda_2) \leq \max(\lambda_1 - \epsilon, \lambda_2) = \lambda_1 - \epsilon.
$$

Since $\lambda_1 \geq \gamma + \mathcal{M}_k^*(\sigma_2)$, $\sigma = (\sigma_1, \sigma_2)$ satisfies $S_{k+1}$.

Part 5: We now claim (iv) implies (iii). The cases $p = 1$ and $p = 2$ have been dealt with in [4]. Now assume that if $\sigma$ satisfies $S_{p-1}$, then $\sigma$ is C-realisable. The proof of the inductive step is essentially the same as the proof of [4, Theorem 3.7].

Let $\sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If $\sigma$ satisfies $S_p$, then there exists a partition of $\sigma$ of the form (114) into sublists $\sigma_1, \sigma_2, \ldots, \sigma_r$, such that $\sigma_1 \in \mathcal{D}_{S_{p-1}}$ and (115) holds. From (112) and (113), we see that

$$
\sigma_1' := (\lambda_1^{(1)} - \mathcal{M}_{S_{p-1}}(\sigma_1), \lambda_2^{(1)}, \lambda_3^{(1)}, \ldots, \lambda_n^{(1)}) \in \mathcal{D}_{S_{p-1}}
$$

and

$$
\sigma_i' := (\lambda_1^{(i)} + \mathcal{M}_{S_{p-1}}(\sigma_i), \lambda_2^{(i)}, \lambda_3^{(i)}, \ldots, \lambda_n^{(i)}) \in \mathcal{D}_{S_{p-1}}: \ i = 2, 3, \ldots, r.
$$

By the inductive hypothesis, $\sigma_i'$ is C-realisable for all $i = 1, 2, \ldots, r$. Now, (115) and (116) imply that

$$
d := \mathcal{M}_{S_{p-1}}(\sigma_1) - \sum_{\sigma_i \in \mathcal{D}_{S_{p-1}}} \mathcal{M}_{S_{p-1}}(\sigma_i) \geq 0
$$

and hence, if we increase the Perron eigenvalue of $\sigma_1'$ by $d$ (as in Theorem 6.2.13), we have that

$$
\sigma_1^* := (\lambda^*, \lambda_2^{(1)}, \lambda_3^{(1)}, \ldots, \lambda_n^{(1)})
$$
is C-realisable, where

$$\lambda^* := \lambda_1 - \sum_{\sigma_i \notin D_{S_{p-1}}} \mathcal{A}_{S_{p-1}}(\sigma_i).$$

Hence $$\tau := (\sigma_1^*, \sigma_2^*, \sigma_3^*, \ldots, \sigma_r^*)$$ is C-realisable.

If $$\lambda^*$$ is the Perron eigenvalue of $$\tau$$, then it is clear that we may transform $$\tau$$ into $$\sigma$$ by applying Theorem 6.2.12 several times (once for every $$i$$ such that $$\sigma_i \notin D_{S_{p-1}}$$). Hence $$\sigma$$ is C-realisable in this case. On the other hand, if $$\lambda^*$$ is not the Perron eigenvalue of $$\tau$$, then there exists $$k \in \{2, 3, \ldots, r\}$$ such that $$\lambda_1^{(k)} + \mathcal{A}_{S_{p-1}}(\sigma_k)$$ is the Perron eigenvalue of $$\tau$$. In this case, note that (115) and (116) imply $$\lambda^* - \lambda_1^{(k)} \geq 0$$. Hence we may increase $$\lambda_1^{(k)} + \mathcal{A}_{S_{p-1}}(\sigma_k)$$ by $$\lambda^* - \lambda_1^{(k)}$$ while decreasing $$\lambda^*$$ by $$\lambda^* - \lambda_1^{(k)}$$, i.e. we may replace the eigenvalue pair $$(\lambda_1^{(k)} + \mathcal{A}_{S_{p-1}}(\sigma_k), \lambda^*)$$ with the pair $$(\lambda^* + \mathcal{A}_{S_{p-1}}(\sigma_k), \lambda_1^{(k)})$$ and it is clear that after this replacement, $$\lambda^* + \mathcal{A}_{S_{p-1}}(\sigma_k)$$ is the Perron eigenvalue of the modified list. Applying Theorem 6.2.12 several more times, we see that $$\sigma$$ is C-realisable.

**Part 6:** Finally, we show that (iii) implies (ii). If $$(\lambda_1, \lambda_2, \ldots, \lambda_n)$$ is C-realisable, then by definition, it may be obtained starting from the $$n$$ lists $$(0), (0), \ldots, (0)$$ and using only the operations defined by Observation 6.2.14 and Theorems 6.2.13 and 6.2.12. By Observation 6.3.7, Lemma 6.3.9, Theorem 6.3.10 and the fact that $$(0) \in \mathcal{H}_1$$, we see that $$(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n$$.

**Corollary 6.4.2.** If $$\sigma$$ is C-realisable, then $$\sigma$$ is symmetrically realisable.

We will illustrate the proof of Theorem 6.4.1 by considering a specific example.

**Example 6.4.3.** Consider again the list $$\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (7, 5, -2, -4, -6)$$. We showed in Example 6.2.9 that $$\sigma \in S_5$$, and in Example 6.3.3 that $$\sigma \in \mathcal{H}_5$$.

Given the Soles matrix $$R$$ of Example 6.2.9, let us follow the proof of Theorem 6.4.1 to obtain the decomposition given in Figure 6. In the notation of the proof, we have $$k = 2$$, $$N_{2,1} = \{\gamma_1, \gamma_2\} = \{1, 2\}$$, $$N_{2,2} = \{\delta_1, \delta_2, \delta_3\} = \{3, 4, 5\}$$, $$\alpha_1 = 5$$, $$\beta_1, \beta_2 = (3, 4)$$,

$$S_1 = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad S_2 = \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix},$$

$$\|u\| = \left\|\begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix}\right\| = \frac{1}{\sqrt{2}}, \quad \|v\| = \left\|\begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{4} \end{bmatrix}\right\| = \frac{1}{\sqrt{2}},$$

$$R_1 = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}}{8} & -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$
and \( c = ||v||^2 \lambda_1 + ||u||^2 \lambda_2 = 6 \). Thus, in order to show that \((7; 5, -2, -4, -6) \in \mathcal{H}_5(0, 0, 0, 0, 0)\), it is sufficient to show that

\[
(7; 5, -6) \in \mathcal{H}_3(0, 0, 6) \tag{188}
\]

and

\[
(6; -2, -4) \in \mathcal{H}_3(0, 0, 0). \tag{189}
\]

Since (188) and (189) obey the conditions given in (107), Lemma 6.3.4 implies that (188) and (189) hold, so we may stop here if we wish.

Alternatively, we may follow the above procedure again in order to further break down (188) and (189). To do this, note that \( R_1 \) is a Soules matrix with associated partition sequence \{\{1, 2, 3\}, \{1, 2\}, \{1\}\} and that

\[
R_1 \Lambda_1 R_1^T = \begin{bmatrix}
0 & 6 & \frac{1}{\sqrt{2}} \\
6 & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 6
\end{bmatrix} =: \Lambda_1, \tag{190}
\]

where \( \Lambda_1 = \text{diag}(7, 5, -6) \). Similarly, \( R_2 \) is a Soules matrix with associated partition sequence \{\{1, 2, 3\}, \{1\}\} and

\[
R_2 \Lambda_2 R_2^T = \begin{bmatrix}
0 & \sqrt{6} & \sqrt{6} \\
\sqrt{6} & 0 & 4 \\
\sqrt{6} & 4 & 0
\end{bmatrix} =: \Lambda_2, \tag{191}
\]

where \( \Lambda_2 = \text{diag}(6, -2, -4) \). Applying our procedure again, we arrive at the decomposition given in Figure 6.

Conversely, suppose we have found the decomposition given in Figure 7 (and hence we know that \( \sigma \in \mathcal{H}_5 \)). Let us construct a Soules matrix whose columns are the eigenvectors of the realising matrix. Examining the leaves in Figure 7, we see that we require \( 2 \times 2 \) Soules matrices \( R_1, R_2, R_3 \) and \( R_4 \), such that for matrices \( \Lambda_1 = \text{diag}(7, 5), \Lambda_2 = \text{diag}(6, -2), \Lambda_3 = \text{diag}(4, -4) \) and \( \Lambda_4 = \text{diag}(6, -6) \), the matrices

\[
\begin{align*}
\Lambda_1 &= R_1 \Lambda_1 R_1^T, \quad (192) \\
\Lambda_2 &= R_2 \Lambda_2 R_2^T, \quad (193) \\
\Lambda_3 &= R_3 \Lambda_3 R_3^T, \quad (194) \\
\Lambda_4 &= R_4 \Lambda_4 R_4^T. \quad (195)
\end{align*}
\]

are nonnegative and of the form

\[
\begin{align*}
\Lambda_1 &= \begin{bmatrix} 6 & * \\ * & 6 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} 0 & * \\ * & 4 \end{bmatrix}, \quad \Lambda_3 = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}, \quad \Lambda_4 = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}.
\end{align*}
\]

The required $R_i$ (and hence $A_i$) are easily calculable:

$$R_1 = R_3 = R_4 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad R_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{2}} \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}.$$  

Note that the $A_i$ are unique and the $R_i$ are unique up to scaling their second columns by a factor of $\pm 1$. These $2 \times 2$ matrices are our “building blocks”.

Applying Lemma 6.2.11 to (192) and (193) gives

$$R_{-2} \Lambda_{-2} R_{-2}^T = \begin{bmatrix} 6 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & 0 & 2\sqrt{3} \\ \frac{\sqrt{3}}{2} & 2\sqrt{3} & 4 \end{bmatrix} =: A_{-2}, \quad (196)$$

where

$$R_{-2} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix}, \quad \Lambda_{-2} = \text{diag}(7,5,-2).$$

Applying Lemma 6.2.11 to (196) and (194) gives

$$R_{-1} \Lambda_{-1} R_{-1}^T = \begin{bmatrix} 6 & \frac{1}{2} & \sqrt{3} & \sqrt{3} & \sqrt{3} & \sqrt{3} \\ \frac{1}{2} & 0 & \sqrt{6} & \sqrt{6} & \sqrt{6} & \sqrt{6} \\ \sqrt{3} & \sqrt{6} & 0 & 4 \\ \sqrt{3} & \sqrt{6} & 4 & 0 \end{bmatrix} =: A_{-1},$$

where

$$R_{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix}, \quad \Lambda_{-1} = \text{diag}(7,5,-2,-4).$$

Since 6 is not the $(4,4)$ entry of $A_{-1}$, let us define $\hat{R}_{-1} := PR_{-1}$ and $\hat{A}_{-1} := PA_{-1}P^T$, where

$$P := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$
Then
\[ \hat{R}_{-1} \Lambda_{-1} \hat{R}_{-1}^T = \hat{\Lambda}_{-1} \] (197)
and 6 is the (4,4) entry of \( \hat{\Lambda}_{-1} \). Hence, applying Lemma 6.2.11 to (197) and (195), we have that
\[
R_0 \Lambda_0 R_0^T = \begin{bmatrix}
0 & \sqrt{6} & \sqrt{6} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\
\sqrt{6} & 0 & 4 & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \\
\sqrt{6} & 4 & 0 & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \\
\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & 0 & 6 \\
\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & 6 & 0
\end{bmatrix} =: \Lambda_0,
\]
where
\[
R_0 = \begin{bmatrix}
\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & 0 & 0 \\
\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}}
\end{bmatrix}
\]
and \( \Lambda_0 = \text{diag}(7, 5, -2, -4, -6) \). Note that the matrix \( R_0 \) agrees with the matrix \( R \) of Example 6.2.9 up to row permutation and the matrix \( \Lambda_0 \) agrees with the matrix \( \Lambda \) of Example 6.2.9 up to permutation similarity.

### 6.5 Comparison to the Literature

Over the years, many realisability criteria have been given in the literature which guarantee that a list of real numbers \( \sigma \) be the spectrum of a nonnegative matrix. Consider, for example, the list of sufficient conditions given in Table 1. In this section, we demonstrate that if \( \sigma := (\lambda_1; \lambda_2, \ldots, \lambda_n) \) satisfies any of these criteria, then \( \sigma \in \mathcal{H}_n \).

First let us consider Conditions 1–9 of Table 1. In [40], the authors prove that if \( \sigma \) satisfies any of the conditions 1–9, then \( \sigma \) must satisfy either Condition 7 or Condition 9. Radwan [50] showed that Condition 7 is sufficient for the existence of a symmetric nonnegative matrix with spectrum \( \sigma \) and Soto [56] showed that the same is true of Condition 9. Hence all of the conditions 1–9 are sufficient for the SNIEP. Moreover, in [4], the authors show that if \( \sigma \) obeys either Condition 7 or Condition 9, then \( \sigma \) is C-realisable. Therefore, by Theorem 6.4.1, if \( \sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n) \) satisfies any of the criteria 1–9, then \( \sigma \in \mathcal{H}_n \).

Conditions 10 and 11 are quite different in that they guarantee the existence of a symmetric realising matrix with a certain set of diagonal elements, which, from the point of view of using Lemma 6.2.11, is important. First, we show that if the lists \((\lambda_1, \lambda_2, \ldots, \lambda_n)\) and \((a_1, a_2, \ldots, a_n)\) satisfy the hypotheses of Theorem 6.2.2, then...
\[(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{K}_n(a_1, a_2, \ldots, a_n).\] The proof is similar to the proof of [16, Theorem 4.4].

**Proposition 6.5.1.** Let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) and \( a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \) satisfy the conditions of Theorem 6.2.2. Then \( (\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{K}_n(a_1, a_2, \ldots, a_n) \).

**Proof.** The \( n = 1 \) case is trivial. If \( n = 2 \), then the conditions given in (106) hold and \( (\lambda_1; \lambda_2) \in \mathcal{K}_2(a_1, a_2) \) by definition. Now assume the statement holds for \( n = m - 1 \) and let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \) and \( a_1 \geq a_2 \geq \cdots \geq a_m \geq 0 \) satisfy

\[
\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} a_i : \quad k = 1, 2, \ldots, m - 1,
\]

\[
\sum_{i=1}^{m} \lambda_i = \sum_{i=1}^{m} a_i,
\]

\[
\lambda_k \leq a_{k-1} : \quad k = 2, 3, \ldots, m - 1.
\]

If we define \( \lambda'_2 := \lambda_1 + \lambda_2 - a_1 \), it is not difficult to see that \( \lambda'_2 \geq \lambda_2 \geq \lambda_3 \) and that \( \lambda'_2, \lambda_3, \ldots, \lambda_m \) and \( a_2, a_3, \ldots, a_m \) satisfy the analogous system. Hence, by the inductive hypothesis,

\[
(\lambda'_2; \lambda_3, \ldots, \lambda_m) \in \mathcal{K}_{m-1}(a_2, a_3, \ldots, a_m).
\]  \hspace{1cm} (198)

In addition, since \( \lambda_1 \geq \max(a_1, \lambda'_2) \) and \( \lambda_1 + \lambda_2 = a_1 + \lambda'_2 \), we have that

\[
(\lambda_1; \lambda_2) \in \mathcal{K}_2(a_1, \lambda'_2).
\]  \hspace{1cm} (199)

By Definition 6.3.1, (199) and (198) then imply that

\[
(\lambda_1; \lambda_2, \ldots, \lambda_m) \in \mathcal{K}_m(a_1, a_2, \ldots, a_m).
\]  \hspace{1cm} \( \square \)
Finally, we consider Condition 11. In [25], Holtz characterises the spectra of real symmetric anti-bidiagonal matrices, i.e. real symmetric matrices whose nonzero entries occur only on the anti-diagonal and immediately below it. We state this result below.

**Theorem 6.5.2.** [25] Let \( \sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n) \), where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). Then \( \sigma \) is the spectrum of a matrix of the form

\[
A = \begin{bmatrix}
0 & 0 & \cdots & 0 & b_n \\
0 & 0 & \cdots & b_{n-2} & b_{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & b_{n-2} & \cdots & 0 & 0 \\
b_n & b_{n-1} & \cdots & 0 & 0 \\
\end{bmatrix}, \quad b_1, b_2, \ldots, b_n > 0,
\]

if and only if

\[
\lambda_1 > -\lambda_n > \lambda_2 > -\lambda_{n-1} > \cdots > (-1)^{n-1}\lambda_{[\frac{n}{2}]} > 0. \tag{200}
\]

The realising matrix is unique.

Note that, in and of themselves, spectra of the form (200) are uninteresting from the point of view of the NIEP, since they are trivially realisable. From our point of view, the interesting fact about such spectra is that they may be realised by a nonnegative symmetric matrix with only one nonzero diagonal element.

**Proposition 6.5.3.** Let \( \lambda_1 \geq -\lambda_2 \geq \lambda_3 \geq \cdots \geq (-1)^{n-1}\lambda_n \geq 0 \) and let 
\[
a := \sum_{i=1}^{n} \lambda_i.
\]

Then \( (\lambda_1; \lambda_2, \ldots, \lambda_n) \in H_n(a, 0, 0, \ldots, 0) \).

**Proof.** Let us first assume \( n \) is odd and write \( n = 2m + 1 \). We will show that

\[
(\lambda_1; \lambda_3, \ldots, \lambda_{2m+1}) \in H_{m+1}(a, -\lambda_2, -\lambda_4, -\lambda_6, \ldots, -\lambda_{2m}). \tag{201}
\]

To see this, let \( (b_1, b_2, \ldots, b_{m+1}) \) be the reordering of \( (a, -\lambda_2, -\lambda_4, -\lambda_6, \ldots, -\lambda_{2m}) \) in descending order, i.e. for some \( s \),

\[
b_i = \begin{cases} 
-\lambda_{2i} & : i < s \\
\quad a & : i = s \\
-\lambda_{2(i-1)} & : i > s.
\end{cases}
\]

By Proposition 6.5.1, in order to show (201), it is sufficient to show that

\[
\sum_{i=1}^{k} \lambda_{2i-1} \geq \sum_{i=1}^{k} b_i : \quad k = 1, 2, \ldots, m, \tag{202}
\]

and

\[
\sum_{i=1}^{m+1} \lambda_{2i-1} = \sum_{i=1}^{m+1} b_i. \tag{203}
\]
and
\[ \lambda_{2k-1} \leq b_{k-1} : k = 2, 3, \ldots, m. \quad (204) \]

The fact that (203) holds is obvious and it is also not difficult to see that (204) holds. To see that (202) holds, note that for \( k \in \{1, 2, \ldots, s - 1\} \),
\[
\sum_{i=1}^{k} \lambda_{2i-1} \geq \sum_{i=1}^{k} (-\lambda_{2i}) = \sum_{i=1}^{k} b_i
\]
and for \( k \in \{s, s+1, \ldots, m\} \),
\[
\sum_{i=1}^{k} \lambda_{2i-1} \geq \sum_{i=1}^{k} \lambda_{2i-1} + \sum_{i=2k}^{2m+1} \lambda_i = \sum_{i=1}^{m+1} \lambda_{2i-1} + \sum_{i=k}^{m} \lambda_{2i} = a - \sum_{i=1}^{k-1} \lambda_{2i} = a + \sum_{i=1}^{s-1} (-\lambda_{2i}) + \sum_{i=s+1}^{k} (-\lambda_{2i-1}) = \sum_{i=1}^{k} b_i.
\]

This establishes (201). In addition, it is clear that
\[ (-\lambda_{2i}; \lambda_{2i}) \in \mathcal{H}_2(0,0) : i = 1, 2, \ldots, m. \]

Hence, by Definition 6.3.1, \((\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n(a,0,0,\ldots,0)\).

Now consider the case when \( n \) is even. Define
\[ a' := a - \lambda_n = \sum_{i=1}^{n-1} \lambda_i. \]
Since \( n-1 \) is odd,
\[ (\lambda_1; \lambda_2, \ldots, \lambda_{n-1}) \in \mathcal{H}_n(a',0,0,\ldots,0). \]
Furthermore, it is clear that
\[ (a'; \lambda_n) \in \mathcal{H}_2(a,0). \]

Therefore
\[ (\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n(a,0,0,\ldots,0). \]

Finally, note that \( \mathcal{H}_n \) is more general than any of the Conditions 1–11. The list \( \sigma := (25, 21, 18, 16, -10, -10, -10, -10, -10, -10, -10, -10) \) was shown to be C-realisable in [4] and was shown to satisfy \( S_3 \) in [57] (hence \( \sigma \in \mathcal{H}_{12} \) by Theorem 6.4.1), but \( \sigma \) does not satisfy any of the conditions 1–11.
7.1 INTRODUCTION

In Chapter 6, we showed that if \( \sigma \) obeys any of the previously known sufficient conditions in the SNIEP, then \( \sigma \) must lie in \( \mathcal{H}_n \) (equivalently, \( \mathcal{S}_n \)). In this chapter, we will show how to construct certain families of symmetrically realisable lists which lie outside of \( \mathcal{H}_n \). Let us write

\[
(\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{K}_n(a_1, a_2, \ldots, a_n)
\]

if \( (\lambda_1 \lambda_2, \ldots, \lambda_n) \) is the spectrum of a nonnegative symmetric matrix with Perron eigenvalue \( \lambda_1 \) and diagonal elements \( (a_1, a_2, \ldots, a_n) \).

We first mention a family of symmetrically realisable lists due to Laffey and Šmigoc which are not contained in \( \mathcal{H}_n \):

**Example 7.1.1.** Let us reconsider the classical example discussed in Example 1.3.12. As previously mentioned, the list

\[
\tau(t) := (3 + t, 3 - t, -2, -2, -2)
\]

is the spectrum of a nonnegative (not necessarily symmetric) matrix if and only if \( t \geq \sqrt{16\sqrt{6} - 39} \approx 0.437991 \); however, the minimum \( t \) for which \( \tau(t) \) is realisable by a symmetric matrix is \( t = 1 \). A proof of this fact can be found in [42].

Let us now consider the list

\[
\tau'(t) := (3 + t, 3 - t, 0, -2, -2, -2).
\]

By Theorem 6.3.13, \( \tau'(t) \in \mathcal{K}_6 \) precisely when \( \tau(t) \in \mathcal{K}_5 \) and hence \( \tau'(t) \notin \mathcal{K}_6 \) for \( t < 1 \); however, Laffey and Šmigoc [35] showed that \( \tau'(t) \) is realisable for \( t = 1/3 \). To see this, note that the matrix

\[
\begin{bmatrix}
2 & \sqrt{\frac{8}{3}} & 0 & 0 \\
\sqrt{\frac{8}{3}} & 0 & 4 & 0 \\
0 & 4 & 0 & \sqrt{\frac{8}{3}} \\
0 & 0 & \sqrt{\frac{8}{3}} & 2
\end{bmatrix}
\]

has spectrum \((10/3, 8/3, 0, -2)\). In addition, it is clear that \((2, -2) \in \mathcal{K}_2(0, 0) \) and hence, by Lemma 6.2.11, \((10/3, 8/3, 0, -2, -2)\) is realisable by a nonnegative symmetric matrix with diagonal elements \((2, 0, 0, 0, 0)\). Applying Lemma 6.2.11 a second time yields a nonnegative symmetric matrix with spectrum \( \tau'(1/3) \).

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7.2 A Construction Using 4 × 4 Jacobi Matrices

We aim to find lists \((\lambda_1, \lambda_2, \ldots, \lambda_n)\) and \((a_1, a_2, \ldots, a_n)\) such that
\[(\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathbb{R}_n(a_1, a_2, \ldots, a_n) \setminus \mathcal{H}_n(a_1, a_2, \ldots, a_n).

By Lemma 6.3.4, if we are to find lists of this type, then we will need to take \(n \geq 4\). Example 7.1.1 gives us one such list:
\[
\left(\frac{10}{3}; \frac{8}{3}, 0, -2\right) \in \mathbb{R}_4(2, 2, 0, 0) \setminus \mathcal{H}_4(2, 2, 0, 0).
\]

Drawing inspiration from Example 7.1.1, we will characterise the spectra of 4 × 4 nonnegative symmetric matrices of the form
\[
J := \begin{bmatrix}
a & b_1 & 0 & 0 \\
b_1 & 0 & b_2 & 0 \\
0 & b_2 & 0 & b_3 \\
0 & 0 & b_3 & a
\end{bmatrix}.
\]  

(205)

**Lemma 7.2.1.** Let \(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4\). Then \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) is the spectrum of a nonnegative symmetric matrix of the form (205) if and only if the following conditions are satisfied:

(i) \(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geq 0\);
(ii) \(-\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 \geq 0\);
(iii) \(-\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 \geq 0\);
(iv) \(\lambda_1^2 - \lambda_2^2 + \lambda_3^2 - \lambda_4^2 \geq 0\).

**Proof.** We will first reparameterise the problem:

**Claim 1:** Suppose \(\alpha, \beta\) and \(\gamma\) are real numbers such that
\[
1 + \alpha \geq 1 + \beta \geq \gamma \geq -\alpha - \beta - \gamma.
\]  

(206)

Then \((1 + \alpha, 1 + \beta, \gamma, -\alpha - \beta - \gamma)\) is the spectrum of a nonnegative matrix of the form
\[
J' := \begin{bmatrix}
1 & \sqrt{s} & 0 & 0 \\
\sqrt{s} & 0 & \sqrt{r} & 0 \\
0 & \sqrt{r} & 0 & \sqrt{t} \\
0 & 0 & \sqrt{t} & 1
\end{bmatrix}
\]

if and only if
\[
\beta \geq 0
\]  

(207)

and
\[
-\beta \leq \gamma \leq 1 - \frac{\beta(\alpha + \beta + 2)}{\alpha + \beta}.
\]  

(208)
We will first show that Claim 1 is equivalent to the statement of the lemma. Condition (i) is clearly necessary, so from now on, let us assume \( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geq 0 \). It is clear that the matrix \( J \) given in (205) has spectrum \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) if and only if \( J' \) has spectrum \((\lambda_1/a, \lambda_2/a, \lambda_3/a, \lambda_4/a)\), where \( a = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)/2 \). Hence, setting

\[
\alpha = \frac{\lambda_1}{a} - 1, \quad \beta = \frac{\lambda_2}{a} - 1, \quad \gamma = \frac{\lambda_3}{a},
\]

we see that the condition \( \beta \geq 0 \) is equivalent to (ii) and that the condition \( \gamma \geq -\beta \) is equivalent to (iii). Furthermore, the condition

\[
\gamma \leq 1 - \frac{\beta(\alpha + \beta + 2)}{\alpha + \beta}
\]

is equivalent to

\[
\frac{\lambda_1^2 - \lambda_2^2 + \lambda_3^2 - \lambda_4^2}{-\lambda_3 - \lambda_4} \geq 0;
\]

however, whenever (ii) holds,

\[-\lambda_3 - \lambda_4 \geq \lambda_1 - \lambda_2 \geq 0
\]

and hence, in this case, (209) is equivalent to (iv).

We will now prove Claim 1. Our first step is to note that, after the substitution

\[
s = s_0(s_0 + 1), \\
t = t_0(t_0 + 1),
\]

the characteristic polynomial of \( J' \) becomes

\[
f(x) = (x - s_0 - 1)(x - t_0 - 1)(x + t_0)(x + s_0) - r(x - 1)^2.
\]

Now define the functions

\[
K(\alpha, \beta, \gamma) := 2(\alpha + \beta + \alpha^2 + \beta^2 - \alpha^2\beta - \alpha\beta^2) - 2(\alpha + \beta)^2\gamma - 2(\alpha + \beta)\gamma^2
\]

and

\[
W_1(\alpha, \beta, \gamma) := \alpha - \alpha\beta - \beta + (\alpha + \beta)\gamma, \\
W_2(\alpha, \beta, \gamma) := \alpha - \alpha\beta - \beta(1 + \beta) - (\alpha + \beta)\gamma, \\
W_3(\alpha, \beta, \gamma) := 1 + \alpha + \beta - \alpha\beta - (\alpha + \beta)\gamma - \gamma^2.
\]

By comparing the coefficients of (210) with those of the polynomial

\[
g(x) = (x - 1 - \alpha)(x - 1 - \beta)(x - \gamma)(x + \alpha + \beta + \gamma),
\]

we deduce that, if

\[
r = (1 + \alpha + \beta)(\alpha + \gamma)(\beta + \gamma), \\
s = \frac{1}{4}K(\alpha, \beta, \gamma) + \frac{1}{2}\sqrt{W_1(\alpha, \beta, \gamma)W_2(\alpha, \beta, \gamma)W_3(\alpha, \beta, \gamma)},
\]

then

\[
\lambda_1 = s_0, \quad \lambda_2 = t_0, \quad \lambda_3 = s_0s_0, \quad \lambda_4 = t_0t_0.
\]
Then \( J' \) will have spectrum \((1 + \alpha, 1 + \beta, \gamma, -\alpha - \beta - \gamma)\), as required. Furthermore, it is not difficult to verify that, up to swapping the values of \( s \) and \( t \), \((211)\) is the only such solution.

First, let us suppose that \((207)\) and \((208)\) hold. We will show that the expressions for \( r \), \( s \) and \( t \) given in \((211)\) are nonnegative real numbers.

We see immediately that \((206)\) and \((207)\) imply \(\alpha + \gamma \geq \beta + \gamma \geq 0\) and so \( r \) is nonnegative. Next, we will show that \( K, W_1, W_2 \) and \( W_3 \) are nonnegative.

Consider \( K \) as a quadratic in \( \gamma \). The leading coefficient of this quadratic is negative and its roots are

\[
\theta_{\pm} := -\frac{\alpha + \beta}{2} \pm \frac{1}{2}\sqrt{\frac{(\alpha + \beta + 2)((\alpha - \beta + 1)^2 - 1 + 4\beta)}{\alpha + \beta}}.
\]

In order to show that \( K \) is nonnegative, it is sufficient to show that

\[
\theta_- \leq \gamma \leq \theta_+,
\]

or, considering \((208)\), it suffices to show that

\[
\theta_- \leq -\beta \quad \quad (212)
\]

and

\[
1 - \frac{\beta(\alpha + \beta + 2)}{\alpha + \beta} \leq \theta_+. \quad \quad (213)
\]

To see that \((212)\) holds, observe that

\[
-\beta - \theta_- = \frac{\alpha - \beta}{2} + \frac{1}{2}\sqrt{\frac{(\alpha + \beta + 2)((\alpha - \beta + 1)^2 - 1 + 4\beta)}{\alpha + \beta}} \geq 0.
\]

To see that \((213)\) holds, observe that

\[
\theta_+ = \left(1 - \frac{\beta(\alpha + \beta + 2)}{\alpha + \beta}\right) = \frac{1}{2}\left(\sqrt{\frac{(\alpha + \beta + 2)((\alpha - \beta + 1)^2 - 1 + 4\beta)}{\alpha + \beta}} - (\alpha - \beta)(\alpha + \beta + 2)\right)
\]

and that

\[
\frac{(\alpha + \beta + 2)((\alpha - \beta + 1)^2 - 1 + 4\beta)}{\alpha + \beta} - \left(\frac{(\alpha - \beta)(\alpha + \beta + 2)}{\alpha + \beta}\right)^2 = \frac{8\alpha \beta (\alpha + \beta + 2)}{(\alpha + \beta)^2} \geq 0.
\]

Therefore \( K \) is nonnegative.

Now, as \( W_1 \) is an increasing function of \( \gamma \), we have

\[
W_1(\alpha, \beta, \gamma) \geq W_1(\alpha, \beta, -\beta) = (\alpha - \beta) + (\alpha^2 - \beta^2) \geq 0.
\]
Similarly, as $W_2$ is a decreasing function of $\gamma$, we see that

$$W_2(\alpha, \beta, \gamma) \geq W_2\left(\alpha, \beta, 1 - \frac{\beta(\alpha + \beta + 2)}{\alpha + \beta}\right) = 0.$$ 

As in our analysis of $K$, let us consider $W_3$ as a quadratic in $\gamma$ with negative leading coefficient and roots

$$\phi_{\pm} := \frac{1}{2} \left(-\alpha - \beta \pm \sqrt{(-\alpha - \beta)^2 + 8\beta}\right).$$

To show that $W_3$ is nonnegative, it is sufficient to show that

$$\phi_- \leq \gamma \leq \phi_+,$$

or, bearing in mind (208), it suffices to show that

$$\phi_- \leq -\beta \tag{214}$$

and

$$1 - \frac{\beta(\alpha + \beta + 2)}{\alpha + \beta} \leq \phi_+. \tag{215}$$

Because of (212), in order to demonstrate that (214) holds, we need only show that $\phi_- \leq \theta_-$. Similarly, because of (213), we may demonstrate (214) by showing that $\theta_+ \leq \phi_+$. To see that $\phi_- \leq \theta_- \text{ and } \theta_+ \leq \phi_+$, we note that

$$\theta_- - \phi_- =$$

$$\frac{1}{2} \left(\sqrt{(\alpha - \beta + 2)^2 + 8\beta} - \sqrt{\frac{(\alpha + \beta + 2)((\alpha - \beta + 1)^2 - 1 + 4\beta)}{\alpha + \beta}}\right) = \phi_+ - \theta_+$$

and that

$$(\alpha - \beta + 2)^2 + 8\beta - \frac{(\alpha + \beta + 2)((\alpha - \beta + 1)^2 - 1 + 4\beta)}{\alpha + \beta} = \frac{8\alpha\beta}{\alpha + \beta} \geq 0.$$ 

Hence $W_3$ is nonnegative.

Since $K, W_1, W_2, W_3 \geq 0$, it follows that $s$ is a nonnegative real number. To see that $t$ is nonnegative, it suffices to show that

$$K(\alpha, \beta, \gamma)^2 \geq 4W_1(\alpha, \beta, \gamma)W_2(\alpha, \beta, \gamma)W_3(\alpha, \beta, \gamma),$$

which holds since, by (208),

$$K(\alpha, \beta, \gamma)^2 - 4W_1(\alpha, \beta, \gamma)W_2(\alpha, \beta, \gamma)W_3(\alpha, \beta, \gamma) =$$

$$16\alpha\beta(1 - \gamma)(\alpha + \beta + \gamma + 1) \geq 16\alpha\beta \left(\frac{\beta(\alpha + \beta + 2)}{\alpha + \beta}\right)(\alpha + 1) \geq 0.$$

Now suppose the expressions for $r$, $s$ and $t$ given in (211) are nonnegative real numbers. We will show that this implies (207) and (208) hold.
Consider the characteristic polynomial of $J'$ (210). We have:

$$
\begin{align*}
    f(-s_0) &= -\tau(s_0 + 1)^2 \leq 0, \\
    f(-t_0) &= -\tau(t_0 + 1)^2 \leq 0, \\
    f(1) &= s_0 t_0 (1 + t_0) (1 + s_0) \geq 0, \\
    f(1 + t_0) &= -\tau s_0^2 \leq 0, \\
    f(1 + s_0) &= -rs_0^2 \leq 0.
\end{align*}
$$

Therefore, $f$ has a root in each of the intervals $(-\infty, -s_0), (-t_0, 1), (1, 1 + t_0)$ and $(1 + s_0, \infty)$. In particular, $f$ has two roots in $(1, \infty)$. Hence (207) holds.

Next, note that, since $r \geq 0$, either $\gamma \geq -\beta$ or $\gamma \leq -\alpha$; however, because of our restriction that $\gamma \geq -\alpha - \beta - \gamma$, the latter option is impossible.

It remains to show that

$$
\gamma \leq 1 - \frac{\beta(\alpha + \beta + 2)}{\alpha + \beta}. \tag{216}
$$

From (211), we see that $t \geq 0$ implies $K(\alpha, \beta, \gamma) \geq 0$, which, as we have already shown, implies

$$
\gamma \leq \theta_+ \leq \phi_+
$$

and hence $W_3(\alpha, \beta, \gamma) \geq 0$. In addition, we have shown that $\gamma \geq -\beta$ implies $W_1(\alpha, \beta, \gamma) \geq 0$. Therefore, for $s$ and $t$ to be real, we must have $W_2(\alpha, \beta, \gamma) \geq 0$ also. As previously noted, this occurs precisely when (216) holds. \hfill \Box

**Definition 7.2.2.** If $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is the spectrum of a nonnegative symmetric matrix of the form (205) with Perron eigenvalue $\lambda_1$, then we write $(\lambda_1; \lambda_2, \lambda_3, \lambda_4) \in \mathcal{B}_4(a, a, 0, 0)$, where $a = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)/2$.

### 7.3 Examples

Using $4 \times 4$ matrices of the form (205), together with arbitrary $2 \times 2$ nonnegative symmetric matrices, one can construct (using Lemma 6.2.11) families of symmetrically realisable lists which are not contained in $\mathcal{F}_n$.

We will consider the following generalisations of the lists $\tau(t)$ and $\tau'(t)$ in Example 7.1.1:

$$
\begin{align*}
    \tau_k(t) &:= (k + 3 + t, \underbrace{k + 3, \ldots, k + 3}_{k \text{ times}}, k + 3 - t, \underbrace{-k - 2, \ldots, -k - 2}_{k + 3 \text{ times}}), \\
    \tau'_k(t) &:= (k + 3 + t, \underbrace{k + 3, \ldots, k + 3}_{k \text{ times}}, k + 3 - t, \underbrace{0, -k - 2, \ldots, -k - 2}_{k + 3 \text{ times}}).
\end{align*}
$$

$k = 0, 1, \ldots$
Example 7.3.1 (k = 1). Suppose we seek the minimum t for which
\[ \tau'_1(t) := (4 + t, 4, 4 - t, 0, -3, -3, -3, -3) \]
is symmetrically realisable. By Theorem 6.3.13, \( \tau'_1(t) \in \mathcal{K}_8 \) precisely when
\[ \tau_1(t) = (4 + t, 4, 4 - t, -3, -3, -3, -3) \in \mathcal{K}_7, \]
and, applying the algorithmic methods of Appendix B, one can determine that this happens if and only if \( t \geq 2 \).

It turns out, however, that \( \tau'_1(t) \) is symmetrically realisable for \( t \geq 1 \).
To see this, note that the list \((5, 4, 0, -3)\) satisfies the conditions of Lemma 7.2.1 and hence \((5, 4, 0, -3) \in \mathcal{J}_4(3, 3, 0, 0)\). In addition, note that \( t \geq 1 \) guarantees \((4 + t, 4 - t) \in \mathcal{K}_2(5, 3)\) and hence we may use Lemma 6.2.11 to decompose \( \tau'_1(t) \) as shown in Figure 9. We denote by \( \bar{\theta} \) a list of zeros of appropriate length.

\[
\begin{align*}
(4 + t; 4, 4 - t, 0, -3, -3, -3, -3) &\in \mathcal{R}_8(\bar{\theta}) \\
(4 + t; 4, 4 - t, 0, -3, -3, -3) &\in \mathcal{R}_7(3, \bar{\theta}) \\
(4 + t; 4, 4 - t, 0, -3, -3) &\in \mathcal{R}_6(3, 3, \bar{\theta}) \\
(4 + t; 4 - t, -3) &\in \mathcal{K}_3(5, \bar{\theta}) \\
(4 + t; 4 - t, -3) &\in \mathcal{K}_3(5, \bar{\theta}) \\
(5; 4, 0, -3) &\in \mathcal{J}_4(3, 3, \bar{\theta}) \\
(3; -3) &\in \mathcal{K}_2(\bar{\theta}) \\
(3; -3) &\in \mathcal{K}_2(\bar{\theta}) \\
(3; -3) &\in \mathcal{K}_2(\bar{\theta})
\end{align*}
\]

Figure 9: Decomposition of \( \tau'_1(t) \) for \( t \geq 1 \)

Example 7.3.2 (k = 2). Suppose we seek the minimum t for which
\[ \tau'_2(t) := (5 + t, 5, 5 - t, 0, -4, -4, -4, -4) \]
is symmetrically realisable. By Theorem 6.3.13, \( \tau'_2(t) \in \mathcal{K}_{10} \) precisely when
\[ \tau_2(t) = (5 + t, 5, 5 - t, -4, -4, -4, -4) \in \mathcal{K}_9, \]
and, as before, applying the algorithmic methods of Appendix B, we see that this occurs precisely when \( t \geq 6 \).

It turns out, however, that \( \tau'_2(t) \) is symmetrically realisable for \( t \geq 2 \).
To see this, note that the list \((7, 5, 0, -4)\) satisfies the conditions of Lemma 7.2.1 and hence, from Figure 10, we see that
\[ (5 + t; 5, 5 - t, 0, -4) \in \mathcal{R}_6(4, 4, 4, 4, 0, 0) \]
for all $t \geq 2$. All that remains is to use the four instances of the diagonal element $4$ to append the remaining four instances of the eigenvalue $-4$.

\[
(5 + t; 5, 5, 5 - t, 0, -4) \in \mathcal{R}_6(4, 4, 4, 4, 0, 0)
\]

\[
(5 + t; 5, 5, 0, -4) \in \mathcal{R}_5(3 + t, 4, 4, 0, 0)
\]

\[
(5 + t; 5) \in \mathcal{H}_2(7, 3 + t)
\]

\[
(7; 5, 0, -4) \in \mathcal{J}_4(4, 4, 0, 0)
\]

Figure 10: Decomposition of $\tau'_2(t)$ for $t \geq 2$

7.4 Conclusion

The Soules set (equivalently $\mathcal{H}_n$) has been extensively studied\(^1\) and many of its essential properties were explored in Chapter 6. Moving forward, one can consider finding symmetrically realisable lists which lie outside of this set. This chapter is a first step in this direction.

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\(^1\) See the references in Section 6.1.
Part IV

CONCLUSIONS
The problem of characterising the spectra of nonnegative matrices—first posed nearly seventy years ago—seems unlikely to be solved in the near future; however, in this thesis, we have made advances in several sub-problems in this area.

In chapter 4, we saw the importance of various constructive methods in the NIEP, which, given a realisable list \( \sigma \) (and realising matrix \( A \)), allow for the construction of a new realisable list \( \sigma' \) with realising matrix \( \Lambda' \). In particular, in [65], Šmigoc gives a construction to replace the Perron eigenvalue of a realisable list with a new list \((\mu_1, \mu_2, \ldots, \mu_n)\), and in [66], she shows how to replace the pair \((\rho, \lambda_2)\)—where \( \rho \) is the Perron eigenvalue and \( \lambda_2 \) is real—with a certain triple \((\mu_1, \mu_2, \mu_3)\). We have developed this idea and shown that \((\rho, \lambda_2)\) may be replaced by a list \((\mu_1, \mu_2, \ldots, \mu_n)\) of arbitrary length, assuming \((\mu_1, \mu_2, \ldots, \mu_n)\) obey certain properties. In addition, we have given a method by which one can replace the triple \((\rho, \alpha + i\beta, \alpha - i\beta)\) by a suitable list \((\mu_1, \mu_2, \mu_3, \mu_4)\). These constructions are, in essence, new tools which may be used to prove that a given list is realisable.

When a given list \( \sigma \) is known to be realisable, it is natural to consider the properties (or possible properties) of a matrix which realises \( \sigma \). In particular, we considered the question of characterising the possible diagonal elements of such a matrix, motivated, partly, by the importance of diagonal elements to the constructions mentioned above. Perhaps the most general class of lists for which the NIEP has been solved (by Laffey and Šmigoc in [34]) are lists of the form \((\rho, \lambda_2, \lambda_3, \ldots, \lambda_n)\), where \( \rho \) is nonnegative and \( \lambda_2, \lambda_3, \ldots, \lambda_n \) have nonpositive real parts. For lists of this type, a complete (and remarkably simple) characterisation of the possible diagonal elements was achieved. We saw that this characterisation turns out to be a generalisation of Laffey and Šmigoc’s solution to the NIEP for lists with nonpositive real parts.

In order to establish the diagonal result, it was necessary to first uncover several fundamental facts regarding the coefficients of certain real polynomials—facts which may well be of independent interest.

If the real polynomial

\[
 f(x) := x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n
\]

has nonnegative roots \( x_1, x_2, \ldots, x_n \), then for each \( k \), the product \( b_{k-1} b_{k+1} \) is nonnegative. Furthermore, Newton’s famous inequali-
ties state that $b_k^2 \geq b_{k-1}b_{k+1}$. We have given families of “Newton-like” inequalities for the case when $x_1, x_2, \ldots, x_n$ are complex numbers with nonnegative real parts.

On the other hand, if $x_1$ is nonnegative and $x_2, x_3, \ldots, x_n$ have nonpositive real parts, then the signs of the coefficients $b_i$ are unknown; however, Laffey and Smigoc [34] showed that in this case, $b_1, b_2 \leq 0$ is enough to guarantee that each of the remaining coefficients $b_3, b_4, \ldots$ are nonpositive also. We have generalised this result by proving that if a given even coefficient $b_{2k}$ is nonpositive, then every even coefficient $b_{2k+2}, b_{2k+4}, \ldots$ that follows is nonpositive also (with an analogous statement holding for the odd coefficients). Real polynomials with one positive root are of particular interest in the NIEP, but are also of interest elsewhere, for example in polynomial real root isolation.

The SNIEP is also extensively studied in this thesis. Many sufficient conditions for this problem (and for the RNIEP) have been given over the years, beginning with the work of Fiedler in 1974. We have introduced a new recursive method for constructing symmetrically realisable lists (and named the resulting set of realisable lists $\mathcal{H}_n$). Using our recursive construction, it was possible to unify a great deal of the literature on these subjects: we showed that if a given list $\sigma$ satisfies essentially any previously known sufficient condition for the SNIEP/RNIEP, then $\sigma$ must lie in $\mathcal{H}_n$. In the process, many interesting properties of these lists were revealed, including a complete algorithmic characterisation.

In light of these results, it was natural to seek symmetrically realisable lists which lie outside of $\mathcal{H}_n$. The thesis concludes by giving some families of lists of this type.

### 8.2 Future work

In this thesis, we have characterised the possible diagonal elements of a realising matrix for lists with negative real parts, and, as mentioned above, we have seen that diagonal elements play a crucial role in several constructive methods in the NIEP. Therefore, it is a logical next step to consider how these two ideas might be combined in order to develop new sufficient conditions for the NIEP.

Lists with one positive entry and and $n - 1$ entries with negative real parts have, by their nature, a large spectral gap, and it appears that realisation becomes more challenging as the spectral gap decreases. It would be of great interest to develop this idea more formally and discover precisely how the spectral gap influences realisability. In [37], Laffey and Šmigoc consider lists with two positive entries and $n - 2$ negative entries. Perhaps the next step is to consider lists with two positive entries and $n - 2$ entries with negative real parts.
In the symmetric problem, the set $\mathcal{H}_n$ is now well understood\textsuperscript{1}. Moving forward, we aim to find further examples of structured non-negative matrices whose spectra lie outside $\mathcal{H}_n$, and it is our hope that this work may lead to general sufficient conditions for realisability outside this set. It would also be of interest to consider the size of $\mathcal{H}_n$ in relation to the entire set of symmetrically realisable lists of length $n$ (called $\mathcal{R}_n$ in Chapter 6). It would be interesting to see how the proportion of $\mathcal{R}_n$ occupied by $\mathcal{H}_n$ depends on $n$ (for $n \leq 4$, the sets are equal).

\textsuperscript{1} See Chapter 6 and the references contained therein.
Part V

APPENDICES
Recall the statement of Theorem 2.3.1, which we recall here for convenience:

**Theorem A.0.1.** Consider the polynomial

\[ f(x) := x^n + (a_1 + ib_1)x^{n-1} + \cdots + (a_n + ib_n), \]

where the \(a_i\) and \(b_i\) are real. Suppose \(f\) has \(n_+\) roots with positive imaginary part, \(n_-\) roots with negative imaginary part and \(n_0\) real roots (\(n_+ + n_- + n_0 = n\)). Let

\[ P(x) := x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n, \]
\[ Q(x) := b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n \]

and \(d := n - 2\min(n_+, n_-)\). Then (counting multiplicities) there exist at least \(d\) real roots of \(P\) (say \(\mu_1, \mu_2, \ldots, \mu_d\)) and at least \(d - 1\) real roots of \(Q\) (say \(\nu_1, \nu_2, \ldots, \nu_{d-1}\)) such that

\[ \mu_1 \leq \nu_1 \leq \mu_2 \leq \nu_2 \leq \cdots \leq \nu_{d-1} \leq \mu_d. \]  (217)

If \(n_0 = 0\), then the inequalities in (217) may be assumed to be strict.

In general, if the imaginary parts of the roots of \(f\) are neither all nonnegative nor all nonpositive, then the real roots of \(P\) and \(Q\) do not necessarily interlace. Here, we will find interlacing subsets of the real roots of \(P\) and \(Q\) of the required length, without reference to Cauchy indices. In order to describe this properly, we will require some notation.

Let \(X := (x_1, x_2, \ldots, x_s)\) and \(Z := (z_1, z_2, \ldots, z_t)\) be finite, nondecreasing, disjoint sequences of real numbers. We write \(#(X|Z) := (s, t)\) to denote the lengths of \(X\) and \(Z\). If \(\theta, \phi \in \mathbb{R} \cup \{-\infty, \infty\}\) with \(\theta < \phi\), let \((X|Z)|_{(\theta, \phi)}\) denote the “restriction” of \((X|Z)\) to the interval \((\theta, \phi)\); specifically,

\[ (X|Z)|_{(\theta, \phi)} := (x_i : \theta < x_i < \phi | z_i : \theta < z_i < \phi). \]

If

\[ (X|Z) = (x_1, x_2, \ldots, x_s|z_1, z_2, \ldots, z_t), \]
\[ (X'|Z') = (x'_1, x'_2, \ldots, x'_s|z'_1, z'_2, \ldots, z'_t) \]

and \(\max\{x_s, z_t\} < \min\{x'_1, z'_1\}\), we define

\[ (X|Z) \cup (X'|Z') := \]
An inductive proof of theorem 2.3.1

If \( \mathcal{X} \) and \( \mathcal{X}^* \) are two subsequences of \( \mathcal{X} \) and \( \mathcal{Z} \) and \( \mathcal{Z}^* \) are two subsequences of \( \mathcal{Z} \), we write \((\mathcal{X}|\mathcal{Z}) \sim (\mathcal{X}^*|\mathcal{Z}^*)\) if the following conditions hold:

(i) \( \#(\mathcal{X}|\mathcal{Z}) = \#(\mathcal{X}^*|\mathcal{Z}^*) \);

(ii) if \( (\mathcal{X}|\mathcal{Z}) = (x_{k_1}, x_{k_2}, \ldots, x_{k_n}|z_1, z_2, \ldots, z_t) \)
and \( (\mathcal{X}^*|\mathcal{Z}^*) = (x_{p_1}, x_{p_2}, \ldots, x_{p_m}|z_{q_1}, z_{q_2}, \ldots, z_{q_r}) \),
then for all \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, r \), \( x_{k_i} < z_{l_j} \) precisely when \( x_{p_i} < z_{q_j} \).

Clearly, \( \sim \) is an equivalence relation.

Now, given \( \mathcal{X} \) and \( \mathcal{Z} \), we will construct specific subsequences \( \mathcal{X} \) and \( \mathcal{Z} \) in such a way that \( \mathcal{X} \) and \( \mathcal{Z} \) strictly interlace. The pair \((\mathcal{X}|\mathcal{Z})\) is obtained from \((\mathcal{X}^*|\mathcal{Z}^*)\) via the following algorithm:

1. Consider those elements \( x \) of \( \mathcal{X} \) which occur with multiplicity greater than one. If \( x \) occurs with even multiplicity, then delete all instances of \( x \) from \( \mathcal{X} \). If \( x \) occurs with odd multiplicity, then delete all but one instance of \( x \) from \( \mathcal{X} \). Similarly, consider those elements \( z \) of \( \mathcal{Z} \) which occur with multiplicity greater than one.

2. Arrange the elements of \( \mathcal{X} \cup \mathcal{Z} \) in increasing order, say

\[
\cdots < x_{k_j} < x_{k_{j+1}} < \cdots < x_{k_{j+1}-1} < z_{l_1} < z_{l_{j+1}} < \cdots < z_{l_{j+1}-1} < x_{k_{j+1}} < x_{k_{j+1}+1} < \cdots < x_{k_{j+2}-1} < \cdots .
\]

3. Consider each subsequence \( \mathcal{X}^{(j)} := (x_{k_j}, x_{k_{j+1}}, \ldots, x_{k_{j+1}-1}) \) of \( \mathcal{X} \). If \( \mathcal{X}^{(j)} \) has even length, delete \( x_{k_j}, x_{k_{j+1}}, \ldots, x_{k_{j+1}-1} \) from \( \mathcal{X} \). If \( \mathcal{X}^{(j)} \) has odd length, delete \( x_{k_{j+1}}, x_{k_{j+2}}, \ldots, x_{k_{j+1}-1} \) from \( \mathcal{X} \). Similarly, if \( \mathcal{Z}^{(j)} := (z_{l_j}, z_{l_{j+1}}, \ldots, z_{l_{j+1}-1}) \) has even length, delete \( z_{l_j}, z_{l_{j+1}}, \ldots, z_{l_{j+1}-1} \) from \( \mathcal{Z} \) and if \( \mathcal{Z}^{(j)} \) has odd length, delete \( z_{l_{j+1}}, z_{l_{j+2}}, \ldots, z_{l_{j+1}-1} \) from \( \mathcal{Z} \).

4. Repeat steps 2 and 3 until strictly interlacing subsequences \( \mathcal{X} \) and \( \mathcal{Z} \) have been obtained.

We write

\[ \mathcal{L}(\mathcal{X}|\mathcal{Z}) := (\mathcal{X}|\mathcal{Z}) , \]

where \( \mathcal{X} \) and \( \mathcal{Z} \) are given by the procedure above. Since \( \mathcal{X} \) and \( \mathcal{Z} \) strictly interlace, \( \#(\mathcal{L}(\mathcal{X}|\mathcal{Z})) = (\sigma, \tau) \) implies \( |\sigma - \tau| \leq 1 \).
Example A.0.2.
\[ \mathcal{L}(0, 0, 1, 3, 4, 6, 7, 9, 10, 11) \sim (2, 5, 5, 8) = (1, 9[2]). \]

This can be seen from the following series of steps:

1. \((0, 0, 1), (2), (3, 4), (5, 5, 5), (6, 7), (8), (9, 10, 11)\)

2. \((1), (2), (3, 4), (5, 6, 7), (8), (9, 10, 11)\)

3. \((1), (2, 5, 8), (9)\)

4. \((1), (9)\)

Observation A.0.3. Let \(X := (x_1, x_2, \ldots, x_s)\) and \(Z := (z_1, z_2, \ldots, z_t)\) be finite, nondecreasing, disjoint sequences of real numbers and suppose

\[ \mathcal{L}(X|Z) = (x_{s_1}, x_{s_2}, \ldots, x_{s_k}, z_{t_1}, z_{t_2}, \ldots, z_{t_L}). \]

Then the elements \(x_{k_1}, x_{k_2}, \ldots, x_{k_s}\) each have odd multiplicity in \(X\) and the elements \(z_{l_1}, z_{l_2}, \ldots, z_{l_t}\) each have odd multiplicity in \(Z\). Furthermore, if \(x_{k_1} < z_{l_1}\) and \(x_{k_s} < z_{l_s}\), then

\[ \mathcal{L}\left( (X|Z)|_{(-\infty, x_{k_1})} \right) = \mathcal{L}\left( (X|Z)|_{(x_{k_1}, z_{l_1})} \right) = \mathcal{L}\left( (X|Z)|_{(z_{l_1}, x_{k_2})} \right) = \cdots = \mathcal{L}\left( (X|Z)|_{(z_{l_s}, \infty)} \right) = (\emptyset|\emptyset). \]

If \(z_{l_1} < x_{k_1}\) or \(z_{l_s} < x_{k_s}\), then the analogous statement holds.

Observation A.0.4. Let \(X\) and \(Z\) be finite nondecreasing disjoint sequences of real numbers and let \(\emptyset < \Phi\) with \(\emptyset, \Phi \notin X\) and \(\emptyset, \Phi \notin Z\). Then

\[ \mathcal{L}(X|Z) \sim \mathcal{L}\left( (X|Z)|_{(-\infty, \emptyset)} \cup (X|Z)|_{(\emptyset, \Phi)} \cup (X|Z)|_{(\Phi, \infty)} \right). \]

Observation A.0.4 states that when computing \(\mathcal{L}(X|Z)\), it essentially does not matter in which order the steps of the algorithm are carried out.

We are now ready to give our inductive proof of Theorem A.0.1.

Proof of Theorem A.0.1. We will first prove the result for those polynomials \(f\) for which \(n_0 = 0\) and \(n_+ \geq n_-\). In this case, the proof will essentially be by induction on \(n_-\). If \(n_- = 0\), then the conclusion follows from Lemma 2.2.4. Now assume \(n_- > 0\) and suppose the roots of \(f\) are

\[ (\alpha_1 - i\beta_1, \ldots, \alpha_{n_-} - i\beta_{n_-}, \hat{\alpha}_1 + i\hat{\beta}_1, \ldots, \hat{\alpha}_{n_+} + i\hat{\beta}_{n_+}), \]

where \(\alpha_j, \beta_j \in \mathbb{R}\) and \(\beta_j, \hat{\beta}_j > 0\) for \(j = 1, 2, \ldots\). We may write

\[ f(x) = (x - \alpha_1 + i\beta_1)(P'(x) + iQ'(x)) \]

for some real polynomials

\[ P'(x) = x^{n-1} + a'_1x^{n-2} + a'_2x^{n-3} + \cdots + a'_{n-1}. \]
and

\[ Q'(x) := b_1'x^{n-2} + b_2'x^{n-3} + \cdots + b_{n-1}'. \]

The real part of \( f(x) \) is then given by

\[ P(x) = (x - \alpha_1)P'(x) - \beta_1Q'(x) \]

and the imaginary part is given by

\[ Q(x) = \beta_1P'(x) + (x - \alpha_1)Q'(x). \]

Let \( \mathcal{X} \) be the sequence of real roots of \( P \) and let \( \mathcal{Z} \) be the sequence of real roots of \( Q \), arranged in nondecreasing order. Since \( n_0 = 0 \), it is clear that \( P \) and \( Q \) have no real root in common, i.e. \( \mathcal{X} \) and \( \mathcal{Z} \) are disjoint. Similarly, let \( \mathcal{X}' \) be the sequence of real roots of \( P' \) and let \( \mathcal{Z}' \) be the sequence of real roots of \( Q' \), again arranged in nondecreasing order. For convenience, we also introduce the following notation: if \( \theta, \phi \in \mathbb{R} \cup (-\infty, \infty) \) with \( \theta < \phi \), define

\[ L(\theta, \phi) := \mathcal{L}((\mathcal{X}|\mathcal{Z})|_{(\theta, \phi)}) \]

and

\[ L'(\theta, \phi) := \mathcal{L}((\mathcal{X}'|\mathcal{Z}')|_{(\theta, \phi)}). \]

We now need to establish a series of claims which show how \( L(\theta, \phi) \) depends on \( L'(\theta, \phi) \).

**Claim 1:** Let \( \theta < \phi \) with \( \alpha_1 \not\in [\theta, \phi] \). Suppose \( \theta \) is a root of \( P' \) and \( \phi \) is a root of \( Q' \) (or vice versa) and suppose \( L'(\theta, \phi) = (\theta|\phi) \). If \( P(\theta) \) and \( P(\phi) \) have opposite sign, then \( \#(L(\theta, \phi)) = \{1, 0\} \). Similarly, if \( Q(\theta) \) and \( Q(\phi) \) have opposite sign, then \( \#(L(\theta, \phi)) = \{0, 1\} \).

Without loss of generality, let \( \theta \) be a root of \( P' \) and \( \phi \) be a root of \( Q' \). Suppose \( P(\theta) \) and \( P(\phi) \) have opposite sign. Without loss of generality, suppose further that

\[ P(\theta) = -\beta_1Q'(\theta) < 0, \]
\[ P(\phi) = (\phi - \alpha_1)P'(\phi) > 0 \]

and \( \alpha_1 < \theta \). This implies

\[ Q(\theta) = (\theta - \alpha_1)Q'(\theta) > 0, \]
\[ Q(\phi) = \beta_1P'(\phi) > 0. \]

Since \( L'(\theta, \phi) = (\theta|\theta) \), \( P' \) and \( Q' \) each have an even number of real roots in the interval \((\theta, \phi)\). Suppose that, combined, \( P' \) and \( Q' \) have \( 2k \) roots in \((\theta, \phi)\). The proof of Claim 1 is by induction on \( k \).

First suppose neither \( P' \) nor \( Q' \) have any real roots in \((\theta, \phi)\). We first note that since \( P(\theta) \) and \( P(\phi) \) have opposite sign, \( P \) must have an odd number of roots in \((\theta, \phi)\). Since neither \( P'(x) \) nor \( Q'(x) \) have a root in \((\theta, \phi)\), we must have that \( P'(x) \) and \( Q'(x) \) are strictly positive for all \( \theta < x < \phi \). Hence \( Q(x) \) is strictly positive on this interval. Since
P has an odd number of roots in (θ, φ) and Q has no roots in (θ, φ), it follows that \#(L(θ, φ)) = (1, 0).

For illustrative purposes, let us now consider the case when (θ, φ) contains precisely two roots of \(P'\), say \(x'_j < x'_{j+1}\), and no roots of \(Q'\). Since \(P'(\phi) > 0\), we see that \(P'\) is strictly negative on \((x'_j, x'_{j+1})\). As before, \(Q'\) is strictly positive on \([θ, φ]\). Hence \(P\) is strictly negative on \([x'_j, x'_{j+1}]\). Similarly, since \(P'\) is strictly positive on \([θ, x'_j) ∪ (x'_{j+1}, φ]\), we see that \(Q(x) > 0\) for all \(x \in [θ, x'_j) ∪ [x'_{j+1}, φ]\). Therefore, the interval \([θ, x'_j]\) contains an even number of roots of \(P\) and no roots of \(Q\), the interval \((x'_{j+1}, φ]\) contains no roots of \(P\) and an even number of roots of \(Q\), and the interval \([x'_j, x'_{j+1})\) contains an odd number of roots of \(P\) and no roots of \(Q\). As before, it follows that \#(L(θ, φ)) = (1, 0).

Now assume the statement holds if, combined, \(P'\) and \(Q'\) have \(2(k - 1)\) roots in \((θ, φ)\). We now consider the case of \(2k\) roots. Since \(L'(θ, φ) = (θ|θ)\), there exists either

1. a pair of real roots of \(P'\), say \(x'_j, x'_{j+1}\), such that either \(x'_j = x'_{j+1}\), or \(x'_j < x'_{j+1}\) and \((x'_j, x'_{j+1})\) does not contain a root of \(P'\) or \(Q'\); or
2. a pair of real roots of \(Q'\), say \(z'_j, z'_{j+1}\), such that either \(z'_j = z'_{j+1}\), or \(z'_j < z'_{j+1}\) and \((z'_j, z'_{j+1})\) does not contain a root of \(P'\) or \(Q'\).

Without loss of generality, assume 1 is the case.

Suppose \(x'_j < x'_{j+1}\). Without loss of generality, suppose \(Q'\) is positive on \([x'_j, x'_{j+1}]\). There are four cases to consider:

1. If \(P'\) is negative on \((x'_j, x'_{j+1})\) and \(α_1 < θ\), then \(P\) is negative on \([x'_j, x'_{j+1}]\) and \(Q(x'_j), Q(x'_{j+1}) > 0\).
2. If \(P'\) is positive on \((x'_j, x'_{j+1})\) and \(α_1 > φ\), then \(P\) is negative on \([x'_j, x'_{j+1}]\) and \(Q(x'_j), Q(x'_{j+1}) < 0\).
3. If \(P'\) is positive on \((x'_j, x'_{j+1})\) and \(α_1 < θ\), then \(Q\) is positive on \([x'_j, x'_{j+1}]\) and \(P(x'_j), P(x'_{j+1}) < 0\).
4. If \(P'\) is negative on \((x'_j, x'_{j+1})\) and \(α_1 > φ\), then \(Q\) is negative on \([x'_j, x'_{j+1}]\) and \(P(x'_j), P(x'_{j+1}) < 0\).

In cases 1 and 2, \(P\) has no roots in \([x'_j, x'_{j+1}]\) and \(Q\) has an even number of roots in \([x'_j, x'_{j+1}]\). In cases 3 and 4, \(P\) has an even number of roots in \((x'_j, x'_{j+1})\) and \(Q\) has no roots in \([x'_j, x'_{j+1}]\). Hence, in all four cases, \(L(x'_j, x'_{j+1}) = (θ|θ)\).

We now observe that the parity of the number of roots of \(P\) or \(Q\) in each interlacing block is completely determined by the pattern of sign changes which occur in the polynomials \(P'\) and \(Q'\). Furthermore, \(Q'(x'_j)\) has the same sign as \(Q'(x'_{j+1})\), and for sufficiently small \(ε\), \(P'(x'_j - ε)\) has the same sign as \(P'(x'_{j+1} + ε)\). Therefore, it follows from the inductive hypothesis that

\[(1, 0) = \# \left( \mathcal{L}(S(θ, x'_j), (θ, x'_j) ∪ (θ, x'_{j+1}, φ)) \right)\]
= \# \left( L \left( (\mathcal{X}|Z)(\theta, x_{j}') \cup (\emptyset|\emptyset) \cup (\mathcal{X}|Z)(x_{j+1}', \phi) \right) \right) \\
= \# \left( L \left( (\mathcal{X}|Z)(\theta, x_{j}') \cup L(x_{j}', x_{j+1}') \cup (\mathcal{X}|Z)(x_{j+1}', \phi) \right) \right) \\
= \#(L(\theta, \phi)),

where the final equality follows from Observation A.0.4.

If \( x_j' = x_{j+1}' \), then the inductive hypothesis again implies

\[(1, 0) = \# \left( L \left( (\mathcal{X}|Z)(\theta, x_{j}') \cup (\mathcal{X}|Z)(x_{j}', \phi) \right) \right) \\
= \#(L(\theta, \phi)).\]

If \( Q(\theta) \) and \( Q(\phi) \) have opposite sign, then we may prove that \( \#(L(\theta, \phi)) = (0, 1) \) in exactly the same manner. This establishes Claim 1.

From this point onward, we will make frequent use of Observation A.0.4.

**Claim 2**: Let \( \theta < \phi \) with \( \alpha_1 \in (\theta, \phi) \) and assume \( \alpha_1 \) is not a root of \( P' \) or \( Q' \). Let \( \theta \) be a root of \( P' \) and \( \phi \) be a root of \( Q' \) (or vice versa) and suppose \( L'(\theta, \phi) = (\emptyset|\emptyset) \). If \( P(\theta) \) and \( P(\phi) \) have the same sign, then \( L(\theta, \phi) = (\emptyset|\emptyset) \).

Without loss of generality, suppose \( \theta \) is a root of \( P' \) and \( \phi \) is a root of \( Q' \). Again without loss of generality, suppose

\[ P(\theta) = -\beta_1 Q'(\theta) > 0 \]

and

\[ P(\phi) = (\phi - \alpha_1)P'(\phi) > 0. \]

This implies

\[ Q(\theta) = (\theta - \alpha_1)Q'(\theta) > 0 \]

and

\[ Q(\phi) = \beta_1 P'(\phi) > 0. \]

To prove Claim 2, we consider the intervals \((\theta, \alpha_1)\) and \((\alpha_1, \phi)\) separately. Let us write

\[ L'(\theta, \alpha_1) = (x_{k_1}', x_{k_2}', \ldots, x_{k_{l_1}}', z_1', z_2', \ldots, z_{l_1}'), \]

\[ L'(\alpha_1, \phi) = (x_{p_1}', x_{p_2}', \ldots, x_{p_{l_2}}', z_{q_1}', z_{q_2}', \ldots, z_{l_2}'). \]

Since

\[ L' \left( (\theta, \alpha_1) \cup L'(\alpha_1, \phi) \right) \sim L'(\theta, \phi) = (\emptyset|\emptyset), \]

either \( L'(\theta, \alpha_1) = L'(\alpha_1, \phi) = (\emptyset|\emptyset) \), or \( L'(\theta, \alpha_1) \) and \( L'(\alpha_1, \phi) \) satisfy:

1. \( \sigma = \delta \) and \( \tau = \hat{\tau}; \)
2. \( x_{k_\sigma}' < z_{l_\tau}' \) if and only if \( z_{q_1}' < x_{p_1}'. \)
If \( L'(\theta, \alpha_1) = L'(\alpha_1, \phi) = (0|0) \), then \( P' \) is strictly positive on \((\theta, \phi)\) and \( Q' \) is strictly negative on \((\theta, \phi)\). This implies \( Q \) is strictly positive on \([\theta, \alpha_1]\) and \( P \) is strictly positive on \([\alpha_1, \phi]\). Therefore \((\theta, \alpha_1)\) contains an even number of roots of \( P \) and no roots of \( Q \) and \((\alpha_1, \phi)\) contains no roots of \( P \) and an even number of roots of \( Q \). Hence \( L(\theta, \phi) = (0|0) \).

Now suppose \( L'(\theta, \alpha_1) \) and \( L'(\alpha_1, \phi) \) satisfy 1 and 2. Without loss of generality, assume \( x_{k_\sigma}' < z_{l_\tau}' \) (and \( z_{q_1}' < x_{p_1}' \)). We first consider the case when \( x_{k_1}' < z_{l_1}' \) and \( \sigma \) is even. Note that since \( x_{k_\sigma}' < z_{l_\tau}' \) and \( x_{k_\sigma}' < z_{l_\tau}' \), we must have \( \sigma = \tau \). Furthermore, by Observation A.0.3,

\[
x_{k_1}', x_{k_2}', \ldots, x_{k_\sigma}', x_{p_1}', x_{p_2}', \ldots, x_{p_\sigma}'
\]

are roots of \( P' \) with odd multiplicity,

\[
z_{l_1}', z_{l_2}', \ldots, z_{l_\sigma}', z_{q_1}', z_{q_2}', \ldots, z_{q_\sigma}'
\]

are roots of \( Q' \) with odd multiplicity and

\[
L'(\theta, x_{k_1}') = L'(x_{k_1}', z_{l_1}') = L'(z_{l_1}', x_{k_2}') = \cdots = L'(z_{l_\sigma}', \alpha_1) = L'(\alpha_1, z_{q_1}') = L'(z_{q_1}', x_{p_1}') = L'(x_{p_1}', z_{q_2}') = \cdots = L'(x_{p_\sigma}', \phi) = (0|0).
\]

In particular, this implies each of the intervals

\[
(\theta, x_{k_1}'), (x_{k_1}', z_{l_1}'), \ldots, (z_{l_\sigma}', \alpha_1), (\alpha_1, z_{q_1}'), (z_{q_1}', x_{p_1}'), \ldots, (x_{p_\sigma}', \phi)
\]

contain an even number of roots of \( P' \) and an even number of roots of \( Q' \).

Let us first consider the interval \((\theta, x_{k_1}')\). We claim that \( L(\theta, x_{k_1}') = (0|0) \). To see this, note that since \( Q'(\theta) < 0 \) and \((\theta, x_{k_1}')\) contains an even number of roots of \( Q' \), we must have \( Q'(x_{k_1}') < 0 \) also. Hence \( P(x_{k_1}'), Q(x_{k_1}') > 0 \). Now suppose \((\theta, x_{k_1}')\) contains no roots of \( P' \) or \( Q' \). This implies \( Q' \) is strictly negative on \((\theta, x_{k_1}')\). Furthermore, since \( P'(\phi) > 0 \) and \((x_{k_1}', \phi)\) contains an odd number of roots of \( P' \), we see that \( P' \) is strictly positive on \((\theta, x_{k_1}')\). Hence \( Q \) is strictly positive on \([\theta, x_{k_1}']\). Therefore \((\theta, x_{k_1}')\) contains an even number of roots of \( P \) and no roots of \( Q \) and so \( L(\theta, x_{k_1}') = (0|0) \), as claimed. If \((\theta, x_{k_1}')\) contains roots of \( P' \) or \( Q' \), we may employ inductive step identical to that used in the proof of Claim 1. By the same logic, we see that \( L(\alpha_1, z_{q_1}') = (0|0) \).

We now consider the interval \((x_{k_1}', z_{l_1}')\). It is not difficult to check that \( Q(x_{k_1}') \) and \( Q(z_{l_1}') \) have different sign. Hence Claim 1 implies

\[
\#(L(x_{k_1}', z_{l_1}')) = (0, 1).
\]

Similarly,

\[
\#(L(z_{l_1}', x_{k_2}')) = \#(L(z_{l_2}', x_{k_3}')) = \#(L(z_{l_3}', x_{k_4}')) = \cdots = \#(L(z_{l_\sigma}', \alpha_1)) = \#(L(z_{q_1}', x_{p_1}')) = \#(L(z_{q_2}', x_{p_2}')) = \cdots = \#(L(z_{q_\sigma}', x_{p_\sigma}')) = (1, 0)
\]

and
Therefore
\[ L(\theta, \phi) = L \left( \cdots \cup L(z'_{k-1}, \alpha_k) \cup L(\alpha_k, z'_{q_k}) \cup L(z'_{q_k}, x'_{p_k}) \cup \cdots \right) \]

\[ = L \left( \cdots \cup L(x'_{k_\sigma}, z'_{l_1}) \cup L(z'_{l_1}, x'_{k_\sigma}) \cup \cdots \right) \]

\[ = \cdots \]

\[ = L \left( L(\theta, x'_{k_1}) \cup L(x'_{k_1}, z'_{l_1}) \cup L(x'_{l_1}, \phi) \right) \]

\[ = (\emptyset, \emptyset). \]

If \( z'_{l_1} < x'_{k_1} \) or \( \sigma \) is odd, a similar argument shows that \( L(\theta, \phi) = (\emptyset, \emptyset). \)

This establishes Claim 2.

Claim 3: Suppose \( b_1 < 0 \). Then

(i) if \( P'(\phi) = 0 \), \( L'(-\infty, \phi) = (\emptyset, \emptyset) \) and \( \alpha_1 > \phi \), then \( L(-\infty, \phi) = (\emptyset, \emptyset) \);

(ii) if \( P'(\phi) = 0 \), \( L'(-\infty, \phi) = (\emptyset, \emptyset) \) and \( \alpha_1 < \phi \), then \( #L(-\infty, \phi) = (0, 1) \);

(iii) if \( P'(\theta) = 0 \), \( L'(\theta, \infty) = (\emptyset, \emptyset) \) and \( \alpha_1 < \theta \), then \( L(\theta, \infty) = (\emptyset, \emptyset) \);

(iv) if \( P'(\theta) = 0 \), \( L'(\theta, \infty) = (\emptyset, \emptyset) \) and \( \alpha_1 > \theta \), then \( #L(\theta, \infty) = (0, 1) \).

The proofs of (i) and (iii) are similar to the proof of Claim 1. The proofs of (ii) and (iv) are similar to the proof of Claim 2. Similarly, one can show:

Claim 4: Suppose \( b_1 > 0 \) and \( b'_{l_1} < 0 \). Then

(i) if \( P'(\phi) = 0 \), \( L'(-\infty, \phi) = (\emptyset, \emptyset) \) and \( \alpha_1 > \phi \), then \( #L(-\infty, \phi) = (0, 1) \);

(ii) if \( P'(\phi) = 0 \), \( L'(-\infty, \phi) = (\emptyset, \emptyset) \) and \( \alpha_1 < \phi \), then \( L(-\infty, \phi) = (\emptyset, \emptyset) \);

(iii) if \( P'(\theta) = 0 \), \( L'(\theta, \infty) = (\emptyset, \emptyset) \) and \( \alpha_1 < \theta \), then \( #L(\theta, \infty) = (0, 1) \);

(iv) if \( P'(\theta) = 0 \), \( L'(\theta, \infty) = (\emptyset, \emptyset) \) and \( \alpha_1 > \theta \), then \( L(\theta, \infty) = (\emptyset, \emptyset) \).

Claim 5: Suppose \( b_1, b'_{l_1} > 0 \). Then

(i) if \( Q'(\phi) = 0 \), \( L'(-\infty, \phi) = (\emptyset, \emptyset) \) and \( \alpha_1 > \phi \), then \( L(-\infty, \phi) = (\emptyset, \emptyset) \);

(ii) if \( Q'(\phi) = 0 \), \( L'(-\infty, \phi) = (\emptyset, \emptyset) \) and \( \alpha_1 < \phi \), then \( #L(-\infty, \phi) = (1, 0) \);
(iii) if \( Q'(\theta) = 0 \), \( L'(\theta, \infty) = (\emptyset \emptyset) \) and \( \alpha_1 < \theta \), then \( L(\theta, \infty) = (\emptyset \emptyset) \);

(iv) if \( Q'(\theta) = 0 \), \( L'(\theta, \infty) = (\emptyset \emptyset) \) and \( \alpha_1 > \theta \), then \( \#(L(\theta, \infty)) = (1, 0) \).

We will now consider the cases \( b_1 < 0 \), \( b_1 > 0 \) and \( b_1 = 0 \) separately.

**Claim 6:** Suppose \( n_0 = 0 \), \( n_+ \geq n_- \) and \( b_1 < 0 \). Then \( \#(\mathcal{L}(X|\emptyset)) = (d, d - 1) \).

The proof of Claim 6 is by induction on \( n_- \). If \( n_- = 0 \), then Claim 6 follows from Lemma 2.2.4. Now assume \( n_- > 0 \) and that the statement holds for polynomials with \( n_- - 1 \) roots with negative imaginary part. We note that \( P'(x) + iQ'(x) \) is monic, has degree \( n - 1 \), has \( n_- - 1 \) roots with negative imaginary part and has no real roots. In addition, \( b_1' = b_1 - \beta < 0 \). Hence, by the inductive hypothesis, \( \#(\mathcal{L}(X'|\emptyset')) = (d + 1, d) \).

Suppose

\[
\mathcal{L}(X'|\emptyset') = (x_{k'_1}, x_{k'_2}, \ldots, x_{k'_{d+1}}, |z_{l'_1}, z_{l'_2}, \ldots, z_{l'_d}|).
\]

By Observation A.0.3, \( x'_{k'_1}, x'_{k'_2}, \ldots, x'_{k'_{d+1}} \) are roots of \( P' \) with odd multiplicity, \( z'_{l'_1}, z'_{l'_2}, \ldots, z'_{l'_d} \) are roots of \( Q' \) with odd multiplicity and

\[
L'(\infty, x'_{k'_1}) = L'(x'_{k'_1}, z'_{l'_1}) = L'(z'_{l'_1}, x'_{k'_2}) = \cdots = L'(x'_{k'_{d+1}}, \infty) = (\emptyset \emptyset).
\]

Without loss of generality, assume \( n \) is even. Let us first suppose there exists \( r \) such that \( x'_{k'_r} < \alpha_1 < z'_{l'_r} \) and (without loss of generality) assume \( r \) is even. Since each of the intervals

\[
(-\infty, x'_{k'_1}), (x'_{k'_1}, z'_{l'_1}), (z'_{l'_1}, x'_{k'_2}), \ldots, (x'_{k'_{d+1}}, \infty)
\]

contain an even number of roots of \( P' \) and an even number of roots of \( Q' \), it is not difficult to deduce that

\[
\begin{align*}
P(x'_{k'_{2j-1}}) &> 0 : \quad j = 1, 2, \ldots, d/2 + 1, \\
P(x'_{k'_{2j}}) &< 0 : \quad j = 1, 2, \ldots, d/2, \\
P(z'_{l'_{2j-1}}) &< 0 : \quad j = 1, 2, \ldots, r/2, \\
P(z'_{l'_{2j}}) &> 0 : \quad j = r/2 + 1, r/2 + 2, \ldots, d/2, \\
P(z'_{l'_1}) &< 0 : \quad j = 1, 2, \ldots, r/2 - 1, \\
P(z'_{l'_2}) &> 0 : \quad j = r/2, r/2 + 1, \ldots, d/2, \\
Q(x'_{k'_{2j-1}}) &> 0 : \quad j = 1, 2, \ldots, r/2, \\
Q(x'_{k'_{2j}}) &< 0 : \quad j = r/2 + 1, r/2 + 2, \ldots, d/2 + 1, \\
Q(x'_{k'_{2j+1}}) &< 0 : \quad j = 1, 2, \ldots, r/2, \\
Q(x'_{k'_{2j+2}}) &> 0 : \quad j = r/2 + 1, r/2 + 2, \ldots, d/2, \\
Q(z'_{l'_{2j-1}}) &> 0 : \quad j = 1, 2, \ldots, d/2, \\
Q(z'_{l'_2}) &< 0 : \quad j = 1, 2, \ldots, d/2.
\end{align*}
\]
Therefore, by Claim 1,
\[ \#(L(x_{k_1}', z_{1_1}')) = \#(L(x_{k_2}', z_{1_2}')) = \cdots = \#(L(x_{k_{t-1}}', z_{1_{t-1}}')) = \#(L(z_{1_1}', x_{k_1}')) = \cdots = \#(L(z_{1_{d-1}}', x_{k_{d-1}}')) = (1, 0) \]

and
\[ \#(L(z_{1_1}', x_{k_1}')) = \#(L(z_{1_2}', x_{k_2}')) = \cdots = \#(L(z_{1_{t-1}}', x_{k_{t-1}}')) = \#(L(x_{k_{t+1}}', z_{1_{t+1}}')) = \cdots = \#(L(x_{k_{d-1}}', z_{1_{d-1}}')) = (0, 1). \]

Furthermore, by Claim 2, \( L(x_{k_1}', z_{1_1}') = (\emptyset|0) \) and by Claim 3,
\[ L(-\infty, x_{k_1}') = L(x_{k_{d+1}}', \infty) = (\emptyset|0). \]

It follows that
\[ \#(\mathcal{L}'(X|Z)) = \#(\mathcal{L}'(L(-\infty, x_{k_1}') \cup L(x_{k_1}', z_{1_1}') \cup \cdots \cup L(x_{k_{d+1}}', \infty))) = (d, d-1). \]

If \( \alpha_1 < x_{k_1}' \) or \( \alpha_1 > x_{k_{d+1}}' \), or if there exists \( r \) such that \( z_{1_r}' < \alpha_1 < x_{k_{r+1}}' \), then the inductive step is performed in exactly the same manner.

Finally, we must accomplish the inductive step when \( \alpha_1 = x_{k_j} \) for some \( j \) or \( \alpha_1 = z_{1_j} \) for some \( j \). In this case, let us consider the polynomial
\[ f_\epsilon(x) := (x - \alpha_1 - \epsilon + i\beta_1)(P'(x) + iQ'(x)). \]

For all sufficiently small \( \epsilon, \alpha_1 + \epsilon \) does not coincide with \( x_{k_j} \) or \( z_{1_j} \) for any \( j \) and hence the statement holds for \( f_\epsilon \). Since
\[ \lim_{\epsilon \to 0} f_\epsilon(x) = f(x) \]

and the roots of a polynomial depend continuously on its coefficients, it follows that the statement also holds for \( f \). This establishes Claim 6.

**Claim 7:** Suppose \( n_0 = 0, n_+ \geq n_- \) and \( b_1 > 0 \). Then \( \#(\mathcal{L}'(X|Z)) = (d, d+1) \).

Let
\[ m := \min \left\{ j : \sum_{i=1}^{j} \beta_1 \geq b_1 \right\}. \]

The proof of Claim 7 is by induction on \( m \).

Since \( b_1 > 0 \), it follows that \( m > 0 \). Let us consider the base case, \( m = 1 \). We begin by assuming that \( \beta_1 > b_1 \) (and hence \( b_1' < 0 \)). In this case, Claim 6 implies \( \#(\mathcal{L}'(X'|Z')) = (d+1, d) \). Let us write \( \mathcal{L}'(X'|Z') \) as in (218) and assume that \( \alpha_1 \) does not coincide with \( x_{k_j}' \) or \( z_{1_j}' \) for any \( j \). Using logic similar to that used in the proof of Claim 6, it is not difficult to show that
1. if $x_{k_1}' < \alpha_1 < x_{k+d+1}'$, then $\#(L(-\infty, x_{k_1}')) = \#(L(x_{k_d+1}', \infty)) = (0, 1)$ and $\#(L(x_{k_1}', x_{k_d+1}')) = (d, d-1)$;

2. if $\alpha_1 < x_{k_1}'$, then $L(-\infty, x_{k_1}') = (\emptyset, 0)$, $\#(L(x_{k_d+1}', \infty)) = (0, 1)$ and $L(x_{k_1}', x_{k_d+1}')$ is of the form

\[(x_{p_1}, x_{p_2}, \ldots, x_{p_d}, z_{q_1}, z_{q_2}, \ldots, z_{q_d}), \quad (219)\]

where $z_{q_1} < x_{p_1}$;

3. if $\alpha_1 > x_{k_d+1}'$, then $\#(L(-\infty, x_{k_1}')) = (0, 1)$, $L(x_{k_d+1}', \infty) = (0, 0)$ and $L(x_{k_1}', x_{k_d+1}')$ is of the form (219), where $x_{p_1} < z_{q_1}$.

In all three cases, it follows that $\#(L'(X, \mathbb{Z})) = (d, d+1)$. If $\beta_1 = b_1$, or if $\alpha_1$ coincides with $x_{k_1}'$ or $z_{q_j}'$ for some $j$, then we may employ a continuity argument similar to that used in the proof of Claim 6. This establishes the claim if $m = 1$.

Now suppose $m \geq 2$. We observe that the polynomial $P'(x) + iQ'(x)$ satisfies

\[\min \left\{ j : \sum_{i=1}^{j} \beta_{l+1} \geq b_1' \right\} = m - 1.\]

Hence, the inductive hypothesis implies $\#(L'(X, \emptyset)) = (d+1, d+2)$. Using now-familiar arguments, it follows that $\#(L'(X, \emptyset)) = (d, d+1)$. This completes the proof of Claim 7.

If $n_0 = 0$, $n_+ \geq n_-$ and $b_1 \neq 0$, then Claims 6 and 7 are enough to guarantee the conclusion of the theorem. In addition, during the proof of the $m = 1$ case of Claim 7, we noted that either $\#(L(x_{k_1}', x_{k_d+1}')) = (d, d-1)$ or $\#(L(x_{k_1}', x_{k_d+1}')) = (d, d)$. This was independent of the fact that $b_1 > 0$. Hence the conclusion of the theorem also holds if $b_1 = 0$.

Let us now consider the case when $n_- > n_+$. In this case, we consider the polynomial $\hat{f}(x) := \hat{P}(x) + i\hat{Q}(x)$, where

\[
\hat{P}(x) := x^n - a_1 x^{n-1} + a_2 x^{n-2} - a_3 x^{n-3} + \cdots + (-1)^n a_n,
\]

\[
\hat{Q}(x) := -b_1 x^{n-1} + b_2 x^{n-2} - b_3 x^{n-3} + \cdots + (-1)^n b_n.
\]

We note that $\hat{f}$ has $n_-$ roots with positive imaginary part, $n_+$ roots with negative imaginary part and no real roots. Therefore, from the above, we see that (counting multiplicities) there exist $d$ real roots of $\hat{P}$ (say $\mu_1, \mu_2, \ldots, \mu_d$) and $d-1$ real roots of $\hat{Q}$ (say $\nu_1, \nu_2, \ldots, \nu_{d-1}$) such that

\[\mu_1 < \nu_1 < \mu_2 < \nu_2 < \cdots < \nu_{d-1} < \mu_d.\]

It follows that $-\mu_1, -\mu_2, \ldots, -\mu_d$ are roots of $P$, $-\nu_1, -\nu_2, \ldots, -\nu_{d-1}$ are roots of $Q$ and

\[\mu_d < -\nu_{d-1} < -\mu_{d-1} < -\nu_{d-2} < \cdots < -\nu_1 < -\mu_1.\]
Finally, suppose \( n_0 > 0 \). Let us label the real roots of \( f \) as \( \eta_1, \eta_2, \ldots, \eta_{n_0} \). Writing

\[
f(x) = \left( \prod_{j=1}^{n_0} (x - \eta_j) \right) \left( \tilde{P}(x) + i \tilde{Q}(x) \right),
\]

we note that the polynomial \( \tilde{P}(x) + i \tilde{Q}(x) \) has \( n_+ \) roots with positive imaginary part, \( n_- \) roots with negative imaginary part and no real roots. Hence, from the above, there exist \( d - n_0 \) real roots of \( \tilde{P} \) (say \( \mu_1, \mu_2, \ldots, \mu_{d-n_0} \)) and \( d - n_0 - 1 \) real roots of \( \tilde{Q} \) (say \( \nu_1, \nu_2, \ldots, \nu_{d-n_0-1} \)) such that

\[
\mu_1 < \nu_1 < \mu_2 < \nu_2 < \cdots < \nu_{d-n_0-1} < \mu_{d-n_0}.
\]

All that remains is to note that the sequences

\[
\mu_1, \mu_2, \ldots, \mu_{d-n_0}, \eta_1, \eta_2, \ldots, \eta_{n_0}
\]

and

\[
\nu_1, \nu_2, \ldots, \nu_{d-n_0-1}, \eta_1, \eta_2, \ldots, \eta_{n_0}
\]

interlace (though not strictly). \( \square \)

Note that Claims 6 and 7 in the above proof actually give more information about the structure of the real roots of \( P \) and \( Q \) than the statement of Theorem A.0.1 requires. In fact, it is not difficult to see that, together with this additional information, Theorem A.0.1 becomes equivalent to Theorem 2.3.8.

To see this, note that if the real roots of \( P \) and \( Q \) interlace, then either \( Q(x)/P(x) \) jumps from \(-\infty\) to \( \infty \) at every root of \( P \), or \( Q(x)/P(x) \) jumps from \( \infty \) to \(-\infty \) at every root of \( P \) (see Figure 11). In this case, \( \Gamma_\infty^\infty(Q(x)/P(x)) \) is equal to \( \pm \) (the number of real roots of \( P \)).

In addition, if \((x_1, x_2)\) is a pair of adjacent real roots of \( P \) with no real root of \( Q \) in between, then either \( Q(x)/P(x) \) jumps from \(-\infty\) to \( \infty \) at \( x_1 \) and from \( \infty \) to \(-\infty \) at \( x_2 \), or vice versa. Hence \((x_1, x_2)\) does not contribute to the Cauchy index of \( Q(x)/P(x) \). It is also clear that adding a pair of adjacent real roots of \( Q \) does nothing to affect the behaviour of \( Q(x)/P(x) \) at its discontinuities. Therefore, adding pairs of adjacent real roots of either \( P \) or \( Q \) does not affect the Cauchy index of \( Q(x)/P(x) \).
Figure 11: The Cauchy index of $\frac{Q(x)}{P(x)}$
Using the results in Chapter 6, we will show how to (algorithmically) determine whether a given list $\sigma := (\lambda_1, \lambda_2, \ldots, \lambda_n)$—where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$—lies in $\mathcal{H}_n$. More specifically, given $(\lambda_2, \lambda_3, \ldots, \lambda_n)$, we will show how to compute the minimal $\lambda_1$ such that $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n$. The algorithm relies on Theorem 6.3.11, which we recall here for convenience:

**Theorem B.0.1.** Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$. Then $(\lambda_1; \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n(\alpha_1, \alpha_2, \ldots, \alpha_n)$ if and only if there exist

$$0 \leq \varepsilon \leq \frac{1}{2}(\lambda_1 - \lambda_2)$$

and two partitions

$$\{3, 4, \ldots, n\} = \{p_1, p_2, \ldots, p_{l-1}\} \cup \{q_1, q_2, \ldots, q_{n-l-1}\},$$

$$\{1, 2, \ldots, n\} = \{r_1, r_2, \ldots, r_l\} \cup \{s_1, s_2, \ldots, s_{n-l}\}$$

such that

$$(\lambda_1 - \varepsilon; \lambda_{p_1}, \lambda_{p_2}, \ldots, \lambda_{p_{l-1}}) \in \mathcal{H}_l(\alpha_{r_1}, \alpha_{r_2}, \ldots, \alpha_{r_l})$$

and

$$(\lambda_2 + \varepsilon; \lambda_{q_1}, \lambda_{q_2}, \ldots, \lambda_{q_{n-l-1}}) \in \mathcal{H}_{n-l}(\alpha_{s_1}, \alpha_{s_2}, \ldots, \alpha_{s_{n-l-1}}).$$

We allow the possibilities $l = 1$, in which case $\{p_1, p_2, \ldots, p_{l-1}\}$ is the empty set, and $l = n - 1$, in which case $\{q_1, q_2, \ldots, q_{n-l-1}\}$ is the empty set.

Recall that, by Lemma 6.3.9, if $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n$, then $(\lambda_1 + t, \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n$ for all $t \geq 0$. Hence, if $\lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$, let us define

$$\Gamma(\lambda_2, \lambda_3, \ldots, \lambda_n) := \min \{\lambda_1 \geq \lambda_2 : (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{H}_n\}. \tag{220}$$

Since $\mathcal{H}_n$ is closed, the minimum in (220) is well-defined.

We will now give a recursive formula for $\Gamma(\lambda_2, \lambda_3, \ldots, \lambda_n)$.

**Proposition B.0.2.** Let $\tau := (\lambda_2, \lambda_3, \ldots, \lambda_n)$, where $\lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$, let $U := \{3, 4, \ldots, n\}$, and for each $S \subseteq U$, define

$$\tau_S := (\lambda_i : i \in S).$$

1 Recall from Theorem 6.4.1 that $\mathcal{H}_n = \mathcal{S}_n$. 

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Then
\[ \Gamma(\tau) = \min_{S \subseteq U} \max \{ \lambda_2 + 2\epsilon_S, \Gamma(\tau_S) + \epsilon_S \}, \]  
where
\[ \epsilon_S := \max \{ 0, \Gamma(\tau_{U \setminus S}) - \lambda_2 \}. \]

We assume \( \Gamma(\tau_{\emptyset}) = 0. \)

Proof. Let
\[ \rho := \min_{S \subseteq U} \max \{ \lambda_2 + 2\epsilon_S, \Gamma(\tau_S) + \epsilon_S \}. \]
Suppose the minimum in (223) is achieved by some subset \( S^* \subseteq U \) and suppose \( |S^*| = k. \) Let us rewrite \( \rho \) as
\[ \rho = \max \{ \lambda_2 + 2\epsilon_{S^*}, \Gamma(\tau_{U \setminus S^*}) + \epsilon_{S^*} \}. \]

We will first show that \( \Gamma(\tau) \leq \rho. \) To see this, consider the lists
\[ \sigma_1 := (\rho - \epsilon_{S^*}, \tau_{S^*}) \]
and
\[ \sigma_2 := (\lambda_2 + \epsilon_{S^*}, \tau_{U \setminus S^*}). \]
By (224), \( \epsilon_{S^*} \leq \frac{1}{2}(\rho - \lambda_2) \) and \( \rho - \epsilon_{S^*} \geq \Gamma(\tau_{S^*}). \) In addition, by (222), \( \epsilon_{S^*} \geq 0 \) and \( \lambda_2 + \epsilon_{S^*} \geq \Gamma(\tau_{U \setminus S^*}). \) Hence, our assumptions guarantee \( \sigma_1 \in \mathcal{H}_{k+1}, \sigma_2 \in \mathcal{H}_{n-k-1} \) and \( 0 \leq \epsilon_{S^*} \leq \frac{1}{2}(\rho - \lambda_2). \) It follows from Theorem B.0.1 that \( (\rho, \tau) \in \mathcal{H}_n, \) or, equivalently, \( \rho \geq \Gamma(\tau). \)

We will now show that \( \Gamma(\tau) \geq \rho. \) Since \( (\Gamma(\tau), \tau) \in \mathcal{I}_n, \) Theorem B.0.1 implies there exist
\[ 0 \leq \epsilon \leq \frac{\Gamma(\tau) - \lambda_2}{2} \]
and \( S \subseteq U \) such that
\[ (\Gamma(\tau) - \epsilon, \tau_S) \in \mathcal{H}_{k+1} \]
and
\[ (\lambda_2 + \epsilon, \tau_{U \setminus S}) \in \mathcal{H}_{n-k-1}, \]
where \( k := |S|. \) Note that (226) and (227) imply
\[ \Gamma(\tau) \geq \Gamma(\tau_S) + \epsilon \]
and
\[ \epsilon \geq \Gamma(\tau_{U \setminus S}) - \lambda_2, \]
respectively.

Next, we note that, by (229) and the fact that \( \epsilon \geq 0, \)
\[ \epsilon \geq \max \{ 0, \Gamma(\tau_{U \setminus S}) - \lambda_2 \} = \epsilon_S, \]
and hence, combining (228) with (230), we see that
\[ \Gamma(\tau) \geq \Gamma(\tau_S) + \epsilon_S. \]
Furthermore, combining (225) with (230), we see that
\[ \Gamma(\tau) \geq \lambda_2 + 2\epsilon_S. \]  

(232)

Hence, by (231) and (232),
\[ \Gamma(\tau) \geq \max\{\lambda_2 + 2\epsilon_S, \Gamma(\tau_S) + \epsilon_S\} \]
\[ \geq \max\{\lambda_2 + 2\epsilon_S, \Gamma(\tau_S^*) + \epsilon_S^*\} \]
\[ \geq \rho. \]

Note that, in the statement of Proposition B.0.2, the lists \( \tau_S \) and \( \tau_{U \setminus S} \) are shorter than \( \tau \). Therefore, provided \( n \) is not too large, (221) may be used to recursively calculate \( \Gamma(\tau) \) (by computer). See Listing 1 for an implementation of this algorithm in Mathematica.

Listing 1: Computation of \( \Gamma(\tau) \) in Mathematica

```mathematica
gamma[tau_] := Module[{l2, Tau, u, U, minmax, i, S, Sc, e},
  If[tau == {}, 0,
    l2 = First[tau];
    Tau = Rest[tau];
    u = Length[tau] - 1;
    U = Range[u];
    minmax = Total[Abs /@ tau];
    For[i = 0, i <= Power[2, u] - 1, i++,
      S = Flatten[Position[IntegerDigits[i, 2, u], 1]];
      Sc = Complement[U, S];
      e = Max[0, gamma[Tau[[Sc]]] - l2];
      minmax = Min[minmax, Max[l2 + 2 e, gamma[Tau[[S]]] + e]];]
    minmax
  ]]
```

There are possible avenues to increase the efficiency of this algorithm that we do not address here.

Finally, suppose we wish to determine whether \((\lambda_1; \lambda_2, \ldots, \lambda_n) \in H_n(a_1, a_2, \ldots, a_n)\), where \( a_1, a_2, \ldots, a_n \) are specified. In this instance, we may use Theorem 6.3.5 recursively. At each step, we need only consider those \( s \) and \( t \) such that
\[(\lambda_1; \lambda_2, \ldots, \lambda_{n-1}) \text{ and } (a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{t-1}, a_{t+1}, \ldots, a_n, c)\]
satisfy Fiedler’s necessary conditions (see Theorem 6.2.1). For an implementation of the latter algorithm, see Listing 2.
Listing 2: Determining whether \( \sigma \in \mathcal{H}_n(a_1, a_2, \ldots, a_n) \)

\[
\text{Fiedler}[\lambda_, b_] := \text{Module}[[\text{result} = \text{True}, n, a, s, k], \\
n = \text{Length}[\lambda]; \\
a = \text{Sort}[b, \text{Greater}]; \\
\text{If}[\lambda[[1]] < a[[1]], \text{result} = \text{False};, \\
\text{For}[s = 1, s <= n - 1, s++, \\
\text{For}[k = s + 1, k <= n, k++, \\
\text{If}[\text{Total}[\lambda[[1 ;; s]]] + \lambda[[k]] < \\
\text{Total}[a[[1 ;; s - 1]]] + a[[k - 1]] + a[[k]], \\
\text{result} = \text{False};; \\
\text{Break}[]; \\
\text{If}[! \text{result}, \text{Break}[]]; \\
\text{For}[c = a[[s]] + a[[t]] - \lambda[[n]]; \\
\text{If}[\text{H}[\text{Most}[\lambda], \text{Append}[\text{Delete}[a, \{\{s\}, \{t\}\}], c]], \\
\text{result} = \text{True};; \\
\text{Break}[]; \\
\text{If}[! \text{result}, \text{Break}[]]; \\
\text{result} 
]
]}

\[
\text{H}[\lambda_, a_] := \text{Module}[[\text{result} = \text{False}, n, s, t, c], \\
n = \text{Length}[\lambda]; \\
\text{If}[\text{Fiedler}[\lambda, a], \\
\text{If}[n <= 3, \text{result} = \text{True}, \\
\text{For}[s = 1, s <= n - 1, s++, \\
\text{For}[t = s + 1, t <= n, t++, \\
\text{c = a[[s]] + a[[t]] - \lambda[[n]]; \\
\text{If}[\text{H}[\text{Most}[\lambda], \text{Append}[\text{Delete}[a, \{\{s\}, \{t\}\}], c]], \\
\text{result} = \text{True};; \\
\text{Break}[]; \\
\text{If}[! \text{result}, \text{Break}[]]; \\
\text{result} 
]
]}
]}

AN ALGORITHMIC TREATMENT OF THE SOULES SET
Example B.0.3. Consider the list $\tau := (5,5,5,-4,-4,-4,-4)$. Let us compute the minimal value of $\lambda_1$ for which $(\lambda_1, \tau) \in \mathcal{H}_9$. Inserting these numbers into the function $\text{gamma}[^\tau]$ given in listing 1,

\[ \text{gamma}[[5, 5, 5, -4, -4, -4, -4]] \]

returns

8

From this, we know that $(8,5,5,5,-4,-4,-4,-4,-4) \in \mathcal{H}_n(a_1, a_2, \ldots, a_9)$ for some $a_i$ with $\sum_i a_i = 3$; however, using the function $\text{H}[^\lambda, a]$ of listing 2, we can determine that it is not possible to choose $a_1 = 3, a_2 = a_3 = \cdots = a_9 = 0$:

\[ \text{H}[[8, 5, 5, 5, -4, -4, -4, -4, -4], [3, 0, 0, 0, 0, 0, 0, 0, 0]] \]

returns

False

On the other hand, we see that

\[ (8,5,5,5,-4,-4,-4,-4,-4) \in \mathcal{H}_n(1,1,1,0,0,0,0,0,0), \]

since

\[ \text{H}[[8, 5, 5, 5, -4, -4, -4, -4, -4], [1, 1, 1, 0, 0, 0, 0, 0, 0]] \]

returns

True


DECLARATION

I hereby certify that the submitted work is my own work, was completed while registered as a candidate for the degree stated on the title page, and I have not obtained a degree elsewhere on the basis of the research presented in this submitted work.

Dublin, Ireland, May 2016

___________________________
Richard Ellard