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THE DIRICHLET PROBLEM FOR THE SLAB AND ENTIRE DATA AND A DIFFERENCE EQUATION FOR HARMONIC FUNCTIONS

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ABSTRACT. It is shown that the Dirichlet problem for the slab and entire data has entire solutions. The proof is based on a generalized Schwarz reflection principle. Moreover it is shown that the inhomogeneous difference equation $h(t+1,y) - h(t,y) = g(t,y)$ for a given entire harmonic functions $g(t,y)$ has an entire harmonic solution.

1. INTRODUCTION

It is well known that the Dirichlet problem for *unbounded* domains differs in many respects from the case of bounded domains due to the non-uniqueness of the solutions, and an excellent discussion of the Dirichlet problem for general unbounded domains can be found in [9].

Maybe the simplest example of this kind is the Dirichlet problem for the strip $(a,b) \times \mathbb{R}$ which has been considered by Widder in [20], see also [5]. A discussion of the Dirichlet for half-spaces can be found in [8], and for a cylinder in [16] and [15].

In this paper we are concerned with the harmonic extendibility of the solution of the Dirichlet problem for entire data on the slab (see [4])

$$S_{a,b} := (a,b) \times \mathbb{R}^d$$

We say that a function $f : \mathbb{R}^d \to \mathbb{C}$ is entire if there exists an analytic function $F : \mathbb{C}^d \to \mathbb{C}$ such that $F(x) = f(x)$ for all $x \in \mathbb{R}^d$. Thus an entire function $f : \mathbb{R}^d \to \mathbb{C}$ is real analytic, and it possesses a convergent everywhere power series expansion. It is well known that every harmonic function $h : \mathbb{R}^d \to \mathbb{C}$ is entire.

Our first main result in this paper is the following:

Theorem 1. *Let h be a solution of the Dirichlet problem for the slab $S_{a,b}$ for entire data $f_0, f_1 : \mathbb{R}^d \to \mathbb{C}$, i.e. h is harmonic on S and $\lim_{t\to a} h(t,y) = f_0(y)$ and $\lim_{t\to b} h(t,y) = f_1(y)$. Then h extends to all of $\mathbb{R}^{d+1}$ as a harmonic function.*

A similar result holds for the Dirichlet problem for the ellipsoid: H.S. Shapiro and the first author have established in [14] that for each entire data function there exists a solution of the Dirichlet problem which extends to harmonic function defined on $\mathbb{R}^d$, see also [2] for further extensions. We refer the reader to [13] a discussion of the question which domains Ω allow entire extensions for entire data.

From Theorem 1 we shall derive our second main result:

Theorem 2. *If $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is harmonic then the difference equation*

$$h(t+1, y) - h(t, y) = g(t, y). \tag{1}$$

has a harmonic solution $h : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$.

Let us recall now some notations and definitions. A function $f : \Omega \to \mathbb{C}$ defined on a domain Ω in the euclidean space $\mathbb{R}^d$is called *harmonic* if f is two times continuously differentiable and $\Delta f(x) = 0$ for all $x \in \Omega$ where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_d^2}$$

is the Laplace operator. We also write Δ_x instead of Δ to indicate the variables for differentiation. We say that a function $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is even (odd respectively) at t_0 if

$$g(t_0 + t, y) = g(t_0 - t, y),$$

and $g(t_0 + t, y) = -g(t_0 - t, y)$ respectively, for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^d$.

2. The Dirichlet problem on the slab for entire data

The following result is well known and follows from the Schwarz reflection principle:

Theorem 3. *Let $h : [a, b] \times \mathbb{R}^d \to \mathbb{C}$ be continuous, and harmonic in the open slab $(a, b) \times \mathbb{R}^d$ such that $h(a, y) = h(b, y) = 0$ for all $y \in \mathbb{R}^d$. Then h extends to a harmonic function on $\mathbb{R}^{d+1}$ which is periodic in the variable t with period $2(b-a)$, i.e.*

$$h(t + 2(b-a), y) = h(t, y).$$

In the following we need an extension of the Schwarz reflection principle:

Theorem 4. *Suppose that Ω is a domain in $\mathbb{R}^{d+1}$ such that for each $x = (x_1, ..., x_{d+1}) \in \Omega$ the vector $\widetilde{x} = (-x_1, x_2, ..., x_{d+1}) \in \Omega$, and let $\Omega_+, \Omega_0, \Omega_-$ denote the sets of points $x \in \Omega$ for which x_1 is positive, zero, negative. Suppose that $y \longmapsto F(y)$ for $y = (x_2, ..., x_{d+1}) \in \mathbb{R}^d$ is an entire function and assume that h is harmonic on Ω_- such that $h(x) \to F(y)$ for all $x \to y \in \Omega_0$. Then h has a harmonic extension to Ω.*

Proof. By the Cauchy-Kovalevskaya Theorem applied to the Laplace operator (see [11, p. 80, Example 11.2]), there is a unique entire function H such that $H(0, y) = F(y)$ and $\frac{\partial}{\partial x} H(0, y) = 0$ for all $y \in \mathbb{R}^d$ and $j = 1, ..., d$. Moreover H is even at $t_0 = 0$ since $H(-x, y)$ solves the same Cauchy problem and the solution is unique. Consider the function

$$f(t, y) := h(t, y) - H(t, y)$$

for $(t, y) \in \Omega_-$. Then $f(t, y) \to 0$ for each $t \to 0$, and by the Schwarz reflection principle (see [1, p. 8]) f extends to a harmonic function $\widetilde{f}$ on Ω by the formula

$$\widetilde{f}(t, y) = -\widetilde{f}(-t, y)$$

for all $(t,y)\in\Omega_+$. Then
$$\widetilde{h}(t,y):=\widetilde{f}(t,y)+H(t,y)$$
is a harmonic function on Ω which extends h on Ω_-, and for $t>0$ we have
$$\widetilde{h}(t,y)=\widetilde{f}(t,y)+H(t,y)=-\widetilde{f}(-t,y)+H(t,y)=-h(-t,y)+2H(t,y).$$
□

Now we obtain the following result:

Theorem 5. *Assume that $h\in C\left([a,b]\times\mathbb{R}^d\right)$ is harmonic in the slab $(a,b)\times\mathbb{R}^d$ such that $y\longmapsto h(a,y)$ and $y\longmapsto h(b,y)$ are entire. Then there exists a harmonic extension $\widetilde{h}:\mathbb{R}^{d+1}\to\mathbb{C}$.*

Proof. It suffices to show: (*) there is an extension $\widetilde{h}\in C\left([a,2b-a]\times\mathbb{R}^d\right)$ of h which is harmonic in $(a,2b-a)\times\mathbb{R}^d$ such that $y\longmapsto\widetilde{h}(2b-a,y)$ is entire: by induction one obtains a harmonic extension on $(a,b+n(b-a))\times\mathbb{R}^d$ for each natural number n such that $y\longmapsto\widetilde{h}(a+n(b-a),y)$ is entire. Similarly, one shows that there is harmonic extension on $(a-n(b-a),b)\times\mathbb{R}^d$ for each natural number n, and the proof is accomplished.

For establishing (*) we may assume that $a<b=0$, and let H be harmonic function on $\mathbb{R}^{d+1}$ which extends $y\longmapsto h(b,y)$ as in the last proof. Theorem 4 says that $f(x)=h(x)-H(x)$ can be extended by reflection to continuous function on $[a,-a]\times\mathbb{R}^d$. Hence $y\longmapsto\widetilde{f}(a,y)=-f(-a,y)$ is an entire function since by assumption $y\longmapsto h(a,y)$ is entire and clearly $y\longmapsto H(-a,y)$ is entire. It follows that $y\longmapsto\widetilde{f}(a,y)+H(a,y)$ is entire and therefore the result is proven. □

Corollary 6. *Let $f_0,f_1:\mathbb{R}^d\to\mathbb{C}$ be entire functions. Then any solution h for the Dirichlet problem for the slab $(a,b)\times\mathbb{R}^d$ with $h(a,y)=f_0(y)$ and $h(b,y)=f_1(y)$ extends to a harmonic function $h:\mathbb{R}^{d+1}\to\mathbb{C}$.*

Proof. It is known that the Dirichlet problem for the slab with continuous data has a solution h, see e.g. [9]. By Theorem 5 the function h has an entire extension. □

3. The difference equation for harmonic functions

It is a well-known fact in complex analysis that the inhomogeneous difference equation

$$f(z+1)-f(z)=g(z) \tag{2}$$

for a given entire function $g(z)$ has an entire solution $f(z)$, see [3, p. 407]. By taking real parts it follows that the difference equation

$$h(t+1,y)-h(t,y)=g(t,y) \tag{3}$$

for harmonic function g on $\mathbb{R}^2$ has a harmonic solution h defined on $\mathbb{R}^2$.

In this section we shall provide a generalization for arbitrary dimension of the variable $y\in\mathbb{R}^d$. In our approach the solution is constructed by the solution of the Dirichlet problem for the slab $[0,1/2]\times\mathbb{R}^d$. In [3, p. 407] the difference equation for analytic functions is constructed using Bernoulli polynomials, an idea which goes back to the work of Guichard, Appel, and Hurwitz more than a century ago. ,

It is a remarkable fact that equation (2) can be solved as well for meromorphic functions. It would be interesting to study analogues of these results for the harmonic difference equation (3).

At first we provide a solution of the difference equation for harmonic functions for the case that $g(t,y)$ is even:

Theorem 7. *Let $g:\mathbb{R}\times\mathbb{R}^d\to\mathbb{C}$ be harmonic and even. Then any solution $h(t,y)$ of the Dirichlet problem for the slab $[0,1/2]\times\mathbb{R}^d$ with*

$$h(0,y)=-\frac{1}{2}g(0,y)\ \text{ and } h\left(\frac{1}{2},y\right)=0$$

for all $y\in\mathbb{R}^d$ induces an entire harmonic solution of the difference equation

$$h(t+1,y)-h(t,y)=g(t,y).$$

Proof. By Corollary 6 there exists an entire harmonic function $h(t,y)$ such that $h(0,y)=-\frac{1}{2}g(0,y)$ and $h\left(\frac{1}{2},y\right)=0$. The last equation and the Schwarz reflection principle shows that

$$h\left(\frac{1}{2}+t,y\right)=-h\left(\frac{1}{2}-t,y\right). \tag{4}$$

Inserting $t=\frac{1}{2}$ in equation (4) gives $h(1,y)=-h(0,y)=\frac{1}{2}g(0,y)$. Now we consider the harmonic function

$$F(t,y)=h(t,y)-\frac{1}{2}g(t-1,y).$$

Then $F(1,y)=0$, and by Schwarz's reflection principle, $F(1+t,y)=-F(1-t,y)$ for $y\in\mathbb{R}^d$. Then

$$\begin{aligned} h(1+t,y) &= F(1+t,y)+\frac{1}{2}g(t,y)=-F(1-t,y)+\frac{1}{2}g(t,y)\\ &= -h(1-t,y)+\frac{1}{2}g(-t,y)+\frac{1}{2}g(t,y)\end{aligned}$$

Since g is even we have $\frac{1}{2}g(-t,y)+\frac{1}{2}g(t,y)=g(t,y)$ and

$$h(1-t,y)=h\left(\frac{1}{2}+\frac{1}{2}-t,y\right)=-h\left(\frac{1}{2}-\left(\frac{1}{2}-t\right),y\right)=-h(t,y).$$

It follows that $h(1+t)=h(t,y)+g(t,y)$. □

The next result is surely part of mathematical folklore, and we include an elementary proof in order to hold the paper self-contained.

Theorem 8. *Let $g(t,y)$ be an entire harmonic function. Then there exists an entire harmonic function $u(t,y)$ such that*

$$\frac{\partial}{\partial t}u(t,y)=g(t,y).$$

If $g(t,y)$ is odd then $u(t,y)$ can be choosen to be even.

Proof. Define $h(t,y):=\int_0^t g(\tau,y)\,d\tau$. Then $\frac{\partial}{\partial t}h(t,y)=g(t,y)$ and

$$\Delta_y\frac{\partial}{\partial t}h(t,y)=\Delta_y g(t,y)=-\frac{\partial^2}{\partial t^2}g(t,y).$$

We conclude that $\frac{\partial}{\partial t}\left(\Delta_y h(t,y)+\frac{\partial}{\partial t}g(t,y)\right)=0$ and it follows that

$$f(y):=\Delta_y h(t,y)+\frac{\partial}{\partial t}g(t,y)$$

only depends on y and not on t. It is well known that for the entire function $f(y)$ there exists an entire function $g(y)$ such $\Delta_y g(y)=f(y)$. Then

$$u(t,y):=h(t,y)-g(y)$$

is a solution of the equation $\frac{\partial}{\partial t}u(t,y)=g(t,y)$, and $u(t,y)$ is harmonic since

$$\begin{aligned}\Delta_{t,y}u(t,y) &= \frac{\partial^2}{\partial t^2}h(t,y)+\Delta_y h(t,y)-\Delta_y g(y)\\ &= \frac{\partial^2}{\partial t^2}h(t,y)+\Delta_y h(t,y)-\Delta_y h(t,y)-\frac{\partial}{\partial t}g(t,y)=0.\end{aligned}$$

□

Now we are able to prove our second main result:

Theorem 9. *If $g:\mathbb{R}\times\mathbb{R}^d\to\mathbb{C}$ is harmonic then the difference equation $h(t+1,y)-h(t,y)=g(t,y)$ has a harmonic solution $h:\mathbb{R}\times\mathbb{R}^d\to\mathbb{C}$.*

Proof. Write the harmonic function g as a sum g_0+g_e where g_0 is odd and g_e is even. For g_e there exists a solution $h_e(t,y)$ by Theorem 7. By Theorem 8 there exists a harmonic even function $u(t,y)$ such that

$$\frac{\partial}{\partial t}u(t,y)=g_0(t,y).$$

By Theorem 7 there exists a harmonic entire function $H(t,y)$ such that

$$H(t+1,y)-H(t,y)=u(t,y).$$

Thus we find the solution of the difference equation for $g_0(t,y)$ via differentiation with respect to t. □

The next result follows from Theorem 3:

Proposition 10. *Let $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ be a harmonic function and let $h_j : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ for $j = 1, 2$ be harmonic solutions of the difference equation*

$$h_j(t+1, y) - h_j(t, y) = g(t, y).$$

Then $h(t, y) := h_1(t, y) - h_2(t, y)$ is periodic and the period is 2.

Finally we want to prove the following result:

Theorem 11. *Let $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ be a harmonic function and let $h_j : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ for $j = 1, 2$ be harmonic solutions of the difference equation*

$$h_j(t+1, y) - h_j(t, y) = g(t, y).$$

Assume that for the function $f(t, y)$ equal to $g(t, y)$ and $h_j(t, y)$ for $j = 1, 2$ we have an estimate of the form

$$|f(t, y)| = o\left(|y|^{(1-d)/2} e^{k|y|}\right) \tag{5}$$

for $|y| \to \infty$. Then there exists a harmonic function $r : \mathbb{R}^d \to \mathbb{C}$ such that

$$h_1(t, y) = h_2(t, y) + r(y)$$

for all $y \in \mathbb{R}^d$ and $t \in \mathbb{R}$.

Proof. Define $h(t, y) := h_1(t, y) - h_2(t, y)$. Define $H(t, y) := H_y(t) := h(\pi t, \pi y)$ for $y \in \mathbb{R}^d$, $t \in \mathbb{R}$. Then H_y is a 2π-periodic function which has a Fourier series $\sum_{k=-\infty}^{\infty} a_k(y) e^{ikt}$. Apply the Laplace operator to the Fourier coefficients

$$a_k(y) = \frac{1}{\pi} \int_{-\pi}^{\pi} H_y(t) e^{ikt} dt, \tag{6}$$

then

$$\Delta_y a_k(y) = \frac{1}{\pi} \int_{-\pi}^{\pi} \Delta_y H(t, y) e^{ikt} dt.$$

Since $H(t, y)$ is harmonic we know that

$$\Delta_y H(t, y) = -\frac{\partial^2}{\partial t^2} H(t, y).$$

Partial integration shows that

$$\Delta_y a_k(y) = -\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\partial^2}{\partial t^2} H(t, y) \cdot e^{ikt} dt = k^2 a_k(y).$$

Hence $a_k(y)$ is a solution of the Helmholtz equation $\Delta_y a_k = k^2 a_k$. A classical result in [7, p. 228], which goes back to work of I. Vekua and F. Rellich in the 1940'ies, shows that $a_k = 0$ for $k \neq 0$ since k^2 is positive number k^2 and estimate (5) holds. □

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