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A characterization of the Khavinson-Shapiro conjecture via Fischer operators

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Abstract

The Khavinson-Shapiro conjecture states that ellipsoids are the only bounded domains in euclidean space satisfying the following property (KS): the solution of the Dirichlet problem for polynomial data is polynomial. In this paper we show that property (KS) for a domain Ω is equivalent to the surjectivity of a Fischer operator associated to the domain Ω.

1 Introduction

In the 19th century ellipsoidal harmonics have been used to prove that for any polynomial p of degree $\leq m$ there exists a harmonic polynomial h of degree $\leq m$ such that $h(\xi) = p(\xi)$ for all $\xi \in \partial E$ where ∂E is the boundary of an *ellipsoid* E in the euclidean space $\mathbb{R}^d$. It follows that an ellipsoid satisfies the following property defined for an arbitrary open subset Ω in $\mathbb{R}^d$:

(KS) For any polynomial p with real coefficients there exists a harmonic polynomial h with real coefficients such that $h(\xi) = p(\xi)$ for all $\xi \in \partial\Omega$.

The Khavinson-Shapiro conjecture [12] states that ellipsoids are the only bounded domains Ω in $\mathbb{R}^d$ with property (KS). Obviously a domain Ω has property (KS) if and only if the Dirichlet problem for polynomial data (restricted to the boundary) have polynomial solutions; for the Dirichlet problem we refer to [2] and [7]. The Khavinson-Shapiro conjecture has been confirmed for large classes of domains but it is still unproven in its full generality, and we refer the interested reader to the expositions [8], [17], [14], [10], [15], [11] and for further ramifications in [13] originating from the work [16].

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In this paper we want to characterize the property (KS) by using Fischer operators. In our context we shall mean by a Fischer operator[2] an operator of the form

$$F_{\psi}(q) := \Delta(\psi q) \text{ for } q \in \mathbb{R}[x]$$

where Δ is the Laplace operator $\frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_d^2}$ and ψ is a fixed element in $\mathbb{R}[x]$, the set of all polynomials in d variables with real coefficients. Fischer operators often allows elementary and short proofs of mathematical statements which usually require hard and deep analysis, see [20], [11]. For example, the statement that ellipsoids have property (KS) can be proven in a few lines using Fischer operators and elementary results in Linear Algebra, see [4], [5], [3], and for further generalizations see [1], [12].

In order to formulate our main result we need some technical definitions. The zero-set of a polynomial $f \in \mathbb{R}[x]$ is denoted by $Z(f) = \{x \in \mathbb{R}^d : f(x) = 0\}$. We say that a subset Z of $\mathbb{R}^d$ is an *admissible common zero set* if there exist *non-constant irreducible* polynomials $f, g \in \mathbb{R}[x]$ such that (i) $f \neq \lambda g$ for all $\lambda \in \mathbb{R}$ and (ii) $Z = Z(f) \cap Z(g)$. For dimension $d = 2$ it is well known that an admissible common zero set is finite, see [19, p. 2]. For arbitrary dimension d it is intuitively clear that an admissible common zero set has "dimension" $\leq d - 2$ at each point.

We say that an open set Ω in $\mathbb{R}^d$ is *admissible* if for any $x \in \partial\Omega$, any open neighborhood V of x and for any finite family of admissible common zero sets $Z_1, ..., Z_r$ the set

$$[\partial\Omega \cap V] \setminus \bigcup_{j=1}^{r} Z_j$$

is non-empty. For dimension $d = 2$ it is easy to see that an open set Ω is admissible if each point $x \in \partial\Omega$ is not isolated in $\partial\Omega$. For arbitrary dimension it seems to be difficult to formulate a precise topological condition but it is intuitively clear that a domain Ω is admissible if each point in the boundary $\partial\Omega$ has a neighborhood of dimension $d - 1$.

The following is now the main result of this paper:

Theorem 1 *Let Ω be an open admissible subset of $\mathbb{R}^d$. Then property (KS) holds for Ω if and only if there exists a non-constant polynomial $\psi \in \mathbb{R}[x]$ such that (i) $\partial\Omega \subset \psi^{-1}\{0\}$ and (ii) the Fischer operator $F_{\psi} : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $F_{\psi}(q) := \Delta(\psi q)$ for $q \in \mathbb{R}[x]$ is surjective.*

An immediate consequence of Theorem 1 is that the Khavinson-Shapiro conjecture is true for all admissible bounded domains if the following purely

[2] More generaly one can define a Fischer operator by $q \longmapsto P(D)(\psi q)$ where $P(D)$ is a linear partial differential operator with constant real coefficients. Fischer's Theorem in [6] states that the Fischer operator is a bijection whenever $\psi(x)$ is a homogeneous polynomial equal to the polynomial $P(x)$.

algebraic conjecture of M. Chamberland and D. Siegel formulated in [5] is true:

(CS) The surjectivity of the Fischer operator $F_\psi : \mathbb{R}[x] \to \mathbb{R}[x]$ implies that the degree of ψ is ≤ 2.

We refer to [15] and [18] for more details on conjecture (CS) and related results. It should be emphasized that the polynomial ψ in conjecture (CS) has real coefficients. In [9] it is shown for dimension $d = 3$ that for any non-constant polynomial $\varphi(z) = a_0 + a_1 z + \cdots + a_n z^n$ in the complex variable z the operator $F_\psi : \mathbb{C}[x] \to \mathbb{C}[x]$ defined by

$$F_\psi(q) = \Delta\left[(x_3 - \varphi(x_1 + ix_2))^2 q(x_1, x_2, x_3)\right]$$

for $q \in \mathbb{C}[x]$ is surjective where $\mathbb{C}[x]$ is the set of all polynomials with complex coefficients.

2 Proof of Theorem 1

From [15] we cite the following result which is known as the *Fischer decomposition* of a polynomial.

Proposition 2 *Suppose ψ is a polynomial. Then the operator $F_\psi(q) := \Delta(\psi q)$ is surjective if and only if every polynomial f can be decomposed as $f = \psi q_f + h_f$, where q_f is a polynomial and h_f is harmonic polynomial*

Proof of Theorem 1: The sufficiency is easy: By assumption, there exists $\psi \in \mathbb{R}[x]$ with $\partial\Omega \subset \psi^{-1}\{0\}$ such that F_ψ is surjective. According to Proposition 2 there exists for each polynomial f a polynomial q and a harmonic polynomial u such that $f = \psi q + u$. For $\xi \in \partial\Omega \subset \psi^{-1}\{0\}$ it follows that $f(\xi) = u(\xi)$. Thus u is a polynomial solution to the Dirichlet problem of the domain Ω and property (KS) is satisfied.

Now assume that (KS) holds. Then there exists a harmonic polynomial $u \in \mathbb{R}[x]$ such that $|\xi|^2 = u(\xi)$ for all $\xi \in \partial\Omega$. Define the polynomial $Q(x) = |x|^2 - u(x)$. Then

$$\partial\Omega \subset Q^{-1}(0) \tag{1}$$

and Q is a non-constant polynomial of degree ≥ 2 since $\Delta(Q) \neq 0$. We factorize $Q(x)$ in irreducible factors, so

$$Q(x) = |x|^2 - u(x) = f_1^{m_1} ... f_r^{m_r}$$

where f_k is not a scalar multiple of f_l for $k \neq l$, and $m_k \geq 1$ is the multiplicity of f_k. It follows that

$$\partial\Omega \subset \bigcup_{k=1}^{r} f_k^{-1}(0). \tag{2}$$

Then $Z(f_k) \cap Z(f_l)$ is an admissible common zero set for $k \neq l$. Let Z be the (finite) union of the sets $Z(f_k) \cap Z(f_l)$ with $k \neq l$. As Ω is admissible there exists $x \in \partial\Omega \setminus Z$, and now (2) implies that

$$I := \{k \in \{1, ...r\} : \exists x \in \partial\Omega \setminus Z \text{ with } f_k(x) = 0\}$$

is non-empty. Let us define

$$\psi := \prod_{k \in I} f_k(x).$$

We want to show that F_ψ is surjective. By Proposition 2 it suffices to show that for any polynomial f there is a harmonic polynomial u and a polynomial q such that $f = \psi q + u$. By property (KS) there exists a harmonic polynomial u such that $f(\xi) = u(\xi)$ for all $\xi \in \partial\Omega$. Define $g(x) = f(x) - u(x)$, so g vanishes on $\partial\Omega$. If we can show that each f_k with $k \in I$ divides g then, by irreducibility of f_k and the condition that $f_k \neq \lambda f_j$ for $k \neq j$, we infer that $\psi = \prod_{k \in I} f_k$ divides g, say $g = \psi q$ for some polynomial q. Then $f - u = g = q\psi$ and we are done.

Let $k \in I$ be fixed and g as above. Then there exists $x \in \partial\Omega \setminus Z$ with $f_k(x) = 0$. Then $f_l(x) \neq 0$ for all $l \neq k$ since otherwise x would be in Z. By continuity there is an open neighborhood V of x such that $f_l(y) \neq 0$ for all $y \in V$ and $l \neq k$. Then we conclude from (2) that

$$\partial\Omega \cap V \subset f_k^{-1}(0). \tag{3}$$

Let us write $g = g_1^{m_1} \cdots g_s^{m_r}$ where $g_1, ..., g_s$ are irreducible polynomials such that $g_j \neq \lambda g_l$ for $j \neq l$. If $f_k = \lambda g_j$ for some $j \in \{1, ..., s\}$ we see that f_k divides g. Assume that this is not the case and define Z_k as the (finite) union of the admissible sets $Z(g_j) \cap Z(f_k)$ for $j = 1, ...s$. Since Ω is admissible there exists $y \in \partial\Omega \cap V \diagdown (Z \cup Z_k)$. The inclusion (3) shows that $f_k(y) = 0$, and since g vanishes on $\partial\Omega$ there exists $j \in \{1, .., s\}$ such that $g_j(y) = 0$. Hence $y \in Z(g_j) \cap Z(f_k) \subset Z_k$. Now we obtain a contradiction since $y \in \partial\Omega \cap V \diagdown (Z \cup Z_k)$.

It remains to prove that $\partial\Omega$ is contained in $\psi^{-1}(0)$. If $j \in \{1, ...r\} \setminus I$ then it follows from the definition of I that $f_j(x) \neq 0$ for all $x \in \partial\Omega \setminus Z$. This fact and (2) imply that

$$\partial\Omega \setminus Z \subset \bigcup_{k \in I} f_k^{-1}(0) =: F. \tag{4}$$

Let $x \in \partial\Omega$. Since Ω is admissible there exists for any ball V with center x and radius $1/m$ an element $x_m \in (\partial\Omega \cap V) \diagdown Z$. Then (4) shows that $x_m \in F$. Since x_m converges to x and F is closed we infer that $x \in F$. Thus $\partial\Omega \subset F$.

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