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A SCHWARZ LEMMA AND COMPOSITION OPERATORS

M. MACKEY AND P. MELLON

ABSTRACT. We give an alternative description of the Carathéodory pseudo-distance on a domain D in an arbitrary complex Banach space. This gives a Schwarz lemma for holomorphic maps of the domain. We specialise to the case of a bounded symmetric domain and obtain some applications. In particular, we give the connected components of the space of composition operators with symbol in a bounded symmetric domain. This generalises results for the space of composition operators on $H^\infty(\Delta)$ in [12] and for $H^\infty(B)$, B the unit ball of a Hilbert space or commutative C^* -algebra in [2].

INTRODUCTION

Let D be a domain in a complex Banach space E and let Δ be the open unit disc in \mathbb{C} . We define the following pseudo-distance on D ,

$$d_D(z, w) := \sup\{|f(z) - f(w)| : f : D \rightarrow \Delta \text{ holomorphic}\} \quad \text{for } z, w \in D.$$

We prove that

$$\log \frac{2 + d_D}{2 - d_D}$$

is in fact the Carathéodory pseudo-distance C_D on D . This results in a Schwarz Lemma for holomorphic maps from D to Δ . When we specialise this to B_E , the open unit ball of a Banach space E , we prove firstly that d_{B_E} can be expressed in terms of holomorphic self-maps of B_E , namely,

$$d_{B_E}(z, w) = \sup\{\|f(z) - f(w)\| : f : B_E \rightarrow B_E \text{ holomorphic}\}.$$

Since the Carathéodory distance on B_E satisfies

$$C_{B_E}(z, 0) = \tanh^{-1} \|z\|$$

we obtain, among others, the following Schwarz Lemma for all $f : B_E \rightarrow B_E$ holomorphic:

$$\|f(z) - f(0)\| \leq \frac{2 - 2\sqrt{1 - \|z\|^2}}{\|z\|} \quad \text{for all } z \in B_E.$$

If B is a bounded symmetric domain and $f : B \rightarrow B$ is holomorphic, we get

$$\|f(z) - f(w)\| \leq \frac{2 - 2\sqrt{1 - \|g_{-z}(w)\|^2}}{\|g_{-z}(w)\|} \quad \text{for all } z, w \in B.$$

The description of d_{B_E} in terms of holomorphic self-maps of B_E makes it suited to the study of composition operators on the space $H^\infty(B_E)$ and, indeed, this is the motivation behind the introduction of d_Δ in the one variable case in [12]. The set-up is as follows:

to every $\phi : B_E \rightarrow B_E$ holomorphic we associate a linear map C_ϕ , called a composition operator, on the space $H^\infty(B_E)$ of all bounded holomorphic functions on B_E by

$$C_\phi(f) = f \circ \phi$$

for $f \in H^\infty(B_E)$. The idea is to associate the function theoretic properties of ϕ with the properties of C_ϕ as a linear mapping.

For $B = \Delta$, a survey of the classical theory of composition operators on the Hardy and Bergman spaces is given in [4] and [16]. To extend the classical results where ϕ is taken as a holomorphic function on Δ to the case where ϕ is a function of several or even infinitely many variables, one can head in a variety of directions. For example, if B_n is the open unit ball of \mathbb{C}^n , MacCluer, Shapiro and Luecking, among others have looked at the action of C_ϕ on the Hardy spaces $H^p(B_n)$, $0 < p < \infty$ and the Bergman spaces $A^p(B_n)$, $0 < p < \infty$. Jafari, Li, Russo and others have studied C_ϕ on the Hardy and Bergman spaces of finite dimensional bounded symmetric domains and strongly pseudo-convex domains. We refer to the survey of Russo [15] for references and more information. In the infinite dimensional case, we refer to [1, 2, 7] which study composition operators on the space $H^\infty(B_E)$, for E a complex Banach space.

Our aim is to extend to a bounded symmetric domain B results of MacCluer, Ohno and Zhao for the one variable case in [12] that determine the connected components of the topological space of composition operators on $H^\infty(\Delta)$ with the natural uniform norm topology. These results were extended in [2] when B is the open unit ball of a Hilbert space or commutative C^* -algebra, and in [17] when¹ B is the open unit ball of \mathbb{C}^n .

We recall that every bounded symmetric domain B can be realised as the open unit ball of a Banach space Z , known as a JB^* -triple [8]. The algebraic properties of Z , in particular the properties of the Bergman operator $B(z, w)$ and the quasi-inverse map $z \rightarrow z^a$ are then used, together with the distance d_B , to determine the connected components of the space of composition operators on $H^\infty(B)$. For a general survey and background details on the Poincaré distance, Carathéodory pseudo-distance and JB^* -triples we refer to [5].

1. NOTATION AND BACKGROUND

We let E and F denote complex Banach spaces and D and \tilde{D} domains in E and F respectively. The set of all holomorphic maps from D to \tilde{D} is denoted by $H(D, \tilde{D})$. We write $H^\infty(D)$ for the space of all bounded \mathbb{C} -valued holomorphic functions on D and $\|f\|_\infty := \sup_{z \in D} |f(z)|$ for all $f \in H^\infty(D)$.

Definition 1.1. The Poincaré distance ρ on Δ is

$$\rho(z, w) := \tanh^{-1} \left| \frac{z - w}{1 - \bar{z}w} \right| \quad \text{for } z, w \in \Delta.$$

¹We thank the referee for drawing our attention to this reference.

The Carathéodory pseudo-distance can be defined on any complex manifold [5], although we restrict our attention here to the case of a domain D .

Definition 1.2. The Carathéodory pseudo-distance on a domain D is given by

$$C_D(z, w) := \sup\{\rho(f(z), f(w)) : f \in H(D, \Delta)\} \quad \text{for } z, w \in D.$$

The Carathéodory pseudo-distances form a Schwarz-Pick system (cf. [5]) for which holomorphic functions act as contractions, namely,

$$C_{D_2}(f(z), f(w)) \leq C_{D_1}(z, w) \quad \text{for all } f \in H(D_1, D_2), \quad z, w \in D_1.$$

In fact, this is the smallest of all Schwarz-Pick systems. For bounded domains, cf. [5, chapters 4 and 5], it turns out that C_D is continuous and generates the original topology thus ensuring that it is actually a distance on D .

We now introduce the class of Banach spaces known as the JB^* -triples. We use H and K to denote arbitrary complex Hilbert spaces and $\mathcal{L}(X, Y)$ to denote the space of continuous linear operators from a Banach space X to a Banach space Y . We let $\mathcal{L}(X) = \mathcal{L}(X, X)$ and $\text{GL}(X)$ be all invertible elements in $\mathcal{L}(X)$.

Definition 1.3. A JB^* -triple is a complex Banach space Z with a real trilinear mapping $\{\cdot, \cdot, \cdot\} : Z \times Z \times Z \rightarrow Z$ satisfying

- (i) $\{x, y, z\}$ is complex linear and symmetric in the outer variables x and z , and is complex anti-linear in y .
- (ii) The map $z \rightarrow \{x, x, z\}$, denoted $x \square x$, is Hermitian, $\sigma(x \square x) \geq 0$ and $\|x \square x\| = \|x\|^2$ for all $x \in Z$, where σ denotes the spectrum.
- (iii) The product satisfies the following ‘‘triple identity’’

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$

Let Z be a JB^* -triple. Several types of linear operators on Z arise naturally from the triple product:

$$x \square y \in \mathcal{L}(Z) : z \rightarrow \{x, y, z\},$$

$$Q_x \in \mathcal{L}_{\mathbb{R}}(Z) : z \rightarrow \{x, z, x\},$$

and the important Bergman operators

$$B(x, y) = I - 2x \square y + Q_x Q_y \in \mathcal{L}(Z).$$

Example 1.4. (i) $\mathcal{L}(H, K)$ is a JB^* -triple for the product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$ where y^* denotes the usual adjoint of y and $B(x, y)z = (1 - xy^*)z(1 - y^*x)$.

- (ii) $\mathcal{C}_0(X)$, the continuous \mathbb{C} -valued functions vanishing at infinity on a locally compact Hausdorff space X , is a JB^* -triple for the product $\{x, y, z\} = x\bar{y}z$ and $B(x, y)z = (1 - x\bar{y})^2z$.

As Banach spaces the JB^* -triples are characterised by the fact that their open unit balls are homogeneous. In fact, if we let $\text{Aut}(B)$ denote all biholomorphic maps from B to B then for all a in B , we have $g_a \in \text{Aut}(B)$ defined by

$$g_a(z) = a + B(a, a)^{\frac{1}{2}}(I + z \square a)^{-1}z$$

(cf. [9]) which satisfies $g_a(0) = a$, $g_a^{-1} = g_{-a}$ and $g'_a(0) = B(a, a)^{\frac{1}{2}}$ (defined in terms of a functional calculus). We note the fundamental formula [10]

$$\|B(a, a)^{-\frac{1}{2}}\| = \frac{1}{1 - \|a\|^2} \quad (1)$$

for $a \in B$. For $z, a \in B$, $z^a := (I - z \square a)^{-1}z$ is called the quasi-inverse of z with respect to a and satisfies

$$\|z^a\| \leq \frac{\|z\|}{1 - \|z\|\|a\|}. \quad (2)$$

The quasi-inverse also satisfies $(z^a)^b = z^{a+b}$ whenever both sides of this equation are well defined. For further details see [11, Chapter 7] or [6].

It is known [8] that every bounded symmetric domain is biholomorphically equivalent to the open unit ball of a JB^* -triple and vice versa. From the homogeneity therefore one can easily see that on a bounded symmetric domain B the Carathéodory distance is given by $C_B(z, w) = \tanh^{-1} \|g_{-z}(w)\|$. For a recent survey of JB^* -triples and bounded symmetric domains we refer to [3].

2. THE CARATHÉODORY PSEUDO-DISTANCE

To study composition operators on $H^\infty(\Delta)$ MacCluer et al. [12] introduce the distance d_∞ on Δ

$$d_\infty(z, w) := \sup\{|f(z) - f(w)| : f \in H^\infty(\Delta), \|f\|_\infty \leq 1\}, \quad z, w \in \Delta.$$

It is not too difficult to see [13] that

$$d_\infty(z, w) = \frac{2 - 2\sqrt{1 - \beta(z, w)^2}}{\beta(z, w)} \quad \text{for } \beta(z, w) := \left| \frac{z - w}{1 - \bar{z}w} \right|$$

or in terms of the Poincaré metric ρ on Δ

$$\rho(z, w) = \tanh^{-1} \beta(z, w) = \log \frac{2 + d_\infty(z, w)}{2 - d_\infty(z, w)}. \quad (3)$$

Motivated by this we introduce the following pseudo-distance on an arbitrary domain D

$$d_D(z, w) := \sup\{|f(z) - f(w)| : f \in H(D, \Delta)\}.$$

We note that $d_\Delta = d_\infty$ above. Clearly

$$\begin{aligned} d_D(z, w) &= \sup\{|g(f(z)) - g(f(w))| : g \in H(\Delta, \Delta), f \in H(D, \Delta)\} \\ &= \sup_{f \in H(D, \Delta)} d_\infty(f(z), f(w)) \quad \text{for } z, w \in D. \end{aligned}$$

Since the map $t \rightarrow \log \frac{2+t}{2-t}$ is strictly increasing on $[0, 2)$ it follows that

$$\begin{aligned} \log \frac{2 + d_D(z, w)}{2 - d_D(z, w)} &= \sup_{f \in H(D, \Delta)} \log \frac{2 + d_\infty(f(z), f(w))}{2 - d_\infty(f(z), f(w))} \\ &= \sup_{f \in H(D, \Delta)} \rho(f(z), f(w)) \quad \text{from (3)} \\ &= C_D(z, w). \end{aligned}$$

In other words, $\log \frac{2+d_D}{2-d_D}$ is the Carathéodory pseudo-distance on D , or equivalently for any domain D

$$d_D(z, w) = \frac{2 - 2\sqrt{1 - (\tanh C_D(z, w))^2}}{\tanh C_D(z, w)} \quad \text{for } z, w \in D. \quad (4)$$

Throughout, we use B_E to denote the open unit ball of an arbitrary complex Banach space E and reserve B to denote a bounded symmetric domain.

We now present a series of Schwarz Lemmas arising from (4).

Lemma 2.1. (i) *Let D be an arbitrary domain and $f : D \rightarrow \Delta$ be holomorphic. Then*

$$|f(z) - f(w)| \leq \frac{2 - 2\sqrt{1 - (\tanh C_D(z, w))^2}}{\tanh C_D(z, w)} \quad \text{for } z, w \in D.$$

In particular, if $D = B_E$ is the open unit ball of a Banach space E then

$$|f(z) - f(0)| \leq \frac{2 - 2\sqrt{1 - \|z\|^2}}{\|z\|} \quad \text{for } z \in B_E.$$

(ii) *Let B be a bounded symmetric domain and $f : B \rightarrow \Delta$ be holomorphic. Then*

$$\begin{aligned} |f(z) - f(w)| &\leq \frac{2 - 2\sqrt{1 - \|g_{-z}(w)\|^2}}{\|g_{-z}(w)\|} \\ &= 2 \frac{\sqrt{\|B_w^{-1}B(w, z)B_z^{-1}\|} - 1}{\sqrt{\|B_w^{-1}B(w, z)B_z^{-1}\|} - 1} \quad \text{for } z, w \in B \end{aligned}$$

where g_z is an automorphism of B taking 0 to z and $B_z := B(z, z)^{\frac{1}{2}}$.

Proof. (i) is immediate from (4). The first part of (ii) follows from (i) since on a bounded symmetric domain B

$$C_B(z, w) = \tanh^{-1} \|g_{-z}(w)\| \quad \text{for } z, w \in B \quad (5)$$

where g_z is an automorphism of B taking 0 to z .

For the second part of (ii) we rewrite

$$\frac{2 - 2\sqrt{1 - \|g_{-z}(w)\|^2}}{\|g_{-z}(w)\|}$$

in terms of Bergman operators using the fact [14, Proposition 3.1] that

$$\frac{1}{1 - \|g_{-z}(w)\|^2} = \|B_w^{-1}B(w, z)B_z^{-1}\| \quad \text{for } z, w \in B. \quad (6)$$

□

For the purpose of studying composition operators on $H^\infty(B_E)$ the distance we really need on B_E is written in terms of self-maps of B_E , namely,

$$\tilde{d}_{B_E}(z, w) := \sup\{\|f(z) - f(w)\| : f \in H(B_E)\}.$$

Proposition 2.2. *The distance \tilde{d}_{B_E} coincides with d_{B_E} .*

Proof. Fix z, w in B_E and $f \in H(B_E)$. By the Hahn-Banach theorem there exists $\lambda = \lambda(z, w, f) \in Z^*$, $\|\lambda\| \leq 1$ with

$$\|f(z) - f(w)\| = \lambda(f(z) - f(w))$$

and hence $\tilde{d}_{B_E}(z, w) \leq d_{B_E}(z, w)$. On the other hand, if $g \in H(B_E, \Delta)$ then for any fixed $u \in \partial B_E$ the map $z \rightarrow g(z)u$ is in $H(B_E)$ and this implies $d_{B_E}(z, w) \leq \tilde{d}_{B_E}(z, w)$. □

Proposition 2.2 together with (4), (5) and (6) now easily gives the following.

Corollary 2.3. (i) *Let E be a Banach space and $f : B_E \rightarrow B_E$ be holomorphic. Then*

$$\|f(z) - f(w)\| \leq \frac{2 - 2\sqrt{1 - (\tanh C_{B_E}(z, w))^2}}{\tanh C_{B_E}(z, w)} \quad \text{for } z, w \in B_E.$$

In particular,

$$\|f(z) - f(0)\| \leq \frac{2 - 2\sqrt{1 - \|z\|^2}}{\|z\|} \quad \text{for } z \in B_E.$$

(ii) *Let B be a bounded symmetric domain and $f : B \rightarrow B$ be holomorphic. Then*

$$\begin{aligned} \|f(z) - f(w)\| &\leq \frac{2 - 2\sqrt{1 - \|g_{-z}(w)\|^2}}{\|g_{-z}(w)\|} \\ &= 2 \frac{\sqrt{\|B_w^{-1}B(w, z)B_z^{-1}\|} - 1}{\sqrt{\|B_w^{-1}B(w, z)B_z^{-1}\|} - 1} \quad \text{for } z, w \in B, \end{aligned}$$

where g_z is an automorphism of B taking 0 to z and $B_z := B(z, z)^{\frac{1}{2}}$.

As the Bergman operators play a fundamental role in the holomorphy of B and $B(a, a)^{\frac{1}{2}} = g'_a(0)$, $a \in B$ the inequality [10]

$$\|B(a, a)^{\frac{1}{2}}\| \leq 1$$

is crucial to the geometry of B . We are able to obtain a simple direct proof of this result.

Corollary 2.4. *For $a \in B$, $\|B(a, a)^{\frac{1}{2}}\| \leq 1$.*

Proof. Fix $a \in B$. For all $z \in B$

$$\frac{2 - 2\sqrt{1 - \|z\|^2}}{\|z\|} = d_B(z, 0) \geq \|g_a(z) - g_a(0)\| = \|B(a, a)^{\frac{1}{2}}z^{-a}\|.$$

Since $z^a \in B$ if $\|z\| < \frac{1}{1+\|a\|}$ and $(z^a)^{-a} = z$ this implies that

$$\|B(a, a)^{\frac{1}{2}}z\| \leq \frac{2 - 2\sqrt{1 - \|z^a\|^2}}{\|z^a\|}$$

when $\|z\| < \frac{1}{1+\|a\|}$. Fix $0 < t < \frac{1}{1+\|a\|}$. For $\|z\| \leq t$, we have from (2) that

$$\|z^a\| \leq \frac{\|z\|}{1 - \|z\|\|a\|} \leq \frac{t}{1 - t\|a\|}$$

and since $h(t) = (2 - 2\sqrt{1 - t^2})/t$ is strictly increasing on $[0, 1)$ this gives

$$\|B(a, a)^{\frac{1}{2}}z\| \leq h(\|z^a\|) \leq h\left(\frac{t}{1 - t\|a\|}\right).$$

Then

$$\begin{aligned} \|B(a, a)^{\frac{1}{2}}\| &= \sup_{\|z\|=1} \|B(a, a)^{\frac{1}{2}}z\| = \frac{1}{t} \sup_{\|z\|=t} \|B(a, a)^{\frac{1}{2}}z\| \\ &\leq \frac{1}{t} h\left(\frac{t}{1 - t\|a\|}\right) \\ &= 2 \left(1 - t\|a\| + \sqrt{(1 - t\|a\|)^2 - t^2}\right)^{-1}. \end{aligned}$$

As $t \rightarrow 0$ this gives $\|B(a, a)^{\frac{1}{2}}\| \leq 1$ as required. \square

3. COMPOSITION OPERATORS ON $H^\infty(B)$

In this section we study the connected components of the space of composition operators on $H^\infty(B)$ with the uniform norm topology where B is a bounded symmetric domain. Our motivation was to extend the one variable results in [12] to the case of infinitely many variables. In the case where B is the open unit ball of a Hilbert space or of a commutative C^* -algebra we refer to [2]. The key to this study is the distance d_B which gives a formula for the hyperbolic distance C_B , namely,

$$C_B = \log \frac{2 + d_B}{2 - d_B}.$$

Just as the Möbius maps are crucial when studying Δ , so the analogous automorphisms $\{g_a : a \in B\}$ of B are essential here and we establish some simple identities.

Lemma 3.1. For $a, b \in B$,

$$g_{-a}(a + b) = (B(a, a)^{-\frac{1}{2}}b)^a \quad \text{when } a + b \in B, \quad (7)$$

$$\text{and} \quad g_{-a}(b) = (B(a, a)^{-\frac{1}{2}}(b - a))^a. \quad (8)$$

Proof. Clearly, the two expressions are equivalent. Recall that $g_a(z) = a + B(a, a)^{\frac{1}{2}}z^{-a}$. Since the inverse of $z \rightarrow z^a$ is $z \rightarrow z^{-a}$ and the inverse of g_a is g_{-a} it follows that $g_{-a}(b) = g_a^{-1}(b) = (B(a, a)^{-\frac{1}{2}}(b - a))^a$. \square

For $z, w \in B$, we define $\beta(z, w) := \|g_{-z}(w)\|$.

Definition 3.2. For $\phi, \psi \in H(B)$ we let

$$d_\beta(\phi, \psi) := \sup_{z \in B} \beta(\phi(z), \psi(z)).$$

We note that d_β is a metric on $H(B)$ and, by virtue of the following result, it is the topological structure of $(H(B), d_\beta)$ that interests us.

Proposition 3.3. *Let $\phi, \psi \in H(B)$. Then*

$$\|C_\phi - C_\psi\| = \frac{2 - 2\sqrt{1 - d_\beta(\phi, \psi)^2}}{d_\beta(\phi, \psi)}.$$

In particular, the space of composition operators on $H^\infty(B)$ with the uniform norm topology is homeomorphic as a topological space to $(H(B), d_\beta)$.

Proof. Proposition 2.2 together with (4) and (5) gives

$$d_B(z, w) = \sup\{\|f(z) - f(w)\| : f \in H(B)\} = \frac{2 - 2\sqrt{1 - \beta(z, w)^2}}{\beta(z, w)}.$$

Since $h(t) = 2(1 - \sqrt{1 - t^2})/t$ is an increasing function on $[0, 1)$ we have

$$\begin{aligned} \|C_\phi - C_\psi\| &= \sup\{\|C_\phi(f) - C_\psi(f)\|_\infty : f \in H^\infty(B), \|f\|_\infty \leq 1\} \\ &= \sup\{\|f \circ \phi - f \circ \psi\|_\infty : f \in H(B)\} \\ &= \sup\{\|f(\phi(z)) - f(\psi(z))\| : f \in H(B), z \in B\} \\ &= \sup_{z \in B} d_B(\phi(z), \psi(z)) \\ &= \sup_{z \in B} \frac{2 - 2\sqrt{1 - \beta(\phi(z), \psi(z))^2}}{\beta(\phi(z), \psi(z))} \\ &= \frac{2 - 2\sqrt{1 - d_\beta(\phi, \psi)^2}}{d_\beta(\phi, \psi)}. \end{aligned}$$

\square

Our aim is to determine the connected components of the space of composition operators on $H^\infty(B)$. The above result means that we can now do this by examining the space $(H(B), d_\beta)$. In order to achieve this, we use JB^* -triple tools such as Bergman operators and the quasi-inverse map as a substitute for the algebra structure used when $B = \Delta$ [12] or B is the unit ball of $\mathcal{C}_0(X)$ [2] and as a substitute for the inner product used when B is a Hilbert ball [2].

To begin with we note that the d_β -topology on $H(B)$ is stronger than the $\|\cdot\|_\infty$ topology. Indeed from (8) we have that

$$(g_{-w}(z))^{-w} = B(w, w)^{-\frac{1}{2}}(z - w)$$

for $z, w \in B$ and hence we may write

$$z - w = B(w, w)^{\frac{1}{2}}(g_{-w}(z))^{-w}.$$

Since $\|B(w, w)^{\frac{1}{2}}\| \leq 1$ and

$$\|(g_{-w}(z))^{-w}\| \leq \frac{\|g_{-w}(z)\|}{1 - \|w\|\|g_{-w}(z)\|} \leq \frac{\|g_{-w}(z)\|}{1 - \|g_{-w}(z)\|}$$

this gives that

$$\|z - w\| \leq \frac{\beta(z, w)}{1 - \beta(z, w)}$$

for all $z, w \in B$. Therefore for $\phi, \psi \in H(B)$ we have

$$\sup_{z \in B} \|\phi(z) - \psi(z)\| \leq \sup_{z \in B} \frac{\beta(\phi(z), \psi(z))}{1 - \beta(\phi(z), \psi(z))}$$

and hence $\|\phi - \psi\|_\infty \leq \frac{d_\beta(\phi, \psi)}{1 - d_\beta(\phi, \psi)}$. In particular, if $d_\beta(\phi, \psi_t) \xrightarrow{t} 0$ then $\|\phi - \psi_t\|_\infty \rightarrow 0$.

The converse however is not true. For example, in Δ , $\beta(a, e^{it}a) = |g_{-a}(e^{it}a)| = \left| \frac{(e^{it}-1)a}{1-e^{it}|a|^2} \right|$ which implies $d_\beta(\text{id}, e^{it}\text{id}) = 1$ for all $t \in (0, 2\pi)$, even though $\|\text{id} - e^{it}\text{id}\|_\infty \rightarrow 0$ as $t \rightarrow 0$.

However, the two topologies do agree on the set of holomorphic functions which map B strictly inside B . In other words, if $\|\phi\|_\infty < 1$ then $\|\phi - \psi_t\|_\infty \xrightarrow{t} 0$ if and only if $d_\beta(\phi, \psi_t) \rightarrow 0$. To see this, we note from (8) that

$$\begin{aligned} \|g_{-a}(b)\| &\leq \|(B(a, a)^{-\frac{1}{2}}(b - a))^a\| \\ &\leq \frac{\|b - a\|}{1 - \|a\|^2 - \|a\|\|b - a\|} \end{aligned}$$

from repeated use of (1) and (2) when $\|b - a\|$ is sufficiently small. Therefore if $\|\phi\|_\infty < 1$ we have

$$\begin{aligned} d_\beta(\phi, \psi_t) &= \sup_{z \in B} \|g_{-\phi(z)}(\psi_t(z))\| \\ &\leq \sup_{z \in B} \frac{\|\phi(z) - \psi_t(z)\|}{1 - \|\phi(z)\|^2 - \|\phi(z)\|\|\phi(z) - \psi_t(z)\|} \\ &\leq \frac{\|\phi - \psi_t\|_\infty}{1 - \|\phi\|_\infty^2 - \|\phi\|_\infty\|\phi - \psi_t\|_\infty} \end{aligned}$$

and hence $\|\phi - \psi_t\|_\infty \xrightarrow{t} 0$ implies that $d_\beta(\phi, \psi_t) \xrightarrow{t} 0$ as well.

Given $\phi, \psi \in H(B)$, it is obvious from the definition that $d_\beta(\phi, \psi) \leq 1$. Later results will show the importance of determining whether $d_\beta(\phi, \psi) < 1$. We remark therefore that if ϕ

maps B strictly inside B , then the condition $d_\beta(\phi, \psi) < 1$ is satisfied for every $\psi \in H(B)$ which also maps B strictly inside B . To see this, we use (6) to write

$$\frac{1}{1 - \|g_{-\phi(z)}(\psi(z))\|^2} = \|B_{\phi(z)}^{-1}B(\phi(z), \psi(z))B_{\psi(z)}^{-1}\|$$

and hence $d_\beta(\phi, \psi) = \sup_{z \in B} \|g_{-\phi(z)}(\psi(z))\| < 1$ if and only if

$$\sup_{z \in B} \|B_{\phi(z)}^{-1}B(\phi(z), \psi(z))B_{\psi(z)}^{-1}\| < \infty.$$

Since from (1)

$$\|B_{\phi(z)}^{-1}B(\phi(z), \psi(z))B_{\psi(z)}^{-1}\| \leq \frac{\|B(\phi(z), \psi(z))\|}{(1 - \|\psi(z)\|^2)(1 - \|\phi(z)\|^2)}$$

and $\|B(\phi(z), \psi(z))\| \leq (1 + \|\phi(z)\| \|\psi(z)\|)^2$ for all $z \in B$, it follows that if $\|\phi\|_\infty < 1$ and $\|\psi\|_\infty < 1$ then

$$\sup_{z \in B} \|B_{\phi(z)}^{-1}B(\phi(z), \psi(z))B_{\psi(z)}^{-1}\| \leq \frac{4}{1 - \|\phi\|_\infty^2} \frac{1}{1 - \|\psi\|_\infty^2} < \infty$$

and hence $d_\beta(\phi, \psi) < 1$.

Since $d_\beta(\phi, 0) = \|\phi\|$, the converse is also true. In other words, for $\phi \in H(B)$

$$d_\beta(\phi, \psi) < 1 \text{ for all } \psi \text{ with } \|\psi\|_\infty < 1 \text{ if and only if } \|\phi\|_\infty < 1. \quad (9)$$

Theorem 3.4. *Let $\phi, \psi \in H(B)$ with $d_\beta(\phi, \psi) < 1$. Then the map*

$$t \mapsto \phi_t := t\phi + (1-t)\psi$$

is a d_β -continuous path joining ϕ to ψ .

The proof breaks into two parts proved below. The first part (Lemma 3.6) shows that any convex combination ϕ_t of ϕ and ψ satisfies $d_\beta(\phi_t, \psi) \leq d_\beta(\phi, \psi)$. The second part (Lemma 3.7) then uses this to show that the map $t \mapsto \phi_t$ is d_β -continuous which proves the theorem. As a result of the homeomorphism $\phi \mapsto C_\phi$ guaranteed by Proposition 3.3, this theorem immediately implies the following.

Corollary 3.5. *Let $\phi, \psi \in H(B)$ with $d_\beta(\phi, \psi) < 1$. Then C_ϕ and C_ψ are in the same path connected component in the space of composition operators on $H^\infty(B)$.*

In particular, from (9), we have that the set of composition operators C_ϕ with $\|\phi\|_\infty < 1$ is path connected. This is proved in a more general setting in [2]

Lemma 3.6. *Let $\phi, \psi \in H(B)$ satisfy $d_\beta(\phi, \psi) = \lambda < 1$. Then for any $t \in [0, 1]$ we have $d_\beta(\phi_t, \psi) \leq \lambda$, where $\phi_t = t\phi + (1-t)\psi$.*

Proof. Note first that $\phi_t = \psi + t(\phi - \psi)$. Fix $z \in B$. For ease of notation we write $f(\phi)$ rather than $f(\phi(z))$ whenever f is a function on B . Now we use (7) with $a = \psi(z)$ and $b = t(\phi(z) - \psi(z))$ to obtain

$$\begin{aligned} \beta(\psi(z), \phi_t(z)) &= \|g_{-\psi}(\psi + t(\phi - \psi))\| \\ &= \left\| \left[B(\psi, \psi)^{-\frac{1}{2}}(t(\phi - \psi)) \right]^\psi \right\| \\ &= t \left\| \left[B(\psi, \psi)^{-\frac{1}{2}}(\phi - \psi) \right]^{t\psi} \right\| \quad (\text{since } (tx)^y = tx^{ty}) \\ &= t \left\| \left(\left[B(\psi, \psi)^{-\frac{1}{2}}(\phi - \psi) \right]^\psi \right)^{(t-1)\psi} \right\| \quad (\text{since } (x^y)^z = x^{y+z}) \\ &= t \| (g_{-\psi}(\phi))^{(t-1)\psi}(z) \| \quad \text{from (8)}. \end{aligned}$$

Since $\sup_{z \in B} \|g_{-\psi(z)}(\phi(z))\| = d_\beta(\phi, \psi) = \lambda < 1$ and $\|(t-1)\psi\|_\infty \leq 1$ we use (2) to get

$$d_\beta(\psi, \phi_t) \leq \frac{t\lambda}{1 - (1-t)\lambda} \leq \lambda. \quad (10)$$

□

Lemma 3.7. *Let $\phi, \psi \in H(B)$ satisfy $d_\beta(\phi, \psi) = \lambda < 1$. Then for $t, t + \delta \in [0, 1]$ we have*

$$\lim_{|\delta| \rightarrow 0} d_\beta(\phi_t, \phi_{t+\delta}) = 0.$$

Proof. Assume firstly that $t > 0$. (Again, we write f for $f(z)$ where convenient.) Notice that $\phi_{t+\delta} = (t + \delta)\phi + (1 - t - \delta)\psi = \phi_t + \delta(\phi - \psi)$. We apply (7) with $a = \phi_t(z)$ and $b = \delta(\phi(z) - \psi(z)) = \frac{\delta}{t}(\phi_t - \psi)$ to get

$$\begin{aligned} g_{-\phi_t}(\phi_{t+\delta}) &= \left(B(\phi_t, \phi_t)^{-\frac{1}{2}} \left(\frac{\delta}{t}(\phi_t - \psi) \right) \right)^{\phi_t} \\ &= (\varepsilon B(\phi_t, \phi_t)^{-\frac{1}{2}}(\psi - \phi_t))^{\phi_t} \quad \text{where } \varepsilon := -\frac{\delta}{t} \\ &= \varepsilon \left(\left[B(\phi_t, \phi_t)^{-\frac{1}{2}}(\psi - \phi_t) \right]^{\varepsilon\phi_t} \right) \end{aligned}$$

since $(\alpha x)^y = \alpha x^{\alpha y}$ for $\alpha \in \mathbb{R}$. Now as $(x^y)^z = x^{y+z}$ we have

$$\begin{aligned} g_{-\phi_t}(\phi_{t+\delta}) &= \varepsilon \left(\left[B(\phi_t, \phi_t)^{-\frac{1}{2}}(\psi - \phi_t) \right]^{\phi_t} \right)^{(\varepsilon-1)\phi_t} \quad \text{which from (8)} \\ &= \varepsilon (g_{-\phi_t}(\psi))^{(\varepsilon-1)\phi_t}. \end{aligned} \quad (11)$$

From Lemma 3.6 we have that $d_\beta(\phi_t, \psi) \leq d_\beta(\phi, \psi)$ and hence $\|g_{-\phi_t(z)}(\psi(z))\| \leq \lambda < 1$ for all $z \in B$ and we can choose δ , and hence ε , sufficiently small so that $|\lambda(\varepsilon - 1)| \leq \lambda' < 1$. In particular,

$$\|g_{-\phi_t(z)}(\psi(z))\| \|(\varepsilon - 1)\phi_t(z)\| \leq \lambda' < 1$$

for all $z \in B$. We then have

$$\begin{aligned} d_\beta(\phi_t, \phi_{t+\delta}) &= \sup_{z \in B} \|g_{-\phi_t(z)}(\phi_{t+\delta}(z))\| \\ &= \sup_{z \in B} \|\varepsilon (g_{-\phi_t(z)}(\psi(z)))^{(\varepsilon-1)\phi_t(z)}\| \quad (\text{from (11)}) \end{aligned}$$

$$\leq |\varepsilon| \frac{\lambda}{1 - \lambda'} = \frac{|\delta\lambda|}{t(1 - \lambda')} \quad (\text{from (2)}).$$

In particular, $\lim_{|\delta| \rightarrow 0} d_\beta(\phi_t, \phi_{t+\delta}) = 0$.

If $t = 0$ then $\phi_t = \phi_0 = \psi$ and $\phi_{t+\delta} = \phi_\delta$. Then (10) shows that $d_\beta(\phi_t, \phi_{t+\delta}) \rightarrow 0$ as $|\delta| \rightarrow 0$. \square

To get the following result we can now adapt the one dimensional proof in [12] to the infinite dimensional setting of a bounded symmetric domain.

Theorem 3.8. *Let B be a bounded symmetric domain and let $\phi, \psi \in H(B)$. The following are equivalent.*

- (i) C_ϕ and C_ψ are in the same path connected component of the space of composition operators on $H^\infty(B)$,
- (ii) $d_\beta(\phi, \psi) < 1$,
- (iii) $\|C_\phi - C_\psi\| < 2$.

Proof. (ii) \iff (iii) is immediate from Proposition 3.3. Corollary 3.5 is precisely the statement that (ii) implies (i).

To show (i) implies (ii), let $\phi \in H(B)$ and let $[\phi] = \{\psi \in H(B), d_\beta(\phi, \psi) < 1\}$. We recall that the map $\phi \mapsto C_\phi$ is a homeomorphism. Then, since (ii) implies (i), $[\phi]$ is contained in the path connected component of ϕ in $H(B)$. In fact $[\phi]$ is the path connected component of ϕ . To see this, choose $\omega \in H(B)$ which is not in $[\phi]$. Then $d_\beta(\phi, \omega) = \sup_{z \in B} \beta(\phi(z), \omega(z)) = 1$ which implies that

$$\sup_{z \in B} C_B(\phi(z), \omega(z)) = \sup_{z \in B} \tanh^{-1} \beta(\phi(z), \omega(z)) = \infty.$$

For $\psi \in [\phi]$, $\sup_{z \in B} C_B(\phi(z), \psi(z)) < \infty$ and so the triangle inequality for the Carathéodory distance C_B implies that

$$\sup_{z \in B} C_B(\psi(z), \omega(z)) \geq \sup_{z \in B} (C_B(\phi(z), \omega(z)) - C_B(\phi(z), \psi(z))) = \infty.$$

Thus $d_\beta(\psi, \omega) = 1$. In other words, the d_β -distance between any element of $[\phi]$ and any element not in $[\phi]$ is equal to 1. In particular, there can be no d_β -continuous path from an element of $[\phi]$ to an element not in $[\phi]$. We conclude $[\phi]$ is the path-connected component of ϕ in $H(B)$. Again by the homeomorphism of Proposition 3.3 the path component of C_ϕ in the space of composition operators is $\{C_\psi : \psi \in [\phi]\}$ and thus (i) implies (ii). \square

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