Wave-like modelling of cascaded, lumped, flexible systems with an arbitrarily moving boundary

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**Abstract.** This paper considers cascaded, lumped, flexible systems, which may be short and non-uniform, which are driven by an arbitrarily moving boundary. Such systems exhibit vaguely wavelike behaviour yet defy classical wave analysis. The paper proposes novel ways to analyse and model such systems in terms of waves. It presents two new wave models for non-uniform systems, one series and one shunt, defining their component wave transfer functions, and thereby providing a way to define, identify and measure component waves. Features of the models are compared. The series and shunt configurations are mutually consistent and can be combined into a single composite wave model. The models are exact, but elements within them remain arbitrary to some degree, implying slight differences in the wave decomposition of the system. Some good model choices are proposed and explored. Wave speed and wave impedance are briefly considered, as are ways to measure component waves. Implications are discussed.

**Keywords:** Flexible systems; wave analysis; wave dynamics; lumped systems; moving boundaries; position control; active vibration damping

1. Introduction

There is a long history of studying waves in lumped systems [1]. Frequently harmonic excitation is specified, with a corresponding harmonic propagating wave response, with assumed solutions implying a defined frequency and wavelength [2-4]. Generally wavelengths are also assumed to be sufficiently long to extend over multiple lumped elements, which are assumed to be uniform, at least internally [5]. In some cases, semi-infinite systems are studied [4]. The excitation is generally either harmonic or steady, so arbitrarily changing or dynamically controlled inputs are not considered, even when the analysis and results are in the time domain [5, 6]. Under such assumptions many classical wave concepts are still recognisable and applicable.

This paper considers finite, lumped, cascaded, flexible systems, attached to a moving boundary, which is the prime source of motion and may move arbitrarily. Such systems exhibit some wave-like behaviour, but orthodox definitions and wave concepts frequently do not apply to them, except perhaps as limiting cases, or under quite strong assumptions. Thus, there may be no obvious frequency, phase, wavelength, amplitude, or speed, for example, in cases where the boundary motion is arbitrary (rather than, say, harmonic or uniform); where the lumped system can be short (with few degrees of freedom); or where the system can be arbitrarily non-uniform. Under such circumstances classical wave approaches break down, terminology becomes undefined, and so it can be argued that there is no wave behaviour, at least not in any established sense.

The present study explores if new definitions of waves might be developed, which might still apply when standard wave criteria fail for the reasons indicated above, and yet might be compatible with classical wave definitions. To this end it proposes “wave models” for such systems. These enable the motion of the cascaded lumped system to be resolved into counter-propagating waves which can be defined precisely, even for arbitrary boundary motion, very short systems and very non-uniform systems. The paper shows that the wave model for a given system is not unique, and so the implied wave composition is to some extent arbitrary. The arbitrariness is explored and shown to be limited. There are also proposals as to how the arbitrary aspects can be resolved. Based on this work new definitions of wave speed and wave impedance are also proposed. Also the measurement in real time of the wave components is considered. All of this is believed to be novel and some of the implications are discussed.

Section 2 gives some background and context to the work. In Section 3 a previously presented wave model is considered for a uniform mass-spring chain, with a moving boundary at one end and the other end free. This introduces concepts such as wave model, wave transfer function, and boundary conditions. Section 4 introduces a new series-loop wave model suitable for non-uniform chain systems. In Section 5 it is shown how lumped systems can also be represented by wave models with shunt lines, modelling partial transmission and partial reflection, especially at non-uniformities. Section 6 compares the series and shunt wave models and how they can be combined. Section 7 considers how the notional wave components can be determined from measurements of the real system. Section 8 shows how wave models and wave transfer functions can be used to define a wave speed and wave impedance. Section 9 has a discussion and finally Section 10 draws some conclusions.

1. background

The present work might begin with the following intuition. When the boundary of a lumped flexible system moves, it causes a disturbance to propagate into the system. In many cases, sooner or later this disturbance apparently returns to the boundary, albeit in a modified form. This behaviour is somewhat wave-like, but it defies all classical defining in terms of waves, especially when the lumped system is short or non-uniform, or when the boundary motion is not regular.

Going further, note that at every instant, the effect of subsequent boundary motion on the flexible system depends both on the continuing boundary motion itself and on the current state of the flexible system. For example, if the flexible system is at rest, a given boundary motion will clearly begin to launch a disturbance into the system, inserting momentum and energy. But if the flexible system is already in motion, the boundary motion at any instant might increase or decrease the flexible system’s energy / momentum. If thinking in terms of waves, at any instant the boundary might be considered to be launching a disturbance, or reflecting or absorbing a previously-launched disturbance: indeed it is apparently doing all three simultaneously. But can these actions be distinguished in a quantifiable way? Can the boundary motion be unambiguously resolved, at every instant, into precisely defined components of launching, reflecting and absorbing?

The questions are not just of academic interest. The main motivation for this work is the control of under-actuated flexible systems, in which the moving boundary corresponds to the actuator, and the lumped system might be, for example, an articulated robot arm, or solar panels on a space structure. The number of lumped elements may be small and they can be non-uniform. But more importantly, for rest-to-rest motion, the transient behaviour is all important. Energy must be inserted into the flexible system and then removed in a rapid and efficient way which leaves behind the correct net displacement. To wait for harmonic waves or vibration modes to become established is to wait too long. There is then no obvious benefit in trying to distinguish between AC (oscillatory) and DC (net displacement, “rigid body” or “zero frequency”) components of these transient waves. The resolving that seems most appropriate is into outgoing and returning component motions.

This resolving, which turns out to be easily achieved in real time, leads to powerful, generic, robust strategies for rapidly controlling the motion of a wide class of flexible mechanical systems of great engineering interest [7-12]. Such control systems can eliminate poles from the flexible system response. The control background for this work is also reflected, for example, in the use of the Laplace domain, transfer functions, block diagrams, and feedback loops.

The focus is on chainlike, rectilinear systems with a free boundary at one end and a motion-controlled actuator at the other end. The ideas are not restricted to this combination, but this configuration is selected because of its frequent occurrence, from atomic force microscopes, through disk drive heads, robots, long-arm manipulators, cranes, and up to large space structures. The analysis extends to systems with damping, but for simplicity negligible damping will be assumed below.

Obviously motion at the moving boundary cannot be achieved without a force, but one can imagine the boundary actuator to be designed as a controlled subsystem which takes in a reference position request and supplies the force needed to move the actuator to this position. The actuator dynamics are of engineering interest, but they not of concern in this work, which begins at the output of the actuator, coinciding with one boundary of the flexible system. In this way, the problem of modelling the flexible system can be treated as a purely kinematic one, with displacement taken as the dependent wave variable, both at the boundary and throughout the system.

The wave analysis here presented is not restricted to displacement waves. The dependent wave variable could also be velocity, acceleration, or force, for example, but only displacement (motion) waves will be considered here.

1. series wave model for uniform mass-spring systems

The first and perhaps simplest case to consider is a series of *n* identical masses *m* and springs having coefficient of stiffness *k*. It is shown in the upper part of Fig.1. The moving boundary is *x*0(t) or *X*0(s) in the Laplace domain, and the free end *xn*(t) or *Xn*(s). One way to develop a wave model for this system is to first imagine it extended indefinitely to the right. One can then determine a transfer function, *G*(s), relating the motion of any mass to that of the previous mass, going rightwards, in this semi-infinite system, starting from rest. The equation of motion from rest of any mass can be expressed in the *s*-domain as

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Multiplying by *G*(s) gives a quadratic in *G*, with solutions

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where *n* = √(*k/m*).

Of the two solutions, only that with the negative sign is strictly proper, physically causal, and reflects the assumed passive nature of the system (excluding the moving boundary). It describes motion propagating rightwards, from the boundary, towards infinity, in the direction of the transfer function *G*. This is the solution used in the wave models below. The solution with the positive sign, referred to below as *G\**, does not have these properties. Its significance will be briefly considered below.



Fig.1 A uniform system (above) and its wave model (below), with *Xi=Ai+Bi*, *i=0, 1, …n.*

The finite system of Fig.1 is then modelled by the arrangement in the lower part of the figure which uses these (semi-infinite) transfer functions, *G*, in a negative feedback loop arrangement. This is an example of a “wave model” of a lumped system, in which the 2*n*+1 transfer functions *G* are “wave transfer functions” (*WTF*s). It models the dynamics of the lumped system exactly, in the sense that

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Equation (3) is easily validated by showing that the sums of the components in the wave model obey the same equations of motion as in the original lumped system, for all masses, including the last one at the free boundary.

The sign convention here, and for all wave models in this paper, is that positive wave values indicate positive displacements to the right, both for rightward wave components *Ai* (which are therefore compressive –or pushing– waves travelling rightwards) and leftward propagating components *Bi* (implying that these are extensive –or pulling– waves travelling leftwards).

The questions asked above in Section 2 can now be given a first, affirmative answer. If the system starts from rest, this wave model provides a way to resolve the motion of any mass into a rightwards propagating component, *Ai*, and a leftwards component *Bi*, unambiguously, and at every instant. In particular, the boundary motion, *X*0 can be resolved into two components, *A*0 and *B*0, outgoing and returning, with *A*0*+B*0*=X*0. With the system at rest, a step motion at the boundary, for example, will cause the outgoing wave *A*0 also to jump by the step, and then *B*0 will describe a returning wave, which in theory begins to arrive immediately. In the absence of further boundary motion, the returning wave *B*0 will be reflected, with inversion, at the boundary, to become a continuing *A*0, which will then circulate indefinitely in the model, corresponding to ongoing, undamped vibrations in the lumped system. If the boundary subsequently moves again, depending on the instantaneous values of *A*0 and *B*0, it may increase or decrease the current value of the reflected wave, *A*0. An increase in *A*0 can be interpreted as an additional launched wave, whereas a decrease can be seen as absorption of the returning wave, whether partial or total (total if the reflected contribution to *A*0 is zero).

An objection to speaking about waves in lumped systems is that, in general, a defining characteristic of wave phenomena is local isolation. There is always a clear propagation delay before motion in one part of a system has any effect at another location. In lumped systems, it is argued, every part is immediately interconnected with every other part. So, for example, when the boundary moves, the entire system is involved in all motion from the first instant, so there is no delay before some putative “return wave” becomes evident at the boundary. In response, it is pointed out that the present proposal does not deny this permanent interconnectedness of lumped systems, yet it also offers a way to speak lucidly about separable outgoing and returning motions within such systems. In the wave model, a change in of *A*0 will, in theory, produce a simultaneous change in *B*0, even if initially it is infinitesimal. The time it takes for the change in *B*0 to become significant, say comparable in magnitude to that of *A*0, depends on the flexible system’s length (number of DOFs), system stiffness values and system inertia values, reflected in the number and form of the *WTF*s. In this way, the wave models accurately capture both the theoretical instantaneous coupling and the apparent delay in lumped systems.

Note that the model works for all values of *n*>0, even down to a 1-DOF flexible system (*n*=1). Strange as it may seem therefore, the motion of even a single mass and spring can in this way be resolved into outgoing and returning component motions. So even the simple pendulum, for example, which played a foundational role in the history of mechanics, can now be seen in terms of circulating waves.

Comments on the wave model and WTFs

The *WTF G*(*s*) [or *G*(*s,n*), or simply *G*], has many interesting properties. It has no finite poles or zeros. Its steady state gain is unity and its instantaneous response is zero. The order is not well defined, but is approximately second order. It has an exactly flat magnitude response of unity up to 2*n*, after which it falls off rapidly, giving a low-pass filter characteristic with corner frequency 2*n*. Its phase lag grows with frequency up to 2*n*, after which it remains constant at precisely –180°. The inverse of *G* is obtained by replacing s with –s,

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| *G-1*(s) = *G*(-s). | () |

This inverse also corresponds to the second solution of the quadratic, *G\**, taking the positive radical in Eq.(2). Of interest below, a further property is that the derivative *G′*(s), with dimension of negative time, has a limiting value, as *s* goes to zero, of -1/*n*, whereas the limit of the derivative of *G*-1 is 1/*n*.

Although the absence of poles in *G*(s) implies no resonance frequencies, curiously the wave model nevertheless does have poles, as it must if it is to model the physical system. Thus for example, in Fig.1, the transfer function from the boundary to the last mass is

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The final expression in Eq.(5) has 2*n* purely complex poles, corresponding to *n* natural frequencies, as it should. Similarly, although the *WTF*s individually have a damped transient response (cf. the approximation in Eq.(6) below), when used in a negative feedback loop they can model undamped, non-decaying vibration. It is also curious that, when modelling fixed-free undamped vibration for example, the circulating waves are being continuously “dispersed” by each *WTF*, yet in the looped model they superpose to maintain the cohesion of an undamped vibrating system, with its periodic motion.

[It is probably obvious that in expressions such as Eq.(5), and in most of what follows, the curved brackets no longer indicate arguments of functions, such as in Eqs.(1 - 4), but product components, so the dots in Eq.(5) are superfluous and will be dropped below.]

Although not of direct concern in this paper, note that the *WTF* *G* is not rational and is challenging to convert to the time domain. However, very good approximations to *G* can be achieved by rational transfer functions of increasing order. A good, second order approximation is given by

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where *G* *= √(2k/m)*. The implication is that the response of the next mass in the semi-infinite, passive chain, to the motion of the previous mass, is approximately that of an inertia, *m*, a stiffness due to two springs, 2*k*, and an energy absorption due to the appended string which corresponds to a viscous damper of about half critical. For much of the work below, this approximation could be used without great loss of accuracy.

The claim above that the wave model in Fig.1 provides *unambiguous* resolution of the system’s motion into wave components should now be nuanced. In fact, the choice of wave model is not unique, and it will be seen that wave models other than that of Fig.1 are capable of capturing the system’s motion, also exactly. All such wave models can then claim to provide exact, self-consistent and unambiguous resolution into wave components, yet they may differ in the wave components they then describe, even if only slightly. To the extent that the choice of models is arbitrary therefore, there are arbitrary, ambiguous aspects in the wave analysis of lumped systems. This point will be further explored below in the context of wave models capable of modelling non-uniform systems.

1. Series wave model for non-uniform systems

In Fig.1, the lumped system was assumed to be uniform and all the WTFs in the corresponding wave model were simply assumed to be equal. It is now considered how this wave model might be adapted to represent a cascade of unequal masses and springs, shown in the upper part of Fig.4.

Series *WTF*s

Consider a section of a series of (in general) unequal masses and springs, a portion of which is shown in Fig.2. Let us assume that the only source of motion is somewhere to the left, and that the system extends indefinitely to the right, or equivalently, meets a boundary to the right which simulates such a passive extension to infinity. Under these assumptions the system can be considered one-way, left-to-right, in the following sense. Motion entering this system from the left –however dispersive and complex its propagation– sooner or later leaves to the right, never to return. Also, at steady state, the net displacement of all masses becomes equal. In what follows, where *Xi* indicates general mass displacements, *Ai* designates displacements associated with this one-way motion, propagating rightwards. The corresponding *WTF*s *Gi* are shown in Fig.2, which again imply that each mass acts as an actuator (motion source) for the next mass to its right, the motion of which is determined by its own mass and intervening spring, and by the remaining system to its right. Thus

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| *Ai = Gi Ai-1*, *i*=1,2, …, *n*. | () |



Fig.2 Part of a non-uniform mass spring chain, motion going rightwards, with WTFs indicated.

Under the action of this rightwards motion, the motion equation of mass *mi* is

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Using Eq.(7) twice, the *Ax* terms can be eliminated to give either

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In either case, once any *WTF* *Gi* in the chain is known, all *WTF*s can be determined in a daisy-chain fashion, using the (assumed) known system parameters. Equation (10) suggests working leftwards, determining each *Gi* from *Gi+1*. This is more reasonable physically, because as noted above, each *WTF* is determined by the system to its right.



Fig.3. Defining the *WTF*s *Gi*, *Hi* and *F* for the wave model of Fig.4.

Now consider Fig.3 which shows the original flexible system of *n* masses (shaded), to which a mirror-image system has been appended, through a repeat of the last spring *kn*. For reasons which will soon become apparent, the displacement wave variables denoted *Ai* are denoted *Bi* on arrival in the mirror system. The extended system is terminated in a mirror actuator, or active boundary, whose motion, *B*0, is determined by measuring the motion *B*1 of the mirror mass *m*1, and then ensuring *B*0*=H*1*B*1. If the transfer function *H*1 determining the motion *B*0 is chosen to simulate a passive, indefinite extension of the system to the right, then the system can be considered one-way, left-to-right, as described above: motion *A*0 enters the system at the left and leaves as *B*0 to the right, over time. Once *H*1 has been specified, the entire string of transfer functions can be determined unambiguously, using Eq.(10).

* 1. The series loop model



Fig.4. A series loop model of a non-uniform system using *WTF*s from Fig.3.

Now consider the wave model in the lower part of Fig.4, in which the *WTF*s used are identical to those in Fig.3, but with the mirror part to the right (the *Hi*) swung around to form a return loop. At the far end, *F* becomes a *WTF* interlink between the forward and return loops in the model, corresponding to the free end of the system. At the active boundary end, *B*0 is returned to *A*0 through the summing junction in a negative feedback arrangement.

This arrangement constitutes a series wave model for the non-uniform case. Its component *WTF*s are defined from Fig.3. Again, it exactly models the original, physical system, in the sense of Eq.(3): that is, the sum *Ai+Bi* of the outgoing and returning wave variables, at each part of the model, for *i*=1,2, …*n*, obeys the same equation of motion as the corresponding *Xi*. This is easily verified. It is thus a new physical model which again allows the system motion at every point to be separated into two notional component motions propagating in opposite directions. The summing junction ensures that *X*0 *= A*0 *+ B*0, so that Eq.(3) also applies to the boundary (*i*=0). Thus, at every instant, any boundary motion *X0*, can again be separated into precisely defined outgoing and returning components, *A*0 and *B*0, that is, a precisely defined combination of launching, reflecting and absorbing waves.

But although the superposed wave model is exact, the model is not (quite) uniquely defined. When it comes to evaluating the WTFs, there is one less equation than the number required. Thus, in Figs.3 and 4, there are 2*n* masses, and so 2*n* equations of motion (such as Eqs.(9) or (10)), but there are 2*n*+1 *WTF*s to be determined. One of the *WTF*s is arbitrary, making all of them arbitrary to some extent. If, for example, *H*1 is chosen, all the others will follow, but how best to choose *H*1 is not obvious, as many candidates meet the general modelling requirements. This question will be further considered below.

* 1. The free boundary

The “free” or unconstrained boundary, at the system tip in Fig.(4), is modelled by the *WTF* *F*, which corresponds to a reflection condition, changing the outgoing wave in the tip mass, *An*, into the returning wave component, *Bn*, in the tip mass. There are various ways this matter can be addressed.

As first approach, analogous to that used in distributed systems, one can imagine that the boundary does not exist, but rather that the system extends beyond the tip to infinity. Then imagine a second, returning system, coming from infinity, whose motion on arrival will be superposed on the first, to reproduce the real system dynamics. The boundary condition in the real system is that there should be no force acting on it from the right. So, when the waves in the returning system are superposed, the net force in the first spring in the imaginary, extended system should be zero. This defines what the return wave should be, for any given outgoing wave, that is, it defines the wave boundary condition. In the wave model, this argument can be used to determine the boundary *WTF*, *F*. The form of *F* which then emerges, however, will depend on exactly how the system is imagined to extend to infinity, as seen in the form of *Gn*, as well as how it is imagined to come back from infinity, reflected in the form of *Hn*. Again there are choices here which are arbitrary to a degree.

Alternatively, the equation of motion of the last mass can be used directly. The form of *Gn*, *F* and *Hn* are related by the requirement that

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This again shows that *Gn*, *Hn* and *F* are interdependent, with one equation for three unknown *WTF*s.

By analogy with free boundaries in distributed systems, an obvious thought is to set *F*=1, which amounts to eliminating *F* from the model, leaving a simple loop at the free tip. This implies *Bn=An*, or wave reflection without inversion, a natural and reasonable choice. Equations (9) or (10), with *F*=1, then allow *Hn* to be determined from *Gn*, or vice versa.

Although setting *F*=1 therefore seems attractive and produces a valid wave model, there are two points to note. In Fig.3, setting *F*=1 means eliminating the spring connecting the original and image systems and coalescing the two masses *mn* to form a single mass of 2*mn*. Overall there is then one less free mass, and so one less equation of motion, leaving 2*n* – 1 equations for 2*n* *WTF*s. So the same model ambiguity remains, with one equation too few. Secondly, even in the case of a perfectly uniform system, the wave model with *F*=1 will no longer be uniform as in Fig.1. The same need to double the central mass in Fig.(3) would make the other *WTF*s different from each other when Eq.(10) is applied.

So if, as seems reasonable, the uniform case should consist of identical *WTF*s, then the extra *WTF* *F=G* is needed in Fig.1. This is an indirect argument for retaining it also in the non-uniform case, although the benefit is now less clear as the *WTF*s are already unequal. (In designing the wave model in Fig.1, the ambiguity was resolved by specifying that all the WTFs should be identical. But this is not an essential requirement, not even in the uniform case.)

An interpretation of this discussion is as follows. The presence of *F* at the boundary implies a time delay (classically a “phase lag”) between *Bn* and *An*, whose sum is the tip position, *Xn*. It is as if the returning wave *Bn* takes a little time to get established after the arrival of the outgoing wave *An*. Suppressing *F* removes this delay. This speeding-up effect however is offset by the extra inertia associated with the doubling of the end mass, the effects of which are now incorporated into the *WTF*s *Gn* and *Hn*, and so into the entire string of *WTF*s, slowing them down by just the right amount to preserve the accuracy of the model.

* 1. Resolving the ambiguity in the series wave model

For an *exact* model, the only requirement on *H*1 is that it should affect passive motion absorption without constraining the final position. Simply a viscous damper, for example, will work. Nevertheless, it is natural to ask what is the best choice for *H*1, which in turn raises the question what “best” might mean in this context.

A good choice for *H*1 would be to use Eq.(2), with *n* =*√*(*k*1*/m*1). In terms of Fig.3, this implicitly ensures that, looking forward from the final mirror mass *m*1, the system ahead appears to extend uniformly to infinity. Another good choice would be to use, for *H*1, the approximate *G* of Eq.(6). This is easily implemented, is computationally light, and turns out to work well in the control application. It also makes all the other *WTF*s derived from it of rational order. Such choices however seem rather *ad hoc*: one would like to have more elegant criteria to guide the choice.

Another thought would be to choose *H*1 so that, if transmission and reflection were possible (as in a shunt model to be considered in Section 4 below), there would be no reflection of the returning wave arriving back though mass *m1* in Fig.4. In other words, anticipating results from Section 4, choose *H*1 so that coefficients *TB* and *RB* in Eqs. (20) and (19), corresponding to *i*=1, become 1 and 0 respectively. Unfortunately this argument leads to no new equations, because the series model, and so Eq.(10), already implicitly assume total transmission and zero reflection

It is here proposed that, to resolve the ambiguity, the most elegant and academically satisfying choice of *H*1 is to set *H*1 = *G*1. This choice leads to interesting features which are now considered in the following section. [If the reader prefers, this following section can be skipped without loss of continuity].

* 1. Consequences of setting *H*1 = *G*1 in Figs. 3&4

Viewed from the input actuator at the left, setting *H*1 = *G*1 means that the system in Fig.3 will then appear as extended indefinitely to the right, in a periodic way, with the real system followed by the mirror system, followed by a repeat of these two systems indefinitely. The same string dynamics will then be seen whether looking rightwards from the final mirror spring *k*1 or leftwards (at least back as far as the boundary, but actually to minus infinity, as will be seen below). The input impedance seen by the wave arriving at the mirror actuator in Fig.3 will then be equal to the output impedance of the flexible system delivering this wave. It is conjectured that this gives the optimum passive absorption in Fig.3, achieving steady state with minimum settling time in response to, say, an impulse or step input from the left boundary.

Formally, setting *H*1 = *G*1supplies the missing final equation to define the *WTF*s. But besides presenting a challenging mathematical manipulation problem, at that point something interesting emerges. When *G*1 is set equal to *H*1, combined with repeated application of Eq.(10), the resulting equation for *H*1 (or *G*1) is a quadratic, implying two solutions in general, just as *G* and *G\** arose in the uniform case (Eq.(2)).

The two pairs of quadratically-related solutions will be denoted *G*1 = *H*1 and *G\**1= *H\**1. Only one of these turns out to be strictly proper, remaining finite as *s*🡪∞, with a physically realisable time-domain impulse response, and giving a phase lagging (time delayed) output in response to an harmonic input. This pair will be denoted *G*1, *H*1. The other solutions, *G\**1, *H\**1 have none of these physically important features. There is no doubt, therefore, which pair to choose in the wave model. But what meaning can then be given to the other solutions, *G\**1= *H\**1?

In Fig.3, the existence of two solutions for the *WTF* *G1* implies that, in the most general case, the transfer from *A*0 to *A*1 can be considered as the sum of two component motions. Denoting the first components by subscripts *a*, and the second by *b*, one can formally write

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| *A*1*= A*1*a+A*1*b  = G*1*.A*0*a +G\**1*.A*0*b.* | () |

The first component, associated with *G*1, is a physically causal motion, that is, a component of *A*1 due to the forward wave component in *A*0, namely *A*0*a*, with finite amplitude and phase lag. This was the only motion intended when conceiving Fig.3, with its assumed one-way property. The second, unexpected term is associated with a component of the motion in *A*1 which is leading the corresponding component in *A*0, (*A*0*b*) and so can be interpreted as leftwards-going wave components seen by a rightwards-looking *WTF*, *G*1. This motion was assumed absent from Fig.3, in which the actuator at the left hand end was intended as the motion source, and the right actuator as the motion sink. The mathematics defining the *WTF*s, however, is unconstrained by such intentions: the proposed passive, absorbing nature of *H*1 was nowhere imposed. The mathematics then reveals the possibility of a second component to the solution, which can be understood physically as leftwards-propagating motion seen by rightwards-looking *WTF*s.

If, in Fig.3, *H*1 is indeed passive and causal, as in the wave model, this second, leftwards-travelling component will not exist and can be safely ignored: the second terms in Eq.(12) will be zero. But if the second, leftwards component is allowed to exist (by making *B*0 active), it will appear non-causal, with a time lead, when seen through *G*1. For this reason, the non-causal features are not surprising.

But such a leftwards wave, if it exists, could be described by its own, causal *WTF*, with the correct time delay, but looking leftwards, from *A*1 to *A*0, (corresponding in Fig.3 to *H*1, going from *B*1 to *B*0). Furthermore, if the dynamics are symmetrical about the first (and last) spring *k*1, this causal *WTF* should equal *H*1*= G*1. So, if the imagined motion component *A*0*b* in *A*0 is processed forward in time, non-causally, from *A*0 to *A*1, and simultaneously processed back again from *A*1 to *A*0, by the causal (time delay) *WTF*, one should get back to the same quantity. Thus the product of the causal and non-causal *WTF*s between station 0 and 1 should be unity, and so

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Furthermore, when Eq.(10) or (9) is then used to determine the other *WTF*s, at pairs of points between masses in Fig.3 where the system appears the same when looking forward from one and looking back from the other, the corresponding *WTF*s will have the property that

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So, in Figs.3 and 4, the *Gi* and *Hi* become, not inverses of each other, but inverses of each others’ associated non-causal versions. (These are the same properties which were found in the uniform *WTF*, *G*(s) and *G\**(s)). Finally, the properties above will also extend to the product of the entire string of *WTF*s in Fig.3, so that

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In other words, the product of the entire symmetrical system of *WTF*s of Figs.3 and 4 behaves, in some respects, like macro versions of the uniform *WTF* *G*(s) of Eq.(2).

These further *WTF* properties, resulting from setting *H*1*=G*1, have been verified as existing in wave models of non-uniform systems such as Fig.3. While not essential for a valid wave model, they do reveal a further, deeper symmetry obtained by setting *H*1*=G*1.

This series wave model is *global*, in the sense that, in theory, all system parameters eventually influence some or all of the *WTF*s. A way to make the *WTF*s local is to allow for partial internal reflection and transmission of waves within the wave model. This will now be considered.

1. wave model with partial internal reflections
   1. Transmission and reflection coefficients

From a classical wave perspective, a lumped system is dispersive, having sharp discontinuities between the lumped elements, even when the system is uniform. In the proposed wave models this dispersion effect is modelled mainly by the *WTFs*, but also by the model loop. When the system is not uniform, such discontinuities become even stronger. In non-lumped (i.e. continuous or distributed) systems, dispersion is usually associated with discontinuities in piecewise uniform systems, and is often analyzed as partial transmission and partial reflection at each discontinuity. In this spirit, the lumped wave models of Figs.1 and 3 will now be adapted to allow modelling of partial transmission and partial reflection at each lumped mass.

In Figs.1 and 3, the upper paths, *Ai*, modelled outgoing waves, the lower path, *Bi*, the return waves, with *Xi=Ai+Bi*. Now, to model internal reflection, the single-loop, series-configuration is replaced by the series-and-shunt arrangement, a representative section of which is shown in Fig.5.



Fig.5. Portion of a lumped system (above) and its wave model (below), allowing partial wave transmission and reflection of the wave components of mass displacements.

As before, the upper path with *WTF*s *Gi* is unidirectional to the right, the lower, with *Hi*, unidirectional to the left. If part of the rightwards incident wave, *IA*, arriving at a mass *mi* is reflected, it should therefore be shunted down to the lower, leftwards-propagating branch of the model, while the remainder, the transmitted part continues to propagate rightwards in the upper branch to mass *mi+*1. The opposite happens with wave motion incident from the right, travelling leftwards. The lightly shaded boxes in the wave model in Fig.5 represent transfer functions, with single input and double output, which determine how much of the incident wave is transmitted and how much reflected at each mass. They will now be evaluated.

The partial reflection and transmission is considered to be happening within the motion of each mass. Consider mass *mi*, and, for now, ignore any returning *B* wave, by assuming *IB*=0. At every instant, the component of the displacement of mass *i*, associated with the rightwards propagating wave *Ai*, can be expressed as either the sum of the incident and reflected displacement wave *IA+RA*, or the transmitted displacement wave, *TA*, so that, in keeping with the adopted wave sign convention outlined above,

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| *Ai=* (*IA + RA*)*i =* (*TA*)*i.* | () |

Associated with the component motion *IA* in *mi*, there is a compressive force in the *ki* spring to its left of

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| [*Gi-1IA – IA*]*ki = ki*(*Gi-1 –* 1)*IA.* | () |

acting rightwards on *mi*. Associated with the reflected component *RA* of *mi*’s motion there is an extensive force in the *ki* spring of *ki*(1*–Hi*)*RA*, acting leftwards on *mi*. Finally, with *TA* there is associated a compressive force in the *ki+1*spring of *ki+1*(1*–Gi+1*)*TA*, which acts leftwards on *mi*. The acceleration on *mi* associated with these forces can be expressed as *s2TA*, giving an equation of motion corresponding to these motion components as

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Combining Eqs.(16) and (18) and rearranging gives

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The terms in square brackets can be considered to be reflection and transmission coefficients, to be applied to the incident, rightwards-propagating wave *IA*, which in turn is given by *Gi* times the rightwards transmitted wave from the previous mass *mi-1*. In Fig.5 they are considered as evaluated in the shaded boxes, with a single input and two outputs. In the model the reflected component, *RA*, is added to the leftwards propagating wave in the lower branch which then enters *Hi*, while *TA* is added to the input to *Gi*+1. By inspection of these coefficients, as the mass *mi* becomes larger, or the frequency grows higher, the reflection coefficient approaches –1, indicating reflection with inversion, while the transmission coefficient approaches zero, both as expected. The low frequency limit is harder to visualise, but using l’Hôpital’s rule it can be shown that as s🡪0, *TA*🡪1 and *RA*🡪0.

The above assumed no leftwards propagating wave. When it exists, *IB*, incident on mass *mi* from mass *mi*+1 to its right, will undergo similar transmission and reflection. The corresponding coefficients *TB* and *RB* can be obtained in the same way, and for the *Bi* wave in mass *mi* they are

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Regarding boundaries in this model, they are similar to those of the series model of Fig.4. At the free boundary to the right there is no transmission and so total reflection. Instead of the corresponding transmission and reflection coefficients of Fig.5, with its two-way path, there is a single connecting path, with the output of the last upper *WTF*, *Gn*, passing directly through the tip *WTF*, *F*, to enter *Hn*, the beginning of the return path. The left boundary at the actuator is simply a summing junction with negative feedback, exactly as in Figs.1 and 4.

Although not considered further in this paper, if an external force *fi*, acts on mass *i*, positive when rightwards, the rightwards coefficients become

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and similarly for the leftwards coefficients *TB* and *RB*. (Alternatively, the leftwards coefficients can be left unchanged and *fi/2* in Eqs.(23) & (24) replaced by *fi*.)

* 1. Model under-specified

These transmission and reflection coefficients in the shunt wave model are expressed in terms of the *WTF*s *Gi* and *Hi*. But these primary *WTF*s have not yet been specified. In fact they are arbitrary to a considerable degree, now having less constraints than in the strictly series models above. For any reasonable choice of *Gi* and *Hi*, the transmission and reflection coefficients based on that choice ensure that the model obeys the equations of motion of the physical system exactly.

One reasonable choice for *Gi* and *Hi* is to use the *WTF* of Eq.(2), *G*(*s,n*), making *n* a parameter tuned to the local spring stiffness and mass values at each point, that is with value *i* *= √*(*ki/mi*). In other words, the non-uniform system is then modelled as if it were a series of very short, quasi-uniform systems, with the transmission and reflection transfer functions correcting the unjustified assumptions implicit in using Eq.(2) for the non-uniform case. This approach could be considered the lumped equivalent of a piecewise-uniform distributed system. It also has the merit that, as the system becomes more uniform, the model approaches Fig.1 again, with uniform *WTF*s and (as will be explained below) vanishing shunting.

Alternatively, the freedom to define *Gi* and *Hi* can be availed of to make them simpler to model than Eq.(2), for example using Eq.(6) with*G* *= √*(*2ki/mi*), or *G* *= √*((*ki + ki+1*)*/mi*). The overall wave model now involves only standard, rational polynomial transfer functions throughout, yet it will still be exact.

1. Comparing series and shunt models

In comparing the series and shunt model, the series model appears simpler and less ambiguous than the shunt. In the series model there is no need to evaluate transmissions and reflections, and all the *WTF*s can be determined in a daisy-chain fashion once *H*1 is chosen. On the other hand, if *H*1 is set equal to *G*1, all the *WTF*s, including *H*1, will be transcendental and global, in that each *WTF* will be a function of all the system mass and spring values, even if their dynamics are dominated by the local mass and spring values. Furthermore, even for a simpler choice for *H*1, the order and complexity of the *WTF*s grows with system length.

By contrast, in the case of the shunt model, all the *WTF*s can be of the same low order (and can all be exactly second order, if desired) no matter how long the system. Also they can be purely local, that is, dependent only on the masses and stiffness values in the immediate neighbourhood.

In the shunt model the reflection coefficients, given by Eq.(21), will be zero in the rightwards propagating wave if

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This condition is identical to Eq.(10), which defines the relationship between the *WTF*s in the series model. Thus the series model could be considered a special case of the shunt model in which *Gi* and *Hi* are specified so as to make all the reflection coefficients zero and all the transmission coefficients unity, thereby allowing the shunts to be dispensed with.

The two models are therefore mutually consistent and can be mixed. Thus shunts could be introduced into an otherwise series model, for example where there are strong discontinuities in the dynamic properties. Between the shunts, the *WTF*s of the series sections will then be local to that section, and their order limited by the length of the section. Such a mixture would be particularly appropriate where the system comprises piecewise uniform sections, with shunts introduced only at the changeover between these sections. The limiting case where these “uniform” sections become as short as single mass-spring pairs, supports the idea of using variations of the uniform *G*(s) of Eq.(2) in the non-uniform shunt model.

It has been emphasised that different choices of *WTF*s and of wave models will give different notional wave components. But it should also be emphasised that the divergence is very limited. The fact that the circulating waves always superpose to model the real dynamics exactly implies that there are strong limits to how much waves can diverge in practice. In the first place, the loops in the models (in both series and shunt models) have a self-correcting effect. For example, with a given actuator motion, *X*0, in a given physical system, different wave models will indicate different initial returning wave components *B*0. But, because *A*0*=X*0 *– B*0, the launch wave component *A*0 will also be different by just the right amount to keep the superposition correct, not only at the actuator, but throughout the system. So, for example, if *B*0 is initially larger in one particular wave model, the negative feedback will quickly reduce the value of *A0*, and this in turn will quickly reduce *B*0 when the wave comes around again, to bring things back towards conformity.

As already noted, even when damping is assumed negligible, the *WTF*s themselves are quite dispersive, yet the complete wave model, incorporating the dispersive *WTF*s, remains coherent and energy conserving, again indicating a self-correcting or self-compensating effect within the model.

Another factor to help convergence is that the *WTF*s share information with each other in the way they are defined. For example, in the series mode, if instead of making *H*1 = *G*1, a computationally simpler choice is made, the successive calculation of the other *WTF*s in the chain, using Eq.(10), takes the initial choice into account, adjusting each one in turn so that it still captures the dynamics of each successive mass and spring.

The variations in successive *WTF*s due to variations in the originally assigned *WTF* will fade with distance, due to the filtering and dispersing effects of the intervening dynamics, although without ever disappearing completely, at least not when the system is lossless. The only way to completely terminate this propagating effect in lossless systems is to use a shunt, whether it is consciously introduced at some internal point in the model, or arises in a natural way at a boundary.

As already noted, the series and shunt models can be designed so that they both approach the uniform case of Fig.1 as the system becomes more uniform, which again highlights their mutual consistency. Formally, Eqs (10) or (9) yield Eq.(2) under the assumption of uniformity, and, in the case of the shunt model, the shunts then become redundant. To put it another way, Fig.1 can be considered a special case of the other models.

The series and shunt models become identical for one degree of freedom systems, such as a single mass and spring, or a simple pendulum. These cases dramatically heighten the “connectedness” of the system and the resulting absence of delay before the “returning wave” manifests itself. Indeed it might seem extreme to think of such simple 1-DOF systems in terms of circulating waves, yet it is reassuring that the proposed wave models remain perfectly valid even for such systems.

Any differences that do exist between waves in different models will be greater during transients. By contrast, if the main interest is steady harmonic behaviour, or where only the superposed motion is of interest, any valid wave model can be used. This applies, for example, in wave-based modal analysis, considered in Ref. [10], where the resonances are seen to correspond to the conditions for standing waves to become established in series wave models such as Figs.1 and 4. On the other hand, transients are very important in wave-based control, for rapid rest-to-rest manoeuvres of flexible systems, so the potential effects of model differences must be considered. It transpires, however, that, even for control, the choice of model ultimately does not matter much. Either model can be used successfully, because, the control is being done at the boundary, which acts as a controlled shunt in both cases.

1. Measuring the waves

In the case of waves in distributed media (1-D or plane waves in 2 or 3-D), the motion at any point can be resolved into two component waves. This requires two measurements and a value for one wave parameter (such as wave speed or impedance). In the lumped case, a similar resolving can be achieved using wave models and *WTF*s, again using two measurements at a given point. In this section the chosen point is the active boundary, and for brevity, only the series wave models will be considered. The same ideas will easily extend to all points and to other wave models.

Applying Eq.3 to the boundary and the first mass, gives respectively

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Multiplying Eq. (27) by *H*1, rearranging, and using Eq.(26) gives

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Thus, if the *WTF*s *G*1 and *H*1 are known, or can be modelled, then by taking *X*0 and *X*1 as inputs, *A*0 and *B*0 can be determined. Although Eq. (29) is implicit in *A*0, there is no difficulty in implementing a scheme in the time domain which takes *X*0 and as inputs (or their time domain values) and produces *A*0 and *B*0, or *a*0(t) and *b*0(t). Practical implementations are shown as real-time control systems in [7-12].

For various practical reasons, instead of using *X*1, it is slightly better to take as inputs the measured variables *X*0 and *F*0, where *F*0 is the interface force between the boundary and the flexible system. Then the dimensionless *WTF*s *G*1 and *H*1 are replaced by cross-over wave transfer functions (*xWTF*s), *Ĝ*1*, Ĥ*1, with mixed dimensions crossing from displacement to force and force to displacement respectively. Then Eqs.(28) and (29) are replaced by

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where .

In the continuous case, component waves can be resolved instantaneously, from the current values of the measurements. In the lumped case, the use of *WTF*s can be seen as adding memory to the wave resolving process, integrating information over time within the dynamics of the *WTF*s.

Note that the steady-state gain around the wave model is unity. This means that, for rest-to-rest motion, the final steady value of *B*0 will equal that of *A*0, and both will be half of the net displacement of the boundary *X*0. For rest-to-rest motion through a distance *d*, therefore, a wave of net displacement ½*d* enters the system through the active boundary, travels to the tip, returns, and leaves the system again through the active boundary, eventually leaving *b0*=½*d*, and *xi=d*.

All the above assumes that the system starts from rest, and that, when measurement begins, the inputs and outputs of all *WTF*s are zero, or at least unchanging. The dynamics of the *WTF*s then provide a memory, which gradually fades, of aspects of recent motion in the system, which assist in the process of resolving the waves. The definitions of the *WTF*s in the wave models also assume zero initial conditions, so the waves defined by reference to the wave models implicitly assume motion from rest. If for some reason the system is already in motion when the measuring begins, and if the history of the motion from rest has not been recorded, there is then no clear definition of the waves, at least not at that instant. If the history is available, then the *WTF*s could be initialised as if they had been monitoring the motion from rest. If not, the initialisation of the *WTF*s will be arbitrary to a strong degree (as only the superposed waves will be specified), implying a correspondingly arbitrary wave distribution in the model. Due to dispersion and the fading *WTF* memories, however, resulting differences in wave components would gradually fade and different wave representations would converge. Note that it is only the wave components within the model which may differ for a while: their superposition will still model the overall system dynamics correctly from the instant of initialisation to all future times.

In an engineering context it will usually be possible to arrange to begin all measurements from a steady-state condition, in which case the problem of initialising the states of the *WTF*s will not arise or will be trivial. Also, in the control applications discussed in the references, the actuator works to insert a specified “new” wave into the system and to absorb the “old” wave, so an arbitrary initialisation would become irrelevant even more rapidly than would otherwise be the case.

1. Wave speed and impedance

In a short lumped system, especially one undergoing transient motion, there is no identifiable “phase” by which one might define a phase velocity. Nor are there identifiable wave fronts with definable speeds. Due to the complicated dispersion, defining impedance is likewise challenging. Furthermore, the masses and springs behave differently. Each mass experiences two (generally) different forces in the attached springs on each side. Similarly, each spring has a single force, but experiences two (generally) different motions associated with separate masses at each end. The force discontinuity across mass *mi* is *mis2*. The motion discontinuity across spring *ki* can be expressed in the wave system as either *Ai*-1(*Gi* – 1) or *Ai*(1 – *Gi*-1). Thus at points where the local motion is clearly identifiable, the force is not, and vice-versa. So if wave impedances or speeds are to be specified, there is a need to clarify where and how they are defined.

Depending on whether the force is taken as that in the spring to the left or to the right of a given mass, the local wave impedance, defined as the ratio of force to velocity, has two possible values

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The two impedances are related by

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so that there is an impedance discontinuity across the mass of *mis*. If preferred, one could also take the average of the two impedance values, in which case the impedance would involve two *WTF*s.

In a previous paper [7], the wave speed for a uniform system was inferred from the phase of *G*(s) with frequency, which implicitly assumes harmonic waveforms. A more general and direct approach is now proposed. If *u*(*x,t*) is a wave variable that is a continuous function of time, *t*, and space, *x*, then for systems obeying PDEs a more fundamental definition of the wave speed, *c*, is

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A lumped system obeys ODEs, and has no inherent physical dimensions. But one can assign a length, *l*, to the separation between the lumped mass centres, for example. Near mass *i*, the spatial rate of change of the displacement can be approximated by a finite difference, either to the left as (*Ai –Ai-1*)/*l* or to the right, as (*Ai*+1*– Ai*)/*l*. The time rate of change, centred across the left or right spring, will be s(*Ai*-1*+Ai*)/2 or s(*Ai+Ai*+1)/2. When substituted into Eq.(34), the speeds becomes

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Although the philosophy here was not to rely on cyclic wave ideas, if required these speeds can be expressed as dispersion relationships as a function of frequency, by replacing *s* with *j*.

The low frequency limit can be got as the limit of Eq.(35) as *s* approaches zero, which, by l’Hôpital’s rule is

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The quantity *G′*(s) has dimensions of time, and, for the wave speed to be positive and finite, it should be negative and finite. Equation (36) shows, for example, that its low frequency limit is (the negative of) the time for a low frequency wave to move from one lumped mass to the next.

For *Gi*=*G*(s) the low frequency wave speed is *nl*, where *n* =√(*k/m*) This corresponds to the speed √(E/**) of longitudinal waves in uniform rods of cross sectional area *A* if the density, **, is taken as *mi*/(*Al*) and the equivalent to Young’s modulus, E, is *ki*.*l*/*A*. Thus for sufficiently long wavelengths, the lumped system behaves as if distributed, which is not surprising. But as the wavelength shortens and the frequency becomes higher, the dynamic effects of the lumped elements become more and more important. The wave speed falls until eventually there is a cut-off when *Gi* in Eq.(35) becomes –1 (unit magnitude, phase –180°).

Based on *WTF*s a wave group velocity, *vg*, can also be defined. In dispersive wave theory it is defined as *vg=d/d*, where ** is the frequency and ** is the wave number. The numerator in Eq. (35), with the negative sign, can be taken as the equivalent of **, and the denominator the equivalent of **, with Eq. (35) being the wave speed */*. Then the group velocity will be

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where the subscripts on *G* have been dropped and the primes indicate derivatives. Again the derivative, *G′*(s), with dimension of time, must be finite (non-zero) and negative for the group velocity to remain finite and positive. Again, for the uniform case, the low frequency limit of the group velocity (as *s* approaches zero) is *nl*, the same as for the wave velocity. This suggests that wave energy propagates at the same rate as wave displacement at low frequencies.

Note that these definitions of wave speed and impedance involve *WTF*s. So to the extent that the *WTF*s are not uniquely defined, these quantities are also not uniquely defined, and their values will (very slightly) reflect assumptions made in setting up the wave model.

Although lacking a similar formal justification to the above, a simpler, workable description of the wave speed through a given *WTF* *G*(s) (or within a single *WTF*) is here proposed as

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This definition seems comparable to those above, at least for low frequencies (small *s*), (noting that the magnitude of *G*(s) is unity up to the break frequency,*G*), while it is also clearly valid when *G*(s) is simply a pure (transport) delay.

1. Discussion

The validity of the classical, second-order wave equation assumes a uniform, continuous wave medium. To the extent that the properties of the continuous medium might be discontinuous around a given point, the concepts of wave speed, impedance and component waves at that point become less clear. Lumped models have inherent property discontinuities, even when uniform, so it is not too surprising that ambiguities also arise when they are analyzed in terms of waves, even if, when reasonable assumptions are made, the differences between the various wave models are minor and have few practical consequences.

There is one special case when *WTF* ambiguities completely disappear, namely in an infinite, or semi-infinte, uniform, system. It is interesting to view this imagined extension to infinity of the lumped system as related to the process of taking the limit in the continuous system, as a spatial increment goes to zero, to arrive at the classical wave equation. In a similar vein, one of the proposed series wave model involves imagining the system extended to infinity in a uniformly repeated pattern.

It has been assumed throughout that the values of the masses and springs are known exactly, and references to “exact” system modelling were in this context. In reality, exact values may not be known, and may vary over time. In any case the very concept of lumped masses and springs is an idealisation which is never exact. (A distributed model is likewise an approximation, of course.)

The waves in the lumped system are notional, but then so are the waves in distributed systems, as are the modal components of vibrating lumped systems. If the resolving into components in the latter examples seems more natural, perhaps it is simply because they are more familiar, and so the arbitrary choices involved seem a little less arbitrary.

At first sight *WTF*s might appear similar to “Transfer matrices” [13, 14], but they are significantly different. Transfer matrices relate all state variables between two points in a chain-like structure, whereas *WTF*s refer to a single wave variable, propagating in one direction. Furthermore, *WTF*s model full dynamic effects, both transient and steady-state, completely capturing how an arbitrary wave propagates between two points over time.

It would also appear that the wave analysis here assumes linearity, especially in the use of transfer functions and superposition. However, the theory can cope, for example, with non-linear springs. The details are given in [11]. An alternative approach is given in [12].

Finally, if the wave analysis of short, lumped system seems somewhat obscure, one might ask: why bother? Beyond the purely academic interest, there are already two established applications of practical engineering interest, one involving strongly transient behaviour [e.g. 8, 9], the other steady state [10]. The application involving transients is in the control of under-actuated flexible systems, when the boundary (actuator) is actively controlled using travelling wave concepts. Rather than resolving motion into vibrational and rigid-body modes and then attempting to control these as if uncoupled, a wave model allows net displacements to be treated as a single, transient wave motion, leading to very rapid, simple, robust, and generic control strategies. When the measurement point is collocated with the control point, the wave ambiguities pose no problem: rather they facilitate control even of complex flexible systems.

The steady-state application is for modal analysis [10], when the boundary is either fixed, or moving harmonically, and vibration modes and resonance frequencies are seen to correspond to standing wave conditions, which in turn are seen as superposed steady travelling waves. The waves in this case now have a single identifiable frequency, travel time, and defined phase delay per lumped element.

It would seem straightforward to develop this work in a number of ways. Similar ideas will apply to systems having internal damping, systems with external damping, systems with other external forces, systems undergoing impact, systems with multiple actuators, and systems with multiple branches. The ideas may also lead to a new way to do system identification from boundary measurements. The ideas might also find application in electrical engineering, for example, and in other disciplines.

1. conclusions

Ways have been presented which allow the dynamics of lumped, cascaded, flexible systems, driven by arbitrary boundary motion, to be resolved precisely into component wave motions, even when the systems are highly non-uniform and may have few degrees of freedom. The motion and its wave components do not have to be periodic, and can be entirely transient. If transient, the motion may involve a net change in the DC level of the wave variable. To the author’s knowledge, all this is novel.

The resolving into wave component motions is shown to be precise, in the sense that their superposition recovers the original system motion precisely. But it is not unique. The resolving is achieved using wave models. Two main kinds of models were presented, series and shunt. The series model in simpler, but involves all system component values at every wave measuring point, and in this sense is global. The shunt model is more complex but its features can be local to a point in the system, with local reflection and transmission. Within each model there are also arbitrary decisions, and proposals have been presented of ways to resolve the associated ambiguities. The models can also be combined, with a series model might be appropriate for uniform sections, with shunt paths added where more significant non-uniformities arise. Different wave models lead to different component wave motions, but the differences are necessarily limited, in part because the superposed motion remains precisely correct.

These wave models allow new definitions of traditional wave concepts, such as wave speed and impedance, which are compatible with the standard definitions but remain valid when the latter cease to have meaning.

The work suggests new ways to think about lumped, flexible systems. General vibratory motion can be seen as travelling waves, precisely defined, while resonance corresponds to standing waves. Furthermore, the ability to resolve lumped system motion into component waves, in real time, has led to new ways to control lumped flexible systems which are of wide engineering interest.

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