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**Dynamic analysis of a plate resting on elastic half-space with  
distributive properties**

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## **ABSTRACT**

This work gives a semi-analytical approach for the dynamic analysis of a plate resting on an elastic, half-space with distributive properties. Such calculations have been associated with significant mathematical challenges, often leading to unrealizable computing processes. Therefore, the dynamic analysis of beams and plates interacting with the surfaces of elastic foundations has to date not been completely solved. To advance this work, the deflections of the plate are determined by the Ritz method, and the displacements of the surface of elastic foundation are determined by studying Green's function. The coupling of these two studies is achieved by a mixed method, known in the theory of elasticity as Zhemochkin's method, which allows determination of reactive forces in the contact zone and, hence, the determination of other physical magnitudes. The obtained solutions can be applied to study the dynamic interaction between soils and structures and to assess numerical computations through various numerical methods programs. Natural frequencies, natural shapes, and the dynamic response of a plate due to external harmonic excitation are determined. Validation with a Winkler problem illustrates the distributive property effects on the results of the dynamic analysis.

**KEY WORDS:** Green's function; distributive properties; plate; Eigen-frequencies; Eigen-shapes; dynamic response.

## **1. INTRODUCTION:**

The dynamic behaviour of beams and plates resting on elastic foundation is a topic of high interest throughout the design and construction sectors. However, the dynamic analysis of such structures is associated with significant mathematical difficulties and leads often to impractical computational processes. For these reasons, the problem

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of dynamic analysis of plates resting on elastic foundations is not completely solved.. Thus, there is a need to develop precise methods of analysis of these structures and to provide a more effective methods than those that currently exist. This work provides a semi-analytical approach to determine the natural frequencies and natural shapes of a rectangular plate resting on elastic foundation with distributive properties (Boussinesq's type) and its response to an external harmonic excitation (Fig. 1). The following aspects are neglected: damping, inertia of the elastic foundation, and friction in the contact zone between the plate and the surface of the elastic foundation. The approach is based on the mixed method known as Zhemochkin's method [1], wherein the rectangular plate is divided into a finite number of identical elements, and in the center of each element is placed a rigid link, through which the contact between the plate and the surface of elastic foundation is accomplished. The approach assumes that contact between the structure and the surface of elastic foundation is replaced by a finite contact in rigid links and that the mass of each element is concentrated in its centre.

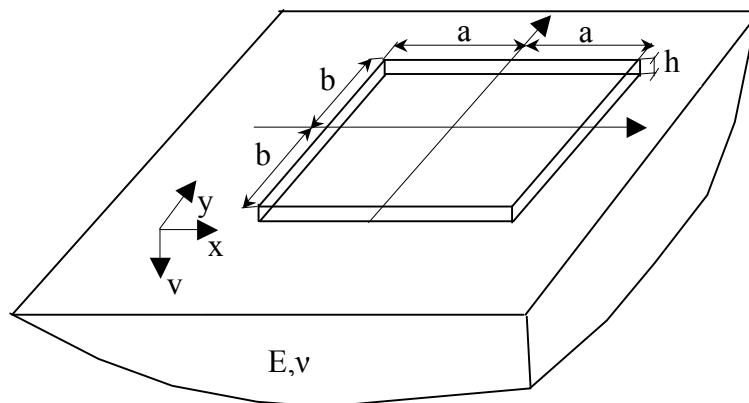


Fig. 1. Plate resting on elastic half space with distributive properties

## 2. PROBLEM ASPECT

### 2. 1. DESCRIPTION OF THE METHOD OF ANALYSIS

In brief, the procedure to study free vibrations of a rectangular plate resting on a surface of elastic foundations of a Boussinesq type [2] is as follows. A rectangular plate with mass  $M$  and cylindrical rigidity  $D$  rests on the surface of an elastic foundation with distributive properties with a modulus of elasticity  $E$  and a Poisson's ratio  $\nu$ . The essential parameters Zhemochkin's method for the study of the dynamic of a plate resting on an elastic foundation are shown in Fig. 2. The inertial forces are applied only on the plate, since the mass of the elastic foundation is not taken into account, while the efforts of connection are applied on the plate and on the surface of the elastic foundation fig. 3.

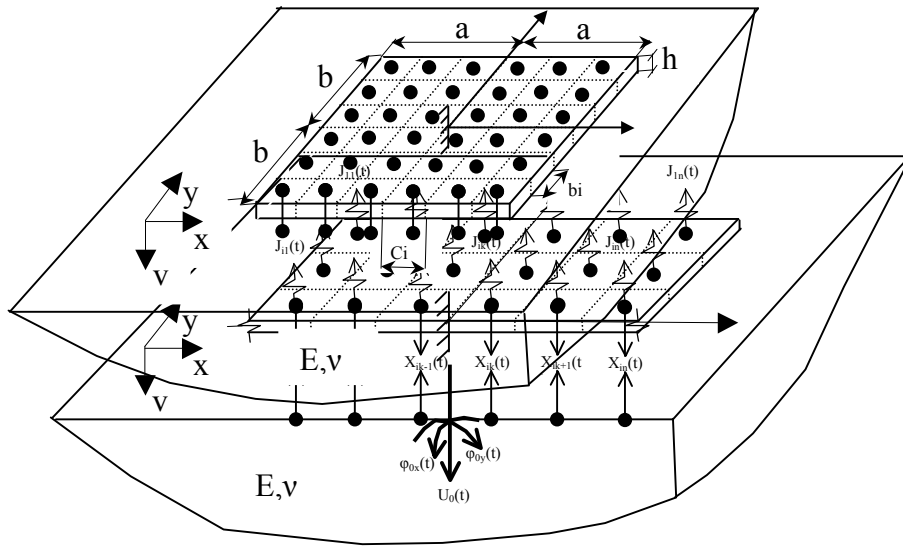


Fig. 3. Illustration of the method of analysis

The canonical system of equations allowing the dynamical study of the rectangular plate resting on the surface of elastic foundation with distributive properties is expressed in equation (1) [1]:

$$\left\{ \begin{array}{l} \vdots \\ \sum_{j=1}^n (v_{ij} + W_{ij}) X_j(t) - \sum_{j=1}^n W_{ij} J_j(t) + \ell_{ix} \varphi_{0x}(t) + \ell_{iy} \varphi_{0y}(t) + u_0(t) + \Delta_{iP} = 0; \quad i = 1, \dots, n \\ \vdots \\ \sum_{j=1}^n [X_j(t) - J_j(t)] \ell_{jx} = I_{0x} \ddot{\varphi}_{0x}(t); \\ \sum_{j=1}^n [X_j(t) - J_j(t)] \ell_{jy} = I_{0y} \ddot{\varphi}_{0y}(t); \\ \sum_{j=1}^n [X_j(t) - J_j(t)] = M \ddot{u}_0(t), \end{array} \right. \quad (1)$$

$X_j$ : connection's force applied on the plate and on the surface of the elastic foundation;  $J_j$ : inertia's force applied only on the plate;  $v_{ik}$ : Green's function defining the displacement of the surface of the elastic foundation at the point  $i$  due to a force  $X_j$  applied in the point  $j$  of the same surface;  $\varphi_{0x}, \varphi_{0y}$ : angle of rotation of the plate relative to the axes  $Ox$  and  $Oy$  at the embedding point;  $u_0$ : initial displacement of the plate at the embedding point;  $W_{ik}$ : deflection of the plate at a point  $i$  due to a force  $X_j$  applied at the point  $j$  of the plate;  $\Delta_{iP}$ : function characterizing the deflection of the plate at a point  $i$  due to an external force  $P_j$  applied at the point  $j$  of the plate (for free vibrations  $\Delta_{iP} = 0$ );  $\ell_{ix}, \ell_{iy}$ : arms of the elements of the plate relative to the

axes of coordinate;  $M$ : total mass of the plate;  $I_{0x}, I_{0y}$ : inertia's moments of the plate relative to the axes of the coordinates.

The relationship for the free vibrations is shown in equation (2) as per [3]:

$$X_k(t) = X_k e^{i\omega t}; \varphi_{0x}(t) = \varphi_{0x} e^{i\omega t}; \varphi_{0y}(t) = \varphi_{0y} e^{i\omega t}; u_0(t) = u_0 e^{i\omega t}; J_k(t) = J_k e^{i\omega t}. \quad (2)$$

Taking into account (2), the system from (1) takes the following form:

$$\begin{cases} \vdots \\ \sum_{j=1}^n (v_{ij} + W_{ij}) X_j - \sum_{j=1}^n W_{ij} J_j + \ell_{ix} \varphi_{0x} + \ell_{iy} \varphi_{0y} + u_0 = 0; \quad i = 1, \dots, n \\ \vdots \\ \sum_{j=1}^n [X_j - J_j] \ell_{jx} = I_{0x} \ddot{\varphi}_{0x}; \\ \sum_{j=1}^n [X_j - J_j] \ell_{jy} = I_{0y} \ddot{\varphi}_{0y}; \\ \sum_{j=1}^n [X_j - J_j] = M \ddot{u}_0. \end{cases} \quad (3)$$

## 2. 2. GREEN FUNCTION DEFINING THE VERTICAL DISPLACEMENT OF THE SURFACE OF HALF SPACE

Since the elastic foundation considered is the elastic half-space with distributive properties (principle of Boussinesq), then the expression of  $v_{ik}$  takes the form of equation (4) as per [2]:

$$v_{ik} = \frac{1-\nu^2}{\pi E} \frac{1}{\Omega} \int_{c_1}^{c_2} \int_{b_1}^{b_2} \frac{d\xi d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}, \quad (4)$$

$\Omega = b_i c_i$ ,  $b_i$  and  $c_i$ : length and width of the element;

$x$  and  $y$ : coordinates of the point  $i$  on the surface where the displacement is determined;

$\xi$  and  $\eta$ : coordinates of the point  $j$  where the force  $X_j$  is applied, fig. 4.

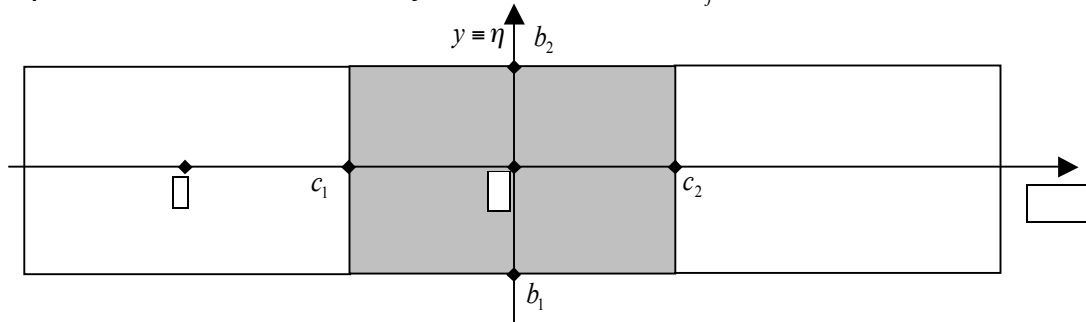


Fig. 4. Geometry of the loaded element

After intégration, expression (4) becomes:

$$v_{ik} = \frac{1-\nu^2}{\pi E} \frac{1}{bc} F_{ik}, \quad (5)$$

$$\begin{aligned} F_{ik} = & (y - c_1) \ln \left( x - b_1 + \sqrt{(x - b_1)^2 + (y - c_1)^2} \right) + (c_1 - y) \ln \left( x - b_2 + \sqrt{(x - b_2)^2 + (y - c_1)^2} \right) + \\ & + (c_2 - y) \ln \left( x - b_1 + \sqrt{(x - b_1)^2 + (y - c_2)^2} \right) + (y - c_2) \ln \left( x - b_2 + \sqrt{(x - b_2)^2 + (y - c_2)^2} \right) + \\ & + (x - b_1) \ln \left( -c_1 + \sqrt{(x - b_1)^2 + (y - c_1)^2} + y \right) + (b_2 - x) \ln \left( -c_1 + \sqrt{(x - b_2)^2 + (y - c_1)^2} + y \right) + \\ & + (b_1 - x) \ln \left( -c_2 + \sqrt{(x - b_1)^2 + (y - c_2)^2} + y \right) + (x - b_2) \ln \left( -c_2 + \sqrt{(x - b_2)^2 + (y - c_2)^2} + y \right) \end{aligned}$$

Figure 5 illustrates the displacements of the surface of half space with distributive properties given by (5) in the contact zone due to a concentrated force applied at the plate's centre.

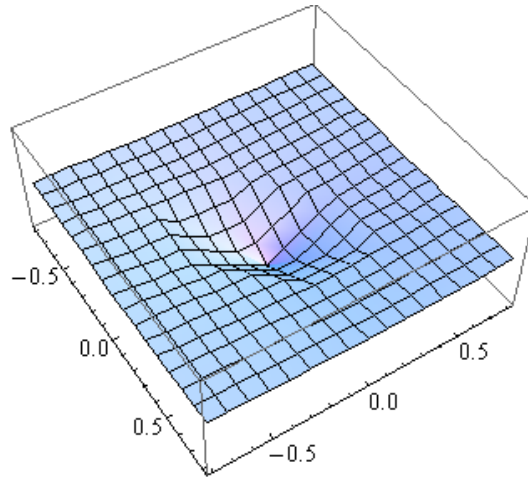


Fig. 5. Surface's deformation of half space due to concentrate force applied in the plate's centre

### 2. 3. FUNCTION OF THE DEFLEXIONS OF PLATE

In equation (6)  $W_{ij}$  is the function defining the deflections of the rectangular plate in a point  $i$  due to the force  $X_j$  applied in a point  $j$ . Based on the Clebsch's solution [4],  $W_{ij}$  takes the following form:

$$W_{ik}(x, y) = W_0(x, y) + \sum_{n=1}^{\infty} A_n W_n(x, y). \quad (6)$$

According to [4] and for the case presented herein, the terms of the expression (6) that satisfy the boundary conditions of the studied plate are as follows:

$$W_{ik}(x, y) = W_0(x, y) + A_{22}W_1(x, y) + B_{22}W_2(x, y) + A_{31}W_3(x, y) + B_{31}W_4(x, y), \quad (7)$$

where:

$$W_0(x, y) = \frac{Pab}{16\pi D} \left\{ \left[ \left( \frac{x-t}{a} - \frac{t}{a} \right)^2 + \left( \frac{y-z}{b} - \frac{z}{b} \right)^2 \right] \ln \left[ \left( \frac{x-t}{a} - \frac{t}{a} \right)^2 + \left( \frac{y-z}{b} - \frac{z}{b} \right)^2 \right] + \right. \\ \left. + 4 \left( \frac{xt}{a^2} + \frac{yz}{b^2} \right) \left( 1 + \ln \left[ \frac{\sqrt{x^2 + y^2} \sqrt{t^2 + z^2}}{ab} \right] \right) - \right. \\ \left. - \left( \frac{t^2}{a^2} + \frac{z^2}{b^2} \right) \ln \left[ \frac{t^2}{a^2} + \frac{z^2}{b^2} \right] - \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right) \ln \left[ \frac{x^2}{a^2} + \frac{y^2}{b^2} \right] \right\};$$

$$W_1(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2}; \quad W_2(x, y) = \frac{2xy}{ab}; \quad W_3(x, y) = \frac{x}{a} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right); \quad W_4(x, y) = \frac{y}{b} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right);$$

$D = \frac{E_0 h^3}{12(1-\nu_0^2)}$ : plate's cylindrical rigid;  $E_0, \nu_0$ : modulus of elasticity and Poisson's ratio

of the material of the plate;  $h$ : plate's thickness;  $\beta = 2(1-\nu_0)$ ;  $x$  and  $y$ : coordinates of the point  $i$  on the plate where the displacement is determined;  $t$  and  $z$ : coordinates of the point  $j$  where the force  $X_j$  is applied, fig. 2.

The coefficients  $A_{22}$ ,  $B_{22}$ ,  $A_{31}$  and  $B_{31}$  are determined by the Ritz's method [5] (i.e. by considering the deformation's energy of the plate). After simplification, these coefficients take the following expressions:

$$A_{22} = \frac{ab}{16D\beta} \left[ \int_{-b-a}^b \int_{-a}^a Q_1(x, y) dx dy - \left( \frac{t^2}{a^2} - \frac{z^2}{b^2} \right) \right]; \quad B_{22} = \frac{ab}{16D\beta} \left[ \int_{-b-a}^b \int_{-a}^a Q_2(x, y) dx dy - \frac{2tz}{ab} \right]; \\ A_{31} = \frac{-3ab}{32D\beta} \left[ \int_{-b-a}^b \int_{-a}^a Q_3(x, y) dx dy - \frac{t}{a} \left( \frac{t^2}{a^2} + \frac{z^2}{b^2} \right) \right]; \quad B_{31} = \frac{-3ab}{32D\beta} \left[ \int_{-b-a}^b \int_{-a}^a Q_4(x, y) dx dy - \frac{z}{b} \left( \frac{t^2}{a^2} + \frac{z^2}{b^2} \right) \right].$$

Expressions of  $Q_1 = (x, y)$ ,  $Q_2 = (x, y)$ ,  $Q_3 = (x, y)$ ,  $Q_4 = (x, y)$  and also the final expressions of these coefficients are very long and, thus, not presented here.

Figure 6 illustrates the displacements of the plate given by (7) due to a vertical load distributed uniformly over the entire plate.

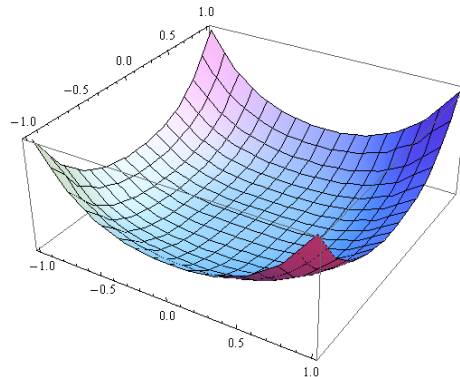


Fig. 6. Plate's deformation due to the uniformly distributed vertical load

## 2. 4. REMARKS

Since the displacements of the surface of a half-space are considered equal to the plate deflection (i.e.  $W_i \equiv v_i$ ), so the inertia force  $J_i$  can be given by the following expression as per [3]:

$$J_i = -M_i \frac{d^2 W_i(t)}{dt^2} = -M_i \frac{d^2 v_i(t)}{dt^2} = M_i \omega^2 v_i = M_i \omega^2 \left( \frac{1-\nu^2}{\pi E} \right) \sum_{j=1}^n X_j F_{ij}, \quad (8)$$

Here  $v_i = \frac{1-\nu^2}{\pi E} \sum_{j=1}^n X_j F_{ij}$ , and  $F_{ij}$  given by (5).

## 3. EXAMPLE OF APPLICATION

The plate is divided into 25 identical elements and considering that the point of embedment coincides with the centre of mass (fig. 7).

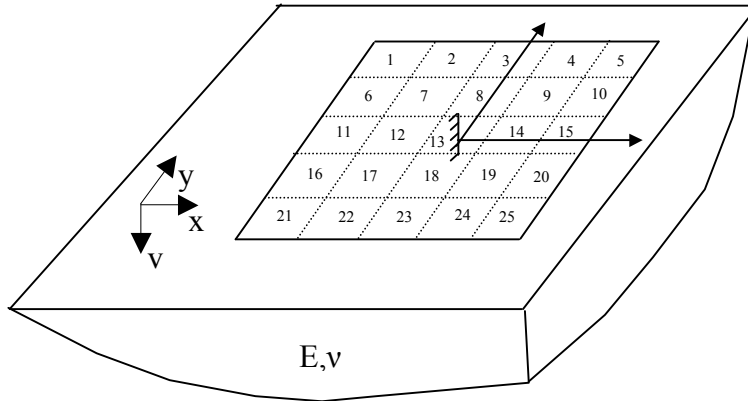


Fig. 7. Elements' numbering of the discretized plate

Also taking into account the dynamic calculation, the following geometrical and mechanical properties of the half-space and plate are specified:

$$a = b = 1 \text{ m}; \quad \nu = 1/3; \quad E = 1 \times 10^8 \text{ N/m}^2; \quad h = 0.2 \text{ m}; \quad b_i = c_i = 0.4 \text{ m}; \quad E_0 = 2 \times 10^{11} \text{ N/m}^2; \quad \nu_0 = 1/3.$$

Introducing such data into the system using (3), and taking into account the expressions of each of its parameters, as well as mathematical simplifications, the following matrix system is obtained:

$$[A] \begin{bmatrix} X_1 \\ \vdots \\ X_{25} \\ \varphi_{0x} \\ \varphi_{0y} \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad (9)$$

The terms of the matrix  $[A]$  are very long expressions resulting from mathematical transformations and, thus, are not included herein.

### 3. 1. DETERMINATION OF THE PLATE’S EIGEN FREQUENCIES

The determination of the plate’s Eigen frequencies resting on the surface of a half-space with distributive properties is done by the resolution of the equation of the determinant of the matrix  $[A]$  of the system (9). Within this long and complicated equation, the roots represent the Eigen frequencies values of the plate.

The calculation of the roots of the equation is executed numerically using “Mathematica”. Figure 8 represents a close up of one of roots of the determinant representing the plate’s Eigen frequencies.

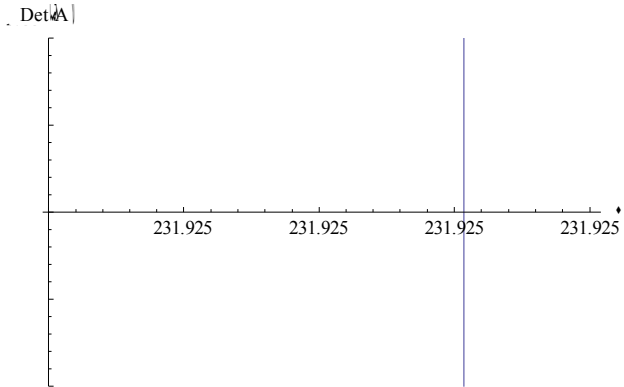


Fig. 8. Graphical close up of the root of determinant representing an Eigen frequency

The spectrum of Eigen frequencies in (Hz) is:  
 $\omega_1 = 231.92528, \omega_2 = 331.46373, \omega_3 = 544.4546, \dots$

### 3. 2. DETERMINATION OF THE PLATE’S NATURAL SHAPES

Determination of the natural shapes of the square plate resting on the surface of a half-space with distributive properties requires determination of the natural shape corresponding to each Eigen frequency. To achieve this, the first equation of the matrix system (9) is excluded, and the system is solved with  $n + 2$  equations and  $n + 2$  unknowns. Subsequently, the natural shapes of the beam are determined by the formula (8) introduced by Zhemochkin and Sinitsyn [1], for a given discretization:  $i = 1, \dots, n; n = 25$ .

The application of these operations allows the determination of the plate’s natural shapes for the first three following forms:

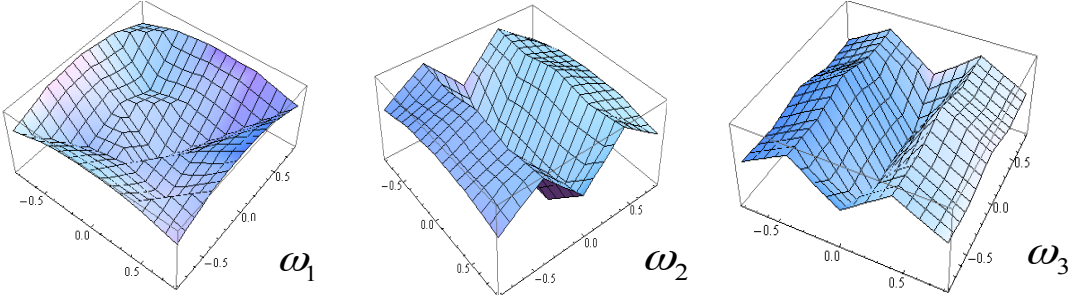


Fig. 9. Natural shapes of the plate corresponding to the first three Eigen frequencies

#### 4. RESPONSE OF THE PLATE RESTING ON THE SURFACE OF A HALF-SPACE WITH DISTRIBUTIVE PROPERTIES DUE TO THE EXTERNAL VERTICAL HARMONIC EXCITATIONS

The forced vibrations of the plate resting on the surface of a half-space with distributive properties concerns its reaction due to the dynamic external vertical loads. The case presented is of the reaction of the plate caused by the vertical harmonic load applied at point  $p$  coinciding with the centre of the element 19 (fig. 7). The external vertical harmonic load is varied according to eq. 10:

$$P_p = P_{19} = P_0 \cos(2 \times \pi \times f \times t), \quad (10)$$

where  $P_0$  : amplitude of excitation;  $f$  : frequency of excitation;  $t$  : time of excitation. The values:  $P_0 = 100000 N$ ,  $f = 150 Hz$  are considered. Obtaining the response requires the resolution of the system (1). In this case, the parameter  $\Delta_{ip}$  is determined by a following formula (11) introduced by Zhemochkin and Sinitsyn [1]:

$$\Delta_{ip} = \sum_{p=1}^n W_{ip} P_p = W_{i19} P_{19}, \quad (11)$$

$W_{ip}$  : plate deflections in the point  $i$  due to the external load  $P_p$  applied on the plate at a point  $p$ . These deflections are due to the external loads  $P_p$  as determined by (7)

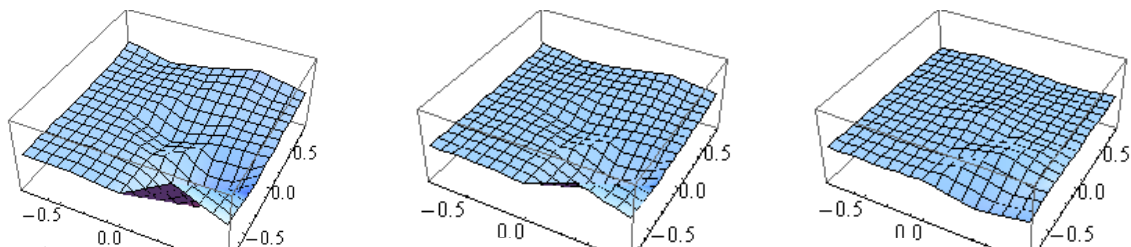
The solution of the system (1) for the forced oscillations gives the unknowns  $X_i$  values varying with the time of excitation. Unknowns  $X_i$  represent reactive efforts in the contact zone. In this case the value's variability is expressed by eq. 12:

$$X_i(t) = X_i(t_0) \cos(2 \times \pi \times f \times t), \quad (i = 1, \dots, 25). \quad (12)$$

Finally, the plate displacements during the time of excitation  $t$ , representing its response are determined by eq. 13:

$$v_i = \frac{1 - \nu^2}{\Omega \pi E} \sum_{j=1}^{25} X_j F_{ij} \quad (13)$$

Figure 10 shows the vertical displacements variability  $v_i$  of the entirety of the plate resting on the surface of a half-space with distributive properties in 3D due to the dynamic external load  $P_{19}$  at each moment  $t_j = t_0 + \Delta t$ . Here,  $t_0 = 0s$ ;  $\Delta t = 0.001s$ . Furthermore, the maximum displacement is obtained at the point in which the external excitation is applied.



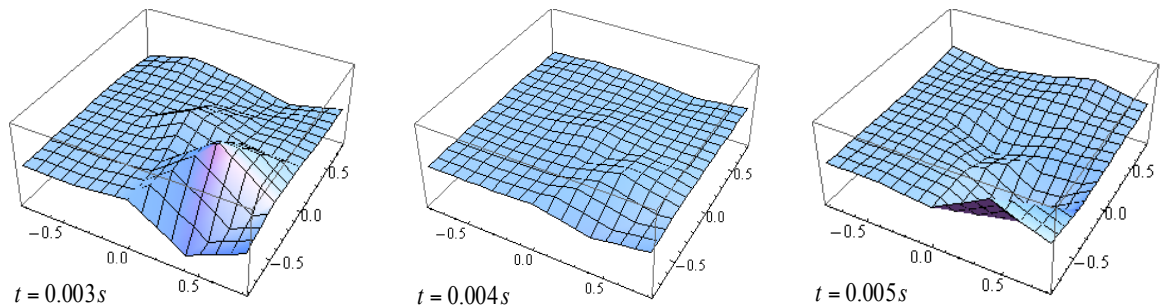


Fig. 10. Response of the plate resting on elastic foundation with distributive properties due to a harmonic load applied at the centre of element 19

## 5. RESULTS COMPARISON

To ensure the reliability of the results, the same square plate resting on two different types of elastic foundations (Boussinesq's model and Winkler's model) are considered. The plates are excited by a harmonic load, expressed by equation (10) and applied to the centre of the element 19 (fig. 7). Figure 11 compares the vertical displacement variability of the plate  $v_{19}$  at the point where the dynamic load is applied, where (I) is the foundation with distributive properties (Boussinesq's model) and (II) the foundation with a spring model (Winkler's model). Of note is that the vertical displacements of the plate for the half-space with distributive properties is always less than the same type of displacement, when the distributive properties of half-space are neglected, thus proving that the half-space distributive properties fundamentally influences the dynamic analysis.

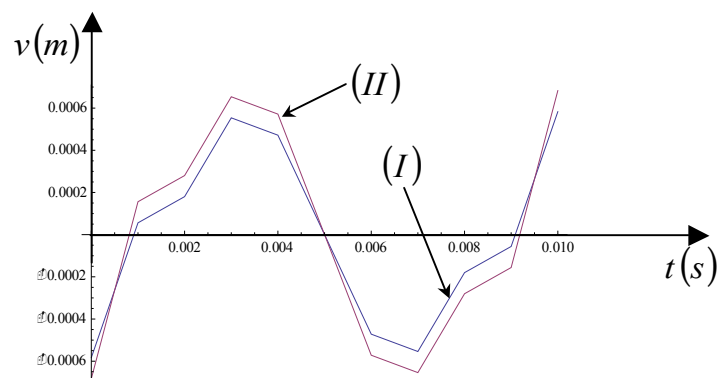


Fig. 11. Comparison of the displacement variability of a point of the plate by two different models

## 6. CONCLUSION

Using a semi-analytic approach, a dynamic analysis of beams and plates resting on the surface of an elastic half-space with different models was achieved to determine the Eigen frequencies, natural shape, plate response to external dynamic loads, and other physical magnitudes. Determination of the reactive forces in the contact zone representing the interaction phenomena between the plate and half-space surface is necessary to find the others physical magnitudes. For this purpose, it is imperative to study Green's function defining the displacements of the contact zone (i.e. contact problems phenomena). The calculated results were compared satisfactorily to the same plate resting on the surface of an elastic foundation using Winkler's model. Additionally, the obtained solution is semi-analytical and can, therefore, be readily computed to be more compatible with engineering applications. As such, this work represents a fundamental advance in the solving of more complicated dynamics problems. The next step is to study the plates resting on elastic foundation with inertial properties.

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