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Analytic content and the isoperimetric inequality in higher dimensions

Stephen J. Gardiner, Marius Ghergu and Tomas Sjödin

Abstract

This paper establishes a conjecture of Gustafsson and Khavinson, which relates the analytic content of a smoothly bounded domain in $\mathbb{R}^N$ to the classical isoperimetric inequality. The proof is based on a novel combination of partial balayage with optimal transport theory.

1 Introduction

Let ω be a bounded domain in the complex plane $\mathbb{C}$ such that $\partial\omega$ is the disjoint union of finitely many simple analytic curves, and let $\mathcal{A}(\omega)$ denote the collection of continuous functions on $\overline{\omega}$ that are analytic on ω. Further, let $\|g\|_S$ denote $\sup_S |g|$ for any bounded function $g : S \to \mathbb{C}$. The *analytic content* of ω is then defined by

$$\lambda(\omega) = \inf\{\|\overline{z} - \phi\|_{\overline{\omega}} : \phi \in \mathcal{A}(\omega)\}.$$

The inequalities for $\lambda(\omega)$ given below, which imply the classical isoperimetric inequality, are due to Alexander [2] and Khavinson [15].

Theorem A. *Let A and P denote the area and perimeter of ω, respectively. Then*

$$\frac{2A}{P} \leq \lambda(\omega) \leq \sqrt{\frac{A}{\pi}}.$$

An exposition of this circle of ideas may be found in Gamelin and Khavinson [10], and a wider survey of related results is provided by Bénéteau and Khavinson [4]. It was shown in [10] that equality with the upper bound occurs if and only if ω is a disc. Recently, Abanov et al [1] have shown that equality with the lower bound occurs if and only if ω is a disc or an annulus.

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Rewriting $\lambda(\omega)$ as $\inf\{\|z-\overline{\phi}\|_{\overline{\omega}} : \phi \in \mathcal{A}(\omega)\}$, it can be seen that a natural generalization of this quantity to smoothly bounded domains Ω in Euclidean space $\mathbb{R}^N$ $(N \geq 2)$ is given by

$$\lambda(\Omega) = \inf\{\|x-f\|_{\overline{\Omega}} : f \in A(\Omega)\},$$

where $A(\Omega)$ denotes the space of *harmonic vector fields* $f = (f_1, ..., f_N) \in C(\overline{\Omega}) \cap C^1(\Omega)$ and

$$\|f\|_S = \sup_S \|f\|, \quad \text{where} \quad \|f\| = \sqrt{f_1^2 + ... + f_N^2}.$$

(Thus f satisfies $\operatorname{div} f = 0$ and $\operatorname{curl} f = 0$, where the latter condition means that

$$\frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j} = 0 \quad \text{for all } j,k \in \{1, ..., N\} \text{ on } \Omega.)$$

If we write

$$\mathcal{H} = \{h \in C^1(\overline{\Omega}) : \Delta h = 0 \ \text{ on } \ \Omega\},$$

then

$$\{\nabla h : h \in \mathcal{H}\} \subset A(\Omega). \tag{1}$$

Let $r_\Omega > 0$ be chosen so that a ball of radius r_Ω has the same volume as Ω. Gustafsson and Khavinson [12] established the following inequalities for $\lambda(\Omega)$ in higher dimensions.

Theorem B. *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with volume V such that $\partial\Omega$ is the disjoint union of finitely many smooth components with total surface area P. Then there exists a constant $c_N > 1$ such that*

$$\frac{NV}{P} \leq \lambda(\Omega) \leq c_N r_\Omega. \tag{2}$$

The lower bound in (2) is sharp, since $\lambda(\Omega) = r$ when Ω is a ball of radius r (see Theorem 3.1 in [12]). Regarding the upper bound, Gustafsson and Khavinson conjectured that the constant c_N may be replaced by 1, in which case (2) would again contain the classical isoperimetric inequality. However, the methods of [12] do not yield such a conclusion. The purpose of this paper is to verify this long-standing conjecture.

Following [12] we define a related domain constant,

$$\begin{aligned} \lambda_1(\Omega) &= \inf\{\|x-\nabla h\|_{\overline{\Omega}} : h \in \mathcal{H}\} \\ &= \inf\{\|\nabla u\|_{\overline{\Omega}} : u \in C^1(\overline{\Omega}) \ \text{ and } \ \Delta u = N \text{ on } \Omega\}, \end{aligned} \tag{3}$$

where we have used the observation that a function u in $C^1(\overline{\Omega})$ satisfies $\Delta u = N$ on Ω if and only if the function h defined by $h(x) = \|x\|^2/2 - u(x)$ belongs to $\mathcal{H}$. For future reference we note that, for such u, it follows from the harmonicity of the partial derivatives of u that $\|\nabla u\|$ is subharmonic on Ω (cf. Theorem 3.4.5 of [3]), and so

$$\|\nabla u\|_{\overline{\Omega}} = \|\nabla u\|_{\partial\Omega} \tag{4}$$

by the maximum principle.

It follows from (1) that

$$\lambda(\Omega) \leq \lambda_1(\Omega).$$

(Equality holds when Ω is simply connected.) Gustafsson and Khavinson actually showed in [12] that $\lambda_1(\Omega) \leq c_N r_\Omega$ for an explicit constant $c_N > 1$. We will establish the following estimate.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ such that $\partial\Omega$ is the disjoint union of finitely many smooth components. Then $\lambda_1(\Omega) \leq r_\Omega$. Further, equality holds if and only if Ω is a ball.*

In the light of Theorem B we immediately arrive at the following conclusion.

Corollary 2. *Let Ω be a bounded domain in $\mathbb{R}^N$ with volume V such that $\partial\Omega$ is the disjoint union of finitely many smooth components with total surface area P. Then*

$$\frac{NV}{P} \leq \lambda(\Omega) \leq r_\Omega.$$

Further, $\lambda(\Omega) = r_\Omega$ if and only if Ω is a ball.

The proof of Theorem 1 combines the technique of partial balayage with results from the theory of optimal transport. Later we will discuss separate necessary and sufficient conditions for a function u in $C^1(\overline{\Omega})$ to be a minimizer for $\lambda_1(\Omega)$ in (3), whenever such minimizers exist.

The purpose of Corollary 2 is to relate analytic content to the isoperimetric inequality. We do not claim that it offers a novel or shorter proof of the latter, since (apart from other more classical proofs) there are already proofs using optimal transport theory as in McCann and Guillen [16], and Cabré [7] had previously provided a short proof of it based on more geometric methods.

2 Proof of Theorem 1

2.1 Tools for the proof

Let m denote Lebesgue measure on $\mathbb{R}^N$ and $O \subset \mathbb{R}^N$ be a bounded domain. Further, let $G_O(\cdot,\cdot)$ denote the Green function of O, and $G_O\mu, G_O\nu$ denote the potentials of (positive) measures μ, ν on O, where $\nu \ll m$. The Green function is normalized so that $-\Delta G_O\gamma = \gamma$ in the sense of distributions for any potential $G_O\gamma$. We define

$$P_\mu^\nu = G_O\nu + \sup\{s : s \text{ is subharmonic on } O \text{ and } s \leq G_O\mu - G_O\nu\} \quad \text{on } O, \tag{5}$$

whence $P_\mu^\nu \leq G_O\mu$, and recall the following facts (see [14] or [11]).

Theorem C. *(a)* $P_\mu^\nu = G_O\eta$ *for some measure* η *on* O *satisfying* $\eta \leq \nu$.
(b) $\eta = \nu|_S + \mu|_{O\backslash S}$, *where* $S = \{G_O\eta < G_O\mu\}$.

We will refer to the measure η in the above theorem as the (*partial*) *balayage of* μ *onto* ν *in* O, and denote it by $\mathcal{B}\mu$, where ν is to be understood from the context. We note that $G_O\mu - G_O\eta$ is the smallest nonnegative lower semicontinuous function w on O satisfying $-\Delta w \geq \mu - \nu$ in the sense of distributions. Thus, if $\mu_1 \geq \mu$, then the set $S(\mu)$ associated with μ is contained in the corresponding set $S(\mu_1)$. It follows that

$$\mu_1 \geq \mu \Longrightarrow \mathcal{B}\mu_1 \geq \mathcal{B}\mu. \tag{6}$$

We will also need the following lemma. Let $B(x,r)$ denote the open ball in $\mathbb{R}^N$ with centre x and radius r.

Lemma 3. *Let* $\nu = Nm|_O$ *and* $\overline{\Omega} \subset \Omega_0 \subset O$, *where* Ω_0 *is another open set. If* τ *is a measure with* $\text{supp}\tau \subset \overline{\Omega}$, *then there exists* $b > 0$ *such that*

$$\text{supp}\mathcal{B}(Nm|_\Omega + b\tau) \subset \Omega_0.$$

Proof. Let Ω' be an open set such that $\overline{\Omega} \subset \Omega'$ and $\overline{\Omega'} \subset \Omega_0$, and let

$$\varepsilon = 2^{-1}\text{dist}(\partial\Omega', \Omega \cup (\mathbb{R}^N\backslash\Omega_0)).$$

Let τ^* be the sweeping (classical balayage) of τ onto $\partial\Omega'$, and define

$$\gamma(x) = \int_{\partial\Omega'} \phi_\varepsilon(x-y)d\tau^*(y) \qquad (x \in \mathbb{R}^N),$$

where ϕ_ε is a non-negative rotationally invariant C^∞ smoothing kernel on $\mathbb{R}^N$ with support $\overline{B}(0,\varepsilon)$ (see, for example, Section 3.3 of [3]). We choose b sufficiently small that $b\gamma \leq N$, whence

$$P^\nu_{G_O(Nm|_\Omega + b\gamma m)} = G_O(Nm|_\Omega + b\gamma m).$$

Since $G_O(\gamma m) \leq G_O\tau^* \leq G_O\tau$, with equality outside $\{x \in O : \text{dist}(x,\Omega') \leq \varepsilon\}$, we see that

$$G_O(Nm|_\Omega + b\gamma m) \leq P^\nu_{Nm|_\Omega + b\tau} \leq G_O(Nm|_\Omega + b\tau),$$

again with equality outside $\{x \in O : \text{dist}(x,\Omega') \leq \varepsilon\}$, and so

$$\text{supp}\mathcal{B}(Nm|_\Omega + b\tau) \subset \{x \in O : \text{dist}(x,\Omega') \leq \varepsilon\}.$$

We recall the following composite result from the theory of optimal transport, in which the existence and smoothness of the function v are due to Brenier [5] and Caffarelli [8], respectively. (See also Chapters 3 and 4 of Villani's book [18].)

Theorem D. *Let* $D \subset \mathbb{R}^N$ *be a bounded open set such that* $m(\partial D) = 0$. *Then there exists a convex function* $v : \mathbb{R}^N \to (-\infty, \infty]$ *which is* C^2 *on* D, *and for which* ∇v *maps* D *into* $B(0, r_D)$ *and is measure-preserving, in the sense that* $m(A) = m((\nabla v)(A))$ *for any Borel set* $A \subset D$.

Lemma 4. *If a measure* γ *on* Ω *has bounded density with respect to* m, *then* $G_\Omega \gamma \in C^1(\overline{\Omega})$, *and*

$$\frac{\partial G_\Omega \gamma}{\partial y_i}(y) = \int_\Omega \frac{\partial G_\Omega}{\partial y_i}(x, y) d\gamma(x) \qquad (y \in \overline{\Omega}; i = 1, ..., N). \tag{7}$$

To see this, we note that standard arguments (cf. Theorem 4.5.3 of [3]) show that $G_\Omega \gamma \in C^1(\Omega)$ and that (7) holds when $y \in \Omega$. We now fix i and note (see Widman [19]) that $(\partial G_\Omega / \partial y_i)(x, \cdot)$ has a continuous extension to $\overline{\Omega}$ for each $x \in \Omega$. Let $y_0 \in \partial \Omega$ and $\varepsilon > 0$, and define

$$\psi_j(y) = \int_{\Omega_j} \frac{\partial G_\Omega}{\partial y_i}(x, y) d\gamma(x) \qquad (y \in \overline{\Omega}; j = 1, 2),$$

where $\Omega_1 = \Omega \backslash B(y_0, \varepsilon)$ and $\Omega_2 = \Omega \cap B(y_0, \varepsilon)$. Then ψ_1 is continuous at y_0, and (by estimates in [19])

$$|\psi_2(y)| \leq C(\Omega) \varepsilon \left\| \frac{d\gamma}{dm} \right\|_{L^\infty(\Omega)} \qquad (y \in B(y_0, \varepsilon/2) \cap \overline{\Omega}).$$

It follows that

$$\frac{\partial G_\Omega \gamma}{\partial y_i}(y) \to \int_\Omega \frac{\partial G_\Omega}{\partial y_i}(x, y_0) d\gamma(x) \qquad (y \to y_0).$$

2.2 Proof of the inequality

Let

$$b_N = \frac{m(\{y \in B(0,1) : y_N \geq 1/2\})}{m(B(0,1))},$$

and let D be a bounded open set such that $\overline{\Omega} \subset D$, $m(\partial D) = 0$ and $m(D \backslash \Omega) < b_N m(\Omega)$. We next choose v as in Theorem D. Since ∇v is measure-preserving on D, the Hessian of v, which is positive semi-definite because v is convex, has determinant equal to 1, and so $\Delta v \geq N$ by the arithmetic-geometric means inequality for the eigenvalues of the Hessian (cf. the argument in Section 1.6 of McCann and Guillen [16]). It will be enough to show that $\lambda_1(\Omega) \leq r_D$, since r_D can be made arbitrarily close to

r_Ω. This inequality trivially holds if $\Delta v \equiv N$ on Ω, so we assume from now on that $(\Delta v - N)m|_\Omega \neq 0$.

Let R be an open set satisfying $\overline{\Omega} \subset R$ and $\overline{R} \subset D$. Since

$$m(D\backslash R) \leq m(D\backslash\Omega) < b_N m(\Omega) < b_N m(D)$$

and ∇v is measure preserving on D, we see that

$$(\nabla v)(R) \cap \{y \in B(r_D) : y \cdot x \geq r_D/2\} \neq \emptyset \qquad (x \in \partial B(0,1)). \tag{8}$$

Also, since v is convex, the function

$$w(x) = \sup\{v(y) + \nabla v(y) \cdot (x-y) : y \in R\} \qquad (x \in \mathbb{R}^N)$$

equals v on R. Clearly w is subharmonic (and indeed convex) on $\mathbb{R}^N$.

We now define $\varepsilon = 2^{-1}\mathrm{dist}(\overline{\Omega}, \mathbb{R}^N\backslash R)$ and

$$w_\varepsilon(x) = \int_{B(\varepsilon)} \phi_\varepsilon(x-y)w(y)dm(y) \qquad (x \in \mathbb{R}^N),$$

where ϕ_ε is a smoothing kernel, as before. Then w_ε is also subharmonic (and convex) on $\mathbb{R}^N$, and $w_\varepsilon \geq w$ (see Theorem 3.3.3 in [3]). We further define

$$O = \{x \in \mathbb{R}^N : w_\varepsilon(x) < C\},$$

where

$$C > \sup\{w_\varepsilon(x) : x \in R\}.$$

The set O clearly contains $\overline{R}$. It is also bounded, since for any $x \in \partial B(0,1)$ we see from (8) that there exists $y_x \in R$ such that $\nabla v(y_x) \cdot x \geq r_D/2$, and so

$$\begin{aligned} w_\varepsilon(tx) &\geq w(tx) \geq v(y_x) + \nabla v(y_x) \cdot (tx - y_x) \\ &= t\nabla v(y_x) \cdot x + v(y_x) - \nabla v(y_x) \cdot y_x \\ &\geq r_D t/2 + \inf\{v(y) - \nabla v(y) \cdot y : y \in R\} \qquad (t > 0). \end{aligned}$$

By Sard's theorem and the smoothness of w_ε, we can choose C such that the set O is smoothly bounded and

$$\Omega_0 \subset O, \text{ where } \Omega_0 = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Omega) < 1\}.$$

Since $w_\varepsilon = C$ on ∂O, the function $C - w_\varepsilon$ is the potential $G_O\mu$ of the measure $\mu = (\Delta w_\varepsilon)m$ on O. Further, $\mu \geq Nm|_\Omega$, since $\Delta w = \Delta v \geq N$ on R, and so

$$\Delta w_\varepsilon(x) = \int_{B(\varepsilon)} \phi_\varepsilon(x-y)(\Delta w)(y)dm(y) \geq N \int_{B(\varepsilon)} \phi_\varepsilon(x-y)dm(y) = N \qquad (x \in \Omega).$$

We next apply Theorem C with $\nu = Nm|_O$. The partial balayage $\eta = \mathcal{B}\mu$ satisfies $\eta = Nm|_S + \mu|_{O\backslash S}$, where $S = \{G_O\eta < G_O\mu\}$. Since $G_O\mu \geq G_O\eta$

and $\mu = (\Delta w_\varepsilon)m \geq Nm = \eta$ on Ω, the function $G_O\mu - G_O\eta$ is nonnegative and superharmonic on Ω. In fact, since Ω is connected and

$$(\mu - \eta)(\Omega) = \int_\Omega (\Delta v - N)dm > 0,$$

it is strictly positive there by the minimum principle, and so $\Omega \subset S$. We note that $G_O\mu, G_O\eta \in C^1(\overline{O})$, by Lemma 4. Since the nonnegative function $G_O\mu - G_O\eta$ achieves its minimum value 0 on $O \backslash S$, we have $\nabla G_O\mu = \nabla G_O\eta$ on $\partial S \cap O$. Also, since $G_O\eta \leq G_O\mu$, we have

$$\|\nabla G_O\eta\| = -\frac{\partial}{\partial n}G_O\eta \leq -\frac{\partial}{\partial n}G_O\mu = \|\nabla G_O\mu\| \quad \text{on} \quad \partial O,$$

where n denotes the outward unit normal to ∂O. Thus

$$\|\nabla G_O\eta\|_{\partial S} \leq \sup\{\|\nabla G_O\mu(x)\| : x \in O\} = \sup\{\|\nabla w_\varepsilon(x)\| : x \in O\} \leq r_D,$$

because w (and hence also w_ε) is Lipschitz on $\mathbb{R}^N$ with Lipschitz constant at most r_D. Finally, since $\Omega \subset S$ and $\eta = Nm$ in S, it follows (see (4)) that

$$\lambda_1(\Omega) \leq \|\nabla G_O\eta\|_{\overline{\Omega}} \leq \|\nabla G_O\eta\|_{\overline{S}} = \|\nabla G_O\eta\|_{\partial S} \leq r_D, \tag{9}$$

as desired.

2.3 The case of equality

The following result strengthens the conclusion of the previous section.

Proposition 5. *Let v be as in Theorem D, with $D = \Omega$. If $(\Delta v - N)m|_\Omega \neq 0$, then there is a domain U containing $\overline{\Omega}$ and a function $u \in C^1(U)$ satisfying $\Delta u = N$ and $\|\nabla u\| \leq r_\Omega$ in U.*

Proof. We may assume that $v(x_0) = 0$ for some $x_0 \in \Omega$. Let

$$D_k = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < \delta_k\} \quad (k \geq 0),$$

where $(\delta_k)_{k\geq 0}$ is a strictly decreasing sequence of positive numbers with limit 0 and $\delta_0 < 1$ is chosen small enough so that $m(D_0 \backslash \Omega) < b_N m(\Omega)$. For each k we choose v_k as in Theorem D, with $D = D_k$, and such that $v_k(x_0) = 0$. We next choose open sets R_k such that $\overline{\Omega} \subset R_k$ and $\overline{R_k} \subset D_k$, and define $\varepsilon_k = 2^{-1}\operatorname{dist}(\overline{\Omega}, \mathbb{R}^N \setminus R_k)$.

For each k we apply the construction of §2.2 with $D = D_k, R = R_k, v = v_k$, and $\varepsilon = \varepsilon_k$. Propositions 3.1 and 3.2 of Brenier [6], applied to the measures

$$\frac{m\,|_{D_k}}{m(D_k)} \quad (k \in \mathbb{N}) \qquad \text{and} \qquad \frac{m\,|_{B(0,r_\Omega)}}{m(B(0, r_\Omega))},$$

show that $v_k \to v$ uniformly on Ω.

The functions w and w_ε in §2.2 will now be denoted, by abuse of notation, w_k and w_{k,ε_k} respectively. Since $\|\nabla w_k\| \leq r_{D_k} \leq r_{D_0}$ on $\mathbb{R}^N$ by construction, we see that $|w_k - w_{k,\varepsilon_k}| \leq r_{D_0}\varepsilon_k$ on $\mathbb{R}^N$. Hence (w_{k,ε_k}) converges uniformly to v on Ω, in view of the fact that $v_k = w_k$ on R_k, which contains Ω.

We now choose numbers C_k so that the set

$$O_k = \{x \in \mathbb{R}^N : w_{k,\varepsilon_k}(x) < C_k\}$$

satisfies $D_0 \subset O_k$ for each k. Since the sequences $\left(\|v_k\|_{L^\infty(D_k)}\right)$ and $\left(\|\nabla v_k\|_{L^\infty(D_k)}\right)$ are bounded, we can furthermore arrange that the set $O = \cup_k O_k$ is bounded.

Let $\gamma_k = (\Delta w_{k,\varepsilon_k})m|_\Omega$ and $\gamma = (\Delta v)m|_\Omega$. These are non-negative measures, and the divergence theorem shows that

$$\|\gamma_k\| = \int_\Omega \Delta w_{k,\varepsilon_k} dm = \int_{\partial\Omega} \frac{\partial w_{k,\varepsilon_k}}{\partial n} d\sigma \leq r_{D_k}\sigma(\partial\Omega) \quad (k \in \mathbb{N}), \tag{10}$$

where σ denotes surface area measure on $\partial\Omega$. Since $\int_\Omega \psi d\gamma_k \to \int_\Omega \psi d\gamma$ for any $\psi \in C_c^2(\Omega)$, we see from (10) and the density of $C_c^2(\Omega)$ in $C_0(\Omega)$ that (γ_k) is weak* convergent to γ on Ω. Further, $\gamma_k \geq Nm|_\Omega$ and $\gamma \geq Nm|_\Omega$, as in §2.2.

Since $(\gamma - Nm)(\Omega) > 0$ by assumption, we can choose a compact set $K \subset \Omega$ such that $\alpha > 0$, where $\alpha = (\gamma - Nm)(K^\circ)$. It follows that $(\gamma_k - Nm)(K^\circ) \geq \alpha/2$ for all sufficiently large k. Let γ_k^* denote the sweeping of $(\gamma_k - Nm)|_{K^\circ}$ onto $\partial\Omega$. Then there exists $b > 0$ such that $\gamma_k^* \geq b\sigma$ for all sufficiently large k.

If we first consider partial balayage in O, then

$$\mathcal{B}(Nm|_\Omega + b\sigma) = Nm|_U \tag{11}$$

for some domain U containing $\overline{\Omega}$. Further, if we choose b sufficiently small, then $\overline{U} \subset D_0$, by Lemma 3. By definition

$$G_O(Nm|_\Omega + b\sigma) \geq G_O(Nm|_U) \text{ with equality in } O \setminus U.$$

Since $\overline{U} \subset D_0 \subset O_k \subset O$, this implies that

$$G_{O_k}(Nm|_\Omega + b\sigma) \geq G_{O_k}(Nm|_U) \text{ with equality in } O_k \setminus U.$$

Thus (11) remains valid if we henceforth consider partial balayage in O_k.

Now let η_k be the measure η constructed as in §2.2. Since

$$\gamma_k \geq Nm|_\Omega + (\gamma_k - Nm)|_{K^\circ},$$

we see from (6) that

$$\begin{aligned} \eta_k &= \mathcal{B}((\Delta w_{k,\varepsilon_k})m|_{O_k}) \geq \mathcal{B}((\Delta w_{k,\varepsilon_k})m|_\Omega) = \mathcal{B}(\gamma_k) \\ &\geq \mathcal{B}(Nm|_\Omega + (\gamma_k - Nm)|_{K^\circ}) = \mathcal{B}(Nm|_\Omega + \gamma_k^*) \\ &\geq \mathcal{B}(Nm|_\Omega + b\sigma) = Nm|_U, \end{aligned}$$

By choosing a suitable subsequence of (δ_k) we can arrange that $(G_{O_k}\eta_k)$ converges locally uniformly on U to a function u in $C^1(U)$ satisfying $\Delta u = N$ in U, and then $(\nabla G_{O_k}\eta_k)$ also converges locally uniformly on U to ∇u. From the final inequality in (9) we see that $\|\nabla u\| \leq \lim_{k\to\infty} r_{D_k} = r_\Omega$ on U, so the proposition is proved.

We can now easily address the case of equality in Theorem 1. We first assume that $\lambda_1(\Omega) = r_\Omega$ and choose v as in Theorem D, with $D = \Omega$. We claim that $\Delta v \equiv N$ on Ω. Indeed, if this were not the case, then there would exist u, U as in the above proposition. Since $r_\Omega \leq \|\nabla u\|_{\partial\Omega}$, the subharmonic function $\|\nabla u\|$ would then achieve its maximum inside U, forcing it to be constant. Thus $\|\text{Hess}(u)\|^2 = \Delta(\|\nabla u\|^2) = 0$, contradicting the fact that $\Delta u = N$ on U. Hence $\Delta v = N$ on Ω. Since the Hessian of v has determinant equal to 1, all its eigenvalues are 1, and so it is the identity matrix. Thus ∇v is a translation, and Ω is a ball (of radius r_Ω).

Conversely, if Ω is a ball of radius r, then (as we noted in the introduction) $\lambda_1(\Omega) = \lambda(\Omega) = r$.

3 Minimizers for $\lambda_1(\Omega)$

In general we do not know whether there exist minimizers u for the definition of $\lambda_1(\Omega)$ in (3). However, we can give separate necessary and sufficient conditions for a function u in $C^1(\overline{\Omega})$ to be a minimizer when such minimizers exist. We begin with a necessary condition.

Proposition 6. *Suppose that* $u \in C^1(\overline{\Omega})$ *and* $\Delta u = N$ *in* Ω. *If* $\|\nabla u\|_{\overline{\Omega}} = \lambda_1(\Omega)$, *then* $\|\nabla u\|$ *is constant on* $\partial\Omega$ *(whence* $\|\nabla u\| = \lambda_1(\Omega)$ *on* $\partial\Omega$*, by (4)).*

Proof. Let $\lambda_1 = \lambda_1(\Omega)$. Suppose that $u \in C^1(\overline{\Omega})$ satisfies $\Delta u = N$ and $\|\nabla u\|_{\overline{\Omega}} = \lambda_1(\Omega)$, and that there exists $y_0 \in \partial\Omega$ such that $\|\nabla u(y_0)\| < \lambda_1$. Then we can choose an open ball B_0 centred at y_0 such that

$$a := \|\nabla u\|_{B_0\cap\overline{\Omega}} < \lambda_1. \tag{12}$$

We choose Ω_0 to be a domain with C^2 boundary such that $\Omega \subset \Omega_0 \subset \Omega \cup B_0$ and $\Omega_0 \backslash \overline{\Omega} \neq \emptyset$.

Let $f = -G_\Omega(Nm)$, let g be a continuous extension of $\left.\dfrac{\partial f}{\partial n}\right|_{\partial\Omega\cap\partial\Omega_0}$ to $\partial\Omega_0$ and $0 < \varepsilon < \lambda_1$. Proposition 4 of Sakai [17] tells us that there is a finite sum μ of point masses (of variable sign) in $\Omega_0\backslash\overline{\Omega}$ satisfying

$$\left|\frac{\partial}{\partial n} G_{\Omega_0}\mu + g\right| < \varepsilon \quad \text{on} \quad \partial\Omega_0.$$

Since

$$\begin{aligned} G_{\Omega_0}\mu &\in C^1(\overline{\Omega}_0\backslash \mathrm{supp}\mu), \\ \nabla G_{\Omega_0}\mu &= \left(-\frac{\partial}{\partial n} G_{\Omega_0}\mu\right) n \quad \text{on} \ \ \partial\Omega_0, \\ \nabla f &= \frac{\partial f}{\partial n} n \quad \text{on} \ \ \partial\Omega, \end{aligned}$$

there exists $\kappa > 0$ such that

$$\|\nabla f - \nabla G_{\Omega_0}\mu\| < \varepsilon \ \text{ on } \ U_\kappa, \tag{13}$$

where

$$U_\kappa = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega \cap \partial\Omega_0) < \kappa\}.$$

Now let $\delta > 0$ and

$$u_\delta = (1-\delta)u + \delta\,(f - G_{\Omega_0}\mu) \quad \text{on} \ \ \overline{\Omega}.$$

Clearly $u_\delta \in C^1(\overline{\Omega})$, $\Delta u_\delta = N$ on Ω and

$$\|\nabla u_\delta\| \le (1-\delta)\,\|\nabla u\| + \delta\,\|\nabla f - \nabla G_{\Omega_0}\mu\|\,.$$

Hence, by (13),

$$\|\nabla u_\delta\| < (1-\delta)\,\|\nabla u\| + \delta\varepsilon \le (1-\delta)\lambda_1 + \delta\varepsilon < \lambda_1 \quad \text{on} \ \ U_\kappa. \tag{14}$$

Also,

$$\|\nabla u_\delta\| \le a + \delta\,\|\nabla f - \nabla G_{\Omega_0}\mu\|_{B_0\cap\Omega} \quad \text{on} \ \ B_0 \cap \Omega \tag{15}$$

and, in view of (12), the right hand side above can be made less than λ_1 by choosing δ to be sufficiently small. Combining (14) and (15), we obtain a contradiction to the hypothesis that u achieves the minimum value in (3). Hence $\|\nabla u\| = \lambda_1$ on $\partial\Omega$, as claimed.

The converse to the above proposition is false, as we will now explain. Let us recall that $\lambda_1(B(0,1)) = 1$. We claim that, when $N \ge 3$, there are functions $u \in C^1(\overline{B(0,1)})$ satisfying $\Delta u = N$ on $B(0,1)$ and $\|\nabla u\| = c$ on $\partial B(0,1)$, yet $\lambda_1(B(0,1)) \ne c$. For example, if we define

$$u(x) = \frac{N}{2(N-2)}\left(\sum_{i=1}^{N-1} x_i^2 - x_N^2\right),$$

then $\Delta u = N$ on $B(0,1)$ and $\|\nabla u\| = N/(N-2)$ on $\partial B(0,1)$.

Similarly, if $N = 2$ and $E = \{(x,y) : 4x^2 + y^2 < 1\}$, then the function $u(x,y) = 2x^2 - y^2$ satisfies $\Delta u = 2$ on E and $\|\nabla u\| = 2$ on ∂E. However, as we will see later, the actual value of $\lambda_1(E)$ is $2/3$.

Next we give a sufficient condition for a function $u \in C^1(\overline{\Omega})$ to be a minimizer for the definition of $\lambda_1(\Omega)$ in (3).

Proposition 7. *Let $u \in C^1(\overline{\Omega})$, where $\Delta u = N$ on Ω. If $\|\nabla u\|$ is constant on $\partial\Omega$ and $\nabla u \cdot n \geq 0$ on $\partial\Omega$, then $\lambda_1(\Omega) = \|\nabla u\|_{\overline{\Omega}}$. Further, any two functions satisfying these hypotheses differ only by a constant.*

Proof. Suppose that $\|\nabla u\| = c$ and $\nabla u \cdot n \geq 0$ on $\partial\Omega$. Let $v \in C^1(\overline{\Omega})$, where $\Delta v = N$ on Ω, and define $w = v - u$. Then $w \in \mathcal{H}$. We choose a point $y \in \partial\Omega$ at which w achieves its maximum value. If w is non-constant, then the Hopf boundary point lemma (see Section 6.4.2 of Evans [9]) tells us that $\nabla w(y) \cdot n_y > 0$ and so $\nabla w(y)$ is actually a positive multiple of n_y. Since $\nabla u \cdot n \geq 0$ on $\partial\Omega$, we now see that

$$\|\nabla v\|_{\overline{\Omega}} \geq \|\nabla v(y)\| > \|\nabla u(y)\| = c.$$

It follows that $c = \lambda_1(\Omega)$, as required.

Finally, the preceding argument shows that any two functions satisfying the hypotheses differ only by a constant.

A further useful sufficient condition for a function u to be a minimizer in (3) applies when u is locally convex.

Theorem 8. *Let $u \in C^1(\overline{\Omega})$, where u is locally convex on Ω and satisfies $\Delta u = N$ there. If $\|\nabla u\| = c$ on $\partial\Omega$, then*
(i) $\lambda_1(\Omega) = c$;
(ii) $\nabla u : \Omega \to B(0, \lambda_1(\Omega))$ is surjective.

Proof. Let $v = \|\nabla u\|^2$. Then $\Delta v = 2\|\text{Hess}(u)\|^2 \geq 0$ on Ω. In fact, $\Delta v > 0$ on a dense open subset of Ω, for otherwise ∇u would be constant on a nonempty open set, contradicting the hypothesis that $\Delta u = N$. Let $\Omega_\varepsilon = \{x \in \Omega : \|\nabla u(x)\| < c - \varepsilon\}$, where $\varepsilon > 0$ is sufficiently small that $\Omega_\varepsilon \neq \emptyset$. Clearly $\overline{\Omega}_\varepsilon \subset \Omega$. Further, by the Hopf boundary point lemma, $\partial v/\partial n > 0$ on $\partial\Omega_\varepsilon$, so $n = \nabla v/\|\nabla v\|$ on $\partial\Omega_\varepsilon$. Since $\text{Hess}(u)$ is positive semidefinite on Ω,

$$\nabla u \cdot n = \nabla u \cdot \frac{\nabla v}{\|\nabla v\|} = \frac{2}{\|\nabla v\|} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \geq 0 \quad \text{on } \partial\Omega_\varepsilon.$$

Given any $x \in \partial\Omega$, we choose $B(y,r) \subset \mathbb{R}^N \backslash \overline{\Omega}$ such that $\partial B(y,r) \cap \overline{\Omega} = \{x\}$ and then $x_\varepsilon \in \partial\Omega_\varepsilon$ satisfying $\|x_\varepsilon - y\| = \text{dist}(y, \overline{\Omega}_\varepsilon)$ to see that

$$(\nabla u \cdot n)(x) = \frac{y - x}{\|y - x\|} \cdot \nabla u(x) = \lim_{\varepsilon \to 0} \frac{y - x_\varepsilon}{\|y - x_\varepsilon\|} \cdot \nabla u(x_\varepsilon) \geq 0,$$

so part (i) follows from Proposition 7. We note, for future reference, that

$$\lambda_1(\Omega_\varepsilon) = \lambda_1(\Omega) - \varepsilon. \tag{16}$$

We will now show, further, that $\nabla u \cdot n > 0$ on $\partial\Omega_\varepsilon$. We write $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and choose our co-ordinate system so that $0 \in \partial\Omega_\varepsilon$, that the normal n_0 is in the direction of $(0, ..., 0, 1)$, and that the Hessian of the function $x' \mapsto u(x', 0)$ at $0'$ is a diagonal matrix. We know from the proof of part (i) that $(\partial u/\partial x_N)(0) \geq 0$. Now suppose, for the sake of contradiction, that $(\partial u/\partial x_N)(0) = 0$. Since v is constant on $\partial\Omega_\varepsilon$ we see that

$$\frac{\partial v}{\partial x_i}(0) = 0, \text{ whence } \frac{\partial u}{\partial x_i}(0)\frac{\partial^2 u}{\partial x_i^2}(0) = 0 \quad (i = 1, ..., N-1).$$

We reorder the first $N-1$ coordinates so that, for some $m \in \{1, ..., N\}$,

$$\frac{\partial u}{\partial x_i}(0) = 0 \quad (1 \leq i \leq m-1) \quad \text{and} \quad \frac{\partial^2 u}{\partial x_i^2}(0) = 0 \quad (m \leq i \leq N-1), \quad (17)$$

and define

$$a_N = \frac{\partial^2 u}{\partial x_N^2}(0) \quad \text{and} \quad b_i = \frac{\partial^2 u}{\partial x_i \partial x_N}(0) \quad (1 \leq i \leq N-1).$$

By (17) the Hessian of the function $(x_m, ..., x_N) \mapsto u(0, ..., 0, x_m, ..., x_N)$ at $(0, ..., 0)$ has the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & b_m \\ 0 & 0 & \cdots & 0 & b_{m+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & b_{N-1} \\ b_m & b_{m+1} & \cdots & b_{N-1} & a_N \end{pmatrix}.$$

Hence $b_i = 0$ when $m \leq i \leq N-1$, because this submatrix of Hess(u) is also positive semidefinite. By (17) and the Hopf boundary point lemma, we now arrive at the contradiction

$$0 < \frac{\partial v}{\partial x_N}(0) = 2\sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(0)\frac{\partial^2 u}{\partial x_i \partial x_N}(0) = 0.$$

Thus

$$\nabla u \cdot n > 0 \text{ on } \partial\Omega_\varepsilon \quad (\varepsilon > 0). \quad (18)$$

We will now establish (ii). Let $y \in \partial B(0, 1)$ and define

$$u_t(x) = u(x) - ty \cdot x \quad (x \in \Omega; 0 \leq t < \lambda_1(\Omega_\varepsilon)).$$

Then $\nabla u_t = \nabla u - ty$ and $\Delta u_t = N$ in Ω_ε. Let

$$A = \{t \in [0, \lambda_1(\Omega_\varepsilon)) : \text{there exists } x \in \Omega_\varepsilon \text{ such that } \nabla u_t(x) = 0\}.$$

It follows from (18) that u cannot attain the value $\min_{\overline{\Omega_\varepsilon}} u$ on $\partial\Omega_\varepsilon$, so $0 \in A$. We will now show that A is both open and closed relative to $[0, \lambda_1(\Omega_\varepsilon))$. To see that A is closed, let $(t^{(k)})$ be a sequence in A that converges to some $t \in [0, \lambda_1(\Omega_\varepsilon))$. There exist points $x^{(k)}$ in Ω_ε such that $\nabla u_{t^{(k)}}(x^{(k)}) = 0$ and, by choosing a subsequence, we may arrange that $(x^{(k)})$ converges to some point $x \in \overline{\Omega}_\varepsilon$. Clearly $\nabla u_t(x) = 0$, so $x \notin \partial\Omega_\varepsilon$, because

$$\|\nabla u_t\| \geq \|\nabla u\| - t = \lambda_1(\Omega) - \varepsilon - t = \lambda_1(\Omega_\varepsilon) - t > 0 \quad \text{on} \quad \partial\Omega_\varepsilon$$

by (16). Hence A is closed. To see that A is open, let $t \in A$, choose $x \in \Omega$ such that $\nabla u_t(x) = 0$, and define

$$\Omega' = \{z \in \Omega_\varepsilon : \|\nabla u_t(z)\| < \alpha\}, \quad \text{where} \quad \alpha = \inf_{\partial\Omega_\varepsilon} \|\nabla u_t\|.$$

Then $\alpha > 0$ and $x \in \Omega'$. We can apply the result of the previous paragraph to u_t to see that $\nabla u_t \cdot n > 0$ on $\partial\Omega'$. When $|s|$ is sufficiently small, the function u_{t+s} thus also has a strictly positive normal derivative on $\partial\Omega'$, and so attains the value $\min_{\overline{\Omega'}} u_{t+s}$ at some point $x^{(s)} \in \Omega'$. Since $\nabla u_{t+s}(x^{(s)}) = 0$, we see that the set A is also open relative to $[0, \lambda_1(\Omega_\varepsilon))$.

Hence $A = [0, \lambda_1(\Omega_\varepsilon))$. It follows that, for any $z \in B(0, \lambda_1(\Omega_\varepsilon))$, there exists $x \in \Omega_\varepsilon$ such that $\nabla u(x) = z$, and so $\nabla u : \Omega_\varepsilon \to B(0, \lambda_1(\Omega_\varepsilon))$ is surjective. Finally, we let $\varepsilon \to 0+$ and note from (16) that $\lambda_1(\Omega_\varepsilon) \to \lambda_1(\Omega)$.

Example. Consider the ellipsoid

$$E = \left\{ x \in \mathbb{R}^N : \sum_{i=1}^{N} a_i^2 {x_i}^2 < 1 \right\},$$

where $a_i > 0$ $(i = 1, ..., N)$. The function

$$u(x) = \frac{N}{2\sum_{i=1}^{N} a_i} \sum_{i=1}^{N} a_i x_i^2$$

is clearly convex, satisfies $\Delta u = N$ on E and $\|\nabla u(x)\| = N / \sum_{i=1}^{N} a_i$ on ∂E. Thus

$$\lambda_1(E) = \frac{N}{a_1 + a_2 + \cdots + a_N}.$$

Finally, we remark that minimizers need not be locally convex functions. For example, if $N \geq 3$ and $\Omega = B(0,R) \backslash \overline{B(0,r)}$, where $R > r > 0$, then the function

$$u(x) = \frac{1}{2}\|x\|^2 + \frac{R + r}{(N-2)(R^{1-N} + r^{1-N})}\|x\|^{2-N} \quad (x \in \Omega)$$

satisfies $\Delta u = N$ on Ω and $\nabla u = cn$ on $\partial\Omega$, where

$$c = \frac{R^N - r^N}{R^{N-1} + r^{N-1}}.$$

Thus $\lambda_1(\Omega) = c$, by Proposition 7. However, along the x_N-axis, $\mathrm{Hess}(u)$ is a diagonal matrix in which the first $N-1$ diagonal entries are valued

$$1 - \frac{R+r}{R^{1-N}+r^{1-N}} x_N^{-N},$$

so u is not locally convex near the inner boundary. (An analogous example when $N = 2$ may be obtained by replacing $\|x\|^{2-N}/(N-2)$ with $\log(1/\|x\|)$ in the formula for $u(x)$ above.)

References

[1] A. Abanov, C. Bénéteau, D. Khavinson, and R. Teodorescu, "A free boundary problem associated with the isoperimetric inequality", *J. Anal. Math.,* to appear; *arXiv:*1601.03885.

[2] H. Alexander, "Projections of polynomial hulls", *J. Funct. Anal.* 13 (1973), 13–19.

[3] D. H. Armitage and S. J. Gardiner, *Classical potential theory.* Springer, London, 2001.

[4] C. Bénéteau and D. Khavinson, "The isoperimetric inequality via approximation theory and free boundary problems", *Comput. Methods Funct. Theory* 6 (2006), 253–274.

[5] Y. Brenier, "Décomposition polaire et réarrangement monotone des champs de vecteurs", *C. R. Acad. Sci. Paris Sér. I Math.* 305 (1987), 805–808.

[6] Y. Brenier, "Polar factorization and monotone rearrangement of vector-valued functions", *Comm. Pure Appl. Math.* 44 (1991), 375–417.

[7] X. Cabré, "Elliptic PDE's in probability and geometry: symmetry and regularity of solutions", *Discrete Contin. Dyn. Syst.* 20 (2008), 425–457.

[8] L. A. Caffarelli, "The regularity of mappings with a convex potential", *J. Amer. Math. Soc.* 5 (1992), 99–104.

[9] L. C. Evans, *Partial differential equations.* 2nd Edition. Amer. Math. Soc., Providence, RI, 2010.

[10] T. W. Gamelin and D. Khavinson, "The isoperimetric inequality and rational approximation", *Amer. Math. Monthly* 96 (1989), 18–30.

[11] S. J. Gardiner and T. Sjödin, "Partial balayage and the exterior inverse problem of potential theory". *Potential theory and stochastics in Albac*, pp.111–123, Theta, Bucharest, 2009.

[12] B. Gustafsson and D. Khavinson, “On approximation by harmonic vector fields”, *Houston J. Math.* 20 (1994), 75–92.

[13] B. Gustafsson and D. Khavinson, “On annihilators of harmonic vector fields”, *J. Math. Sci.* (New York) 92 (1998), 3600–3612.

[14] B. Gustafsson and M. Sakai, “Properties of some balayage operators, with applications to quadrature domains and moving boundary problems”, *Nonlinear Anal.* 22 (1994), 1221–1245.

[15] D. Khavinson, “Annihilating measures of the algebra R(X)”, *J. Funct. Anal.* 58 (1984), 175–193.

[16] R. J. McCann and N. Guillen, “Five lectures on optimal transportation: geometry, regularity and applications”, in: *Analysis and geometry of metric measure spaces*, pp.145–180, CRM Proc. Lecture Notes, 56, Amer. Math. Soc., Providence, RI, 2013.

[17] Sakai, M. “Linear combinations of harmonic measures and quadrature domains of signed measures with small supports”. *Proc. Edinburgh Math. Soc. (2)* 42 (1999), 433–444.

[18] C. Villani, *Topics in optimal transportation.* Amer. Math. Soc., Providence, RI, 2003.

[19] K.-O. Widman, “Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations”, *Math. Scand.* 21 (1967), 17–37.

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