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# Universal Taylor series for non-simply connected domains

# Séries universelles de Taylor pour les domaines non-simplement connexes

Stephen J. Gardiner and N. Tsirivas

**Abstract**

It is known that, for any simply connected proper subdomain $\Omega$ of the complex plane and any point $\zeta$ in $\Omega$, there are holomorphic functions on $\Omega$ that have "universal" Taylor series expansions about $\zeta$; that is, partial sums of the Taylor series approximate arbitrary polynomials on arbitrary compacta in $\mathbb{C}\backslash\Omega$ that have connected complement. This note shows that this phenomenon can break down for non-simply connected domains $\Omega$, even when $\mathbb{C}\backslash\Omega$ is compact. This answers a question of Melas and disproves a conjecture of Müller, Vlachou and Yavrian.

**Résumé**

Il est connu que, pour un sous-domaine propre simplement connexe $\Omega$ du plan complexe et un point quelconque $\zeta$ de $\Omega$, il y a des fonctions holomorphes sur $\Omega$ qui possèdent des séries de Taylor «universelles» autour de $\zeta$; c'est-à-dire tout polynôme peut être approximé, sur tout compact de $\mathbb{C}\backslash\Omega$ ayant un complémentaire connexe, par les sommes partielles de la série de Taylor. Cette note montre que ce résultat n'est plus vrai en général pour les domaines non-simplement connexes $\Omega$, même lorsque $\mathbb{C}\backslash\Omega$ est compact. Cela répond à une question de Melas et réfute une conjecture de Müller, Vlachou et Yavrian.

## 1 Introduction

Let $\Omega$ be a proper subdomain of the complex plane $\mathbb{C}$ and let $\zeta \in \Omega$. A function $f$ on $\Omega$ is said to belong to the collection $U(\Omega, \zeta)$, of holomorphic

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functions on $\Omega$ with universal Taylor series expansions about $\zeta$, if the partial sums

$$S_N(f,\zeta)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!}(z-\zeta)^n$$

of the Taylor series have the following property:

*for every compact set $K \subset \mathbb{C}\backslash\Omega$ with connected complement and every function $g$ which is continuous on $K$ and holomorphic on $K^\circ$, there is a subsequence $(S_{N_k}(f,\zeta))$ that converges to $g$ uniformly on $K$.*

Nestoridis [17], [18] has shown that $U(\Omega,\zeta) \neq \emptyset$ for any simply connected domain $\Omega$ and any $\zeta \in \Omega$. (The corresponding result, where $K$ is required to be disjoint from $\overline{\Omega}$, had previously been established by Luh [12] and Chui and Parnes [4].) In fact, Nestoridis showed that possession of such universal Taylor series expansions is a generic property of holomorphic functions on simply connected domains $\Omega$, in the sense that $U(\Omega,\zeta)$ is a dense $G_\delta$ subset of the space of all holomorphic functions on $\Omega$ endowed with the topology of local uniform convergence (see also Melas and Nestoridis [14] and the survey of Kahane [11]).

The situation when $\Omega$ is non-simply connected is much less well understood, despite much recent research: see, for example, [2], [3], [5], [6], [7], [9], [13], [15], [19], [22], [23], [24], [25]. Melas [13] (see also Costakis [5]) has shown that $U(\Omega,\zeta) \neq \emptyset$ for any $\zeta \in \Omega$ whenever $\mathbb{C}\backslash\Omega$ is compact and connected, and has asked if $U(\Omega,\zeta)$ can be empty when $\mathbb{C}\backslash\Omega$ is compact but disconnected. On the other hand, Müller, Vlachou and Yavrian [15] have shown, for non-simply connected domains $\Omega$, that thinness of the set $\mathbb{C}\backslash\Omega$ at infinity is necessary for $U(\Omega,\zeta)$ to be non-empty, and have conjectured that this condition is also sufficient. There is clearly a large gap between the results of [13] and [15]. Also there has been no known example of a domain $\Omega$ and points $\zeta_1, \zeta_2 \in \Omega$ such that $U(\Omega,\zeta_1) \neq \emptyset$ and $U(\Omega,\zeta_2) = \emptyset$.

The purpose of this note is to establish the following result. We denote by $D(a,r)$ the open disc of centre $a$ and radius $r$, and write $\mathbb{D} = D(0,1)$. By a *non-degenerate continuum* we mean a connected compact set containing more than one element.

**Theorem 1** *Let $\Omega$ be a domain of the form $\mathbb{C}\backslash(L \cup \{1\})$, where $L$ is a non-degenerate continuum in $\mathbb{C}\backslash\overline{\mathbb{D}}$. Then $U(\Omega,0) = \emptyset$.*

The conjecture of Müller, Vlachou and Yavrian is thus disproved. Also, if we take $L$ to be $\overline{D}(-5/3,1/3)$, then $U(\Omega,0) = \emptyset$ by Theorem 1 and yet a result of the second author [22] tells us that $U(\Omega,-1/2) \neq \emptyset$ (see also Costakis and Vlachou [7]). Thus we now have an example of a domain where the existence of functions with universal Taylor series depends on the chosen centre for expansion. The result of Melas, that $U(\Omega,0) \neq \emptyset$ if $\mathbb{C}\backslash\Omega$

is compact and connected, is now seen to be sharp in the sense that, by Theorem 1, it can fail with the removal of one additional point from the domain. Theorem 1 fails if $L$ is allowed to be a singleton [13].

## 2 Proof

Let $\Omega$ be as in the statement of Theorem 1, and suppose, for the sake of contradiction, that there exists a function $f$ in $U(\Omega, 0)$. We can write $f = g + h$, where $g$ is the singular part of the Laurent expansion of $f$ associated with the singularity at 1, and $h$ is holomorphic on $\mathbb{C}\backslash L$. We denote the Taylor coefficients of $g$ and $h$ about 0 by $(a_n)$ and $(b_n)$, respectively. Since $(S_N(f,0)(1))$ is dense in $\mathbb{C}$ and $(S_N(h,0)(1))$ converges, we see that $g$ is non-zero.

Let $\rho = \inf\{|z| : z \in L\}$ and $0 < \delta < \varepsilon < \rho - 1$. The Taylor series for $g$ and $h$ about 0 converge absolutely in $\mathbb{D}$ and $D(0, \rho)$, respectively, so we can define the finite quantities

$$\alpha_\delta = \sum_{n=0}^{\infty} \frac{|a_n|}{(1+\delta)^n} \quad \text{and} \quad \beta_\delta = \sum_{n=0}^{\infty} |b_n| \left( \frac{\rho}{1+\delta} \right)^n .$$

Since $f \in U(\Omega, 0)$, we can choose a strictly increasing sequence $(N_k)$ of natural numbers such that

$$S_{N_k}(g,0)(z) + S_{N_k}(h,0)(z) \to 0 \quad \text{as } k \to \infty, \text{ uniformly on } L. \tag{1}$$

On $\overline{D}(0, \rho(1+\varepsilon))$ we have

$$|S_{N_k}(h,0)(z)| \leq \sum_{n=0}^{N_k} |b_n| \, \rho^n (1+\varepsilon)^n \leq \{(1+\varepsilon)(1+\delta)\}^{N_k} \beta_\delta,$$

so by (1) we can choose $k_0$ such that

$$|S_{N_k}(g,0)(z)| \leq \{(1+\varepsilon)(1+\delta)\}^{N_k} (\beta_\delta + 1) \quad (z \in L \cap \overline{D}(0, \rho(1+\varepsilon)); k \geq k_0).$$

We also have

$$|S_{N_k}(g,0)(z)| \leq \sum_{n=0}^{N_k} |a_n| \, (1+\varepsilon)^n \leq \{(1+\varepsilon)(1+\delta)\}^{N_k} \alpha_\delta \quad (z \in \overline{D}(0, 1+\varepsilon)),$$

so

$$|S_{N_k}(g,0)(z)| \leq \{(1+\varepsilon)(1+\delta)\}^{N_k} \gamma_\delta \quad (z \in A_\varepsilon; k \geq k_0), \tag{2}$$

where $\gamma_\delta = \max\{\alpha_\delta, \beta_\delta + 1\}$ and

$$A_\varepsilon = \overline{D}(0, 1+\varepsilon) \cup \left[ L \cap \overline{D}(0, \rho(1+\varepsilon)) \right].$$

Let $G_\varepsilon$ denote the Green function for the domain $D_\varepsilon = (\mathbb{C} \cup \{\infty\}) \backslash A_\varepsilon$ with pole at infinity. Then

$$G_\varepsilon(z) - \log|z| \to -\log \mathcal{C}(A_\varepsilon) \qquad (|z| \to \infty),$$

where $\mathcal{C}(A)$ denotes the logarithmic capacity of a set $A$ (see Section 5.8 of [1], or Section 5.2 of [21]). Thus we can choose $r_{\delta,\varepsilon} > \max\{|z| : z \in L\}$ such that

$$G_\varepsilon(z) \le \log|z| - \log \mathcal{C}(A_\varepsilon) + \delta \qquad (|z| \ge r_{\delta,\varepsilon}). \tag{3}$$

Bernstein's lemma (Theorem 5.5.7 in [21]) tells us that any polynomial $q$ of degree $n \ge 1$ satisfies

$$\left( \frac{|q(z)|}{\max_{A_\varepsilon} |q|} \right)^{1/n} \le e^{G_\varepsilon(z)} \qquad (z \in D_\varepsilon \backslash \{\infty\}).$$

Applying this inequality to the polynomial $S_{N_k}(g,0)$, and using (2) and then (3), we obtain

$$\begin{aligned} |S_{N_k}(g,0)(z)| &\le \{(1+\varepsilon)(1+\delta)\}^{N_k} \gamma_\delta e^{N_k G_\varepsilon(z)} \\ &\le \left\{ \frac{(1+\varepsilon)(1+\delta)e^\delta |z|}{\mathcal{C}(A_\varepsilon)} \right\}^{N_k} \gamma_\delta \qquad (|z| \ge r_{\delta,\varepsilon}; k \ge k_0). \end{aligned}$$

We next adapt an argument from pp.498,499 of Gehlen [8]. Let $\nu \in (0,1)$. Since

$$\begin{aligned} |a_n|^{1/n} &= \left| \frac{1}{2\pi i} \int_{\{|z|=r_{\delta,\varepsilon}\}} \frac{S_{N_k}(g,0)(z)}{z^{n+1}} dz \right|^{1/n} \\ &\le \left\{ \frac{(1+\varepsilon)(1+\delta)e^\delta}{\mathcal{C}(A_\varepsilon)} \right\}^{N_k/n} \gamma_\delta^{1/n} r_{\delta,\varepsilon}^{N_k/n-1} \qquad (n \le N_k; k \ge k_0), \end{aligned}$$

we obtain

$$\limsup_{k\to\infty} \max_{\nu N_k \le n \le N_k} |a_n|^{1/n} \le \frac{\left\{(1+\varepsilon)(1+\delta)e^\delta\right\}^{1/\nu} r_{\delta,\varepsilon}^{1/\nu-1}}{\mathcal{C}(A_\varepsilon)} = \lambda, \quad \text{say.} \tag{4}$$

Since $L$ is a non-degenerate continuum that intersects $\{|z| = \rho\}$, we have

$$\mathcal{C}(L \cap \overline{D}(0, \rho(1+\varepsilon))) > 0$$

and so

$$\mathcal{C}(A_\varepsilon) > \mathcal{C}(\overline{D}(0, 1+\varepsilon)) = 1+\varepsilon.$$

We can thus choose $\delta$ sufficiently small that $(1+\varepsilon)(1+\delta)e^\delta < \mathcal{C}(A_\varepsilon)$, and then choose $\nu$ sufficiently close to 1 to ensure that $\lambda < 1$.

Finally, we will apply an observation of Müller (see Remark 2 in [16]). Since the function $g$ has its only singularity at 1 and vanishes at $\infty$, Wigert's theorem (Theorem 11.2.2 in Hille [10]) tells us that there is an entire function $F$ of exponential type 0 such that $F(n) = a_n$ for all $n \geq 0$. However, Theorem V of Pólya [20] says that, for any $\mu > 0$, however small, such a function $F$ has the property that the sequence $\{n \in \mathbb{N} : |F(n)| > e^{-\mu n}\}$ is of density 1. This contradicts (4) with $\lambda < 1$. Thus our original assumption, that there exists $f$ in $U(\Omega, 0)$, must be false, and the proof of the theorem is complete. ∎

**Remarks.** 1) The assumption that $L$ is a continuum can be relaxed. It is enough to suppose that $L$ is a compact subset of $\mathbb{C} \backslash \overline{\mathbb{D}}$ such that $\mathcal{C}(D(0, \rho^2) \cap L) > 0$ where $\rho = \inf\{|z| : z \in L\}$.
2) The proof actually shows that there is no holomorphic function $f$ on $\Omega$ such that $(S_N(f, 0))$ is divergent at $z = 1$ and has a subsequence that is uniformly bounded on $L$.

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