Positivity properties for the clamped plate boundary problem on the ellipse and strip

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The positivity preserving property for the biharmonic operator with Dirichlet boundary condition is investigated. We discuss here the case where the domain is an ellipse (that may degenerate to a strip) and the data is a polynomial function. We provide various conditions for which the positivity is preserved.

1 Introduction

Let Ω be a domain in $\mathbb{R}^n$ and $\partial\Omega$ its boundary and let Δ be the Laplace operator defined by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}.$$

The clamped plate boundary value problem is the problem of finding a solution u of the biharmonic equation

$$\Delta^2 u = f \quad \text{in } \Omega \tag{1}$$

satisying the boundary condition

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{2}$$

where $\Delta^2 u = \Delta(\Delta u)$ and $\partial/\partial\nu$ is the outer normal derivative at $\partial\Omega$. From a physical point of view the following conjecture due to T. Boggio and J. Hadamard is very natural:

(P): Positivity of the data function $f \in C^4\left(\overline{\Omega}\right)$ implies positivity of the solution u.

In case of the Laplace operator Δ subject to Dirichlet boundary conditions, the property (P) is true due to the standard maximum principle. Here by f (resp. u) positive on $\overline{\Omega}$ we understand $f \geq 0$ (resp. $u \geq 0$) in $\overline{\Omega}$.

It is not difficult to see that property (P) is equivalent to the statement that the Green function of the domain Ω is positive. In 1905, Boggio [1] proved the positivity of the Green function for the ball and conjectured that this should be true for any reasonable domain. In 1908, Hadamard disproved the conjecture for an annulus but only after 1949 numerous counterexamples to the conjecture of Boggio have been found, the most striking by Garabedian [2] showing that the ellipse with ratio of half axes ≈ 1.6 does not have a positive Green function. For details we refer the reader to the excellent book [3] as well as [4]-[10].

In this paper we investigate the question whether for certain subclasses of positive data functions f on the ellipsoid the solution u of (1)-(2) is positive expecting that additional properties at the data function may lead to the positivity of the solution. The first main result of the paper shows that indeed for any *affine* positive function f on an ellipsoid the solution u is positive on the ellipsoid. In case where $f \geq 0$ is a positive polynomial of degree 2, the question of positivity of the solution is more subtle and we discuss only the case of the ellipse

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$E_b := \left\{(x,y) \in \mathbb{R}^2 : x^2 + by^2 < 1\right\}$ which degenerates for $b = 0$ to a strip in $\mathbb{R}^2$. In the latter case, a fairly simple proof shows that for any polynomial f of degree ≤ 2 positivity on the strip implies positivity of the solution u. On the other hand, the positivity property (P) is not preserved on a general elipse. We show in Section 5 that for $b = 19$ and $f = \left(x - \frac{281}{303}\right)^2$ the unique solution u of (1)-(2) is not positive in E_{19}. Indeed, as we shall see later on in this paper, u has the form $u = \left(x^2 + 19y^2 - 1\right)^2 q(x,y)$ where

$$q(x,y) = \frac{1032855739}{10542848322456} - \frac{281}{1468944}x + \frac{1373}{14711904}x^2 - \frac{95}{7355952}y^2 \tag{3}$$

is not positive on E_{19} since $q(1,0) < 0$. Therefore we found a positive polynomial data f of degree two for which the solution u of (1)-(2) is not positive. This seems to be the simplest counterexample to the Boggio conjecture, cf. also [14] where the data function $f(x,y)$ is a polynomial of degree 3.

The paper is organized as follows. In Section 2 we present some properties of the Fischer operator that will be useful in our analysis; we also refer the reader to [11, 12, 13] for a systematic study on this topic. In Section 3 we prove that positivity for the clamped plate problem is preserved for affine functions on the ellipsoids. In Section 4 we show that positivity is preserved for quadratic polynomials on the strip. In the last Section we give certain criteria for polynomials of degree two for which positivity is preserved in case of an ellipse in the plane. These conditions have led us to the simple counterexample (3).

2 Fischer operator

Let $a_1, ..., a_n$ be positive real numbers. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ define

$$\psi(x) = a_1 x_1^2 + ... + a_n x_n^2 - 1 \text{ and } E_\psi := \{x \in \mathbb{R}^n : \psi(x) < 0\}.$$

Let $\mathcal{P}_{\leq m}(\mathbb{R}^n)$ be the set of all polynomials of degree $\leq m$. The Fischer operator (see [11, 12, 13]) is defined by

$$F_\psi(q) := \Delta^2 \left(\psi^2(x) \cdot q(x)\right).$$

Note that F_ψ maps $\mathcal{P}_{\leq m}(\mathbb{R}^n)$ into $\mathcal{P}_{\leq m}(\mathbb{R}^n)$ since ψ^2 has degree 4. By Theorem 3 in [11] the Fischer operator F_ψ is injective since ψ^2 is an elliptic polynomial. Furthermore, we have:

Theorem 2.1 *Let $f \in \mathcal{P}_{\leq m}(\mathbb{R}^n)$ be a polynomial. Then, there exists a unique solution u of (1)-(2) in $\Omega = E_\psi$. Furthermore, we have*

(i) $u \in \mathcal{P}_{\leq m+4}(\mathbb{R}^n)$;

(ii) $u(x) = \psi^2(x) q(x)$, *for some* $q \in \mathcal{P}_{\leq m}(\mathbb{R}^n)$.

P r o o f. The uniqueness follows by applying the Green formula to (1) and using (2). Since

$$F_\psi : \mathcal{P}_{\leq m}(\mathbb{R}^n) \to \mathcal{P}_{\leq m}(\mathbb{R}^n)$$

is injective, it is also bijective so for any $f \in \mathcal{P}_{\leq m}(\mathbb{R}^n)$ there exists a unique $q \in \mathcal{P}_{\leq m}(\mathbb{R}^n)$ such that $F_\psi(q) = f$. Hence $u := \psi^2 \cdot q$ is the unique solution of (1)-(2). $\square$

The Fischer operator can now be used for computing the solution of (1)-(2) with polynomial data. Assume in the following that $n = 2$ and let $E_b = \left\{(x,y) \in \mathbb{R}^2 : x^2 + by^2 < 1\right\}$ be an ellipse; note that the case $b = 0$ corresponds to the strip $(-1,1) \times \mathbb{R}$ in $\mathbb{R}^2$. For $\psi(x,y) = x^2 + by^2 - 1$ we have defined the Fischer operator

$F_\psi(q) := \Delta^2\left(\psi^2 q\right)$. A straightforward computation gives the following information for $T := \frac{1}{24}F_\psi$:

$$\begin{aligned}
T(1) &= 1 + \frac{2}{3}b + b^2,\\
T(x) &= \left(5 + 2b + 1b^2\right)x,\\
T(y) &= \left(1 + 2b + 5b^2\right)y,\\
T\left(x^2\right) &= \left(4b + 15 + b^2\right)x^2 + \left(2b^2 + 2b\right)y^2 - 2 - \frac{2}{3}b,\\
T(xy) &= \left(5 + 6b + 5b^2\right)xy,\\
T\left(y^2\right) &= (2 + 2b)x^2 + \left(1 + 4b + 15b^2\right)y^2 - \frac{2}{3} - 2b.
\end{aligned}$$

Hence $1, x, y$ and xy are eigenvectors of the Fischer operator F_ψ. The matrix associated with the linear map T and the basis $1, x, y, x^2, xy, y^2$ is given by

$$\begin{pmatrix}
1+\frac{2}{3}b+b^2 & 0 & 0 & -2-\frac{2}{3}b & 0 & -2b-\frac{2}{3}\\
0 & 5+2b+b^2 & 0 & 0 & 0 & 0\\
0 & 0 & 1+2b+5b^2 & 0 & 0 & 0\\
0 & 0 & 0 & 4b+15+b^2 & 0 & 2+2b\\
0 & 0 & 0 & 0 & 5+6b+5b^2 & 0\\
0 & 0 & 0 & 2b^2+2b & 0 & 1+15b^2+4b
\end{pmatrix}$$

Let $T^{-1} = 24F_\psi^{-1}$ be the inverse matrix of T. It is not difficult to find that $\left(3b^2+2b+3\right)T^{-1}$ is equal to

$$\begin{pmatrix}
3 & 0 & 0 & \frac{2}{3}\frac{9b^3+41b^2+11b+3}{5b^4+20b^3+78b^2+20b+5} & 0 & \frac{2}{3}\frac{3b^3+11b^2+41b+9}{5b^4+20b^3+78b^2+20b+5}\\
0 & \frac{3b^2+2b+3}{5+2b+b^2} & 0 & 0 & 0 & 0\\
0 & 0 & \frac{3b^2+2b+3}{1+2b+5b^2} & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{3}\frac{\left(3b^2+2b+3\right)\left(1+15b^2+4b\right)}{5b^4+20b^3+78b^2+20b+5} & 0 & -\frac{2}{3}\frac{\left(3b^2+2b+3\right)(b+1)}{5b^4+20b^3+78b^2+20b+5}\\
0 & 0 & 0 & 0 & \frac{3b^2+2b+3}{5+6b+5b^2} & 0\\
0 & 0 & 0 & -\frac{2}{3}\frac{b\left(3b^2+2b+3\right)(b+1)}{5b^4+20b^3+78b^2+20b+5} & 0 & \frac{1}{3}\frac{\left(3b^2+2b+3\right)\left(4b+15+b^2\right)}{5b^4+20b^3+78b^2+20b+5}
\end{pmatrix}$$

For computational convenience we introduce

$$S := \left(5b^4 + 20b^3 + 78b^2 + 20b + 5\right)\cdot\left(3b^2 + 2b + 3\right)\cdot 24\cdot F_\psi^{-1}. \tag{4}$$

Let $f(x,y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2$ be a polynomial of degree ≤ 2. Then we can write

$$S(f)(x,y) = A_0 + A_1x + A_2y + A_3x^2 + A_4xy + A_5y^2. \tag{5}$$

where the coefficients $A_0, ..., A_5$ can be read off from the matrix representation:

$$\begin{aligned}
A_0 &= 3\left(5b^4 + 20b^3 + 78b^2 + 20b + 5\right)a_0 + \frac{2}{3}\left(9b^3 + 41b^2 + 11b + 3\right)a_3\\
&\quad + \frac{2}{3}\left(3b^3 + 11b^2 + 41b + 9\right)a_5, &(6)\\
A_1 &= \frac{C_b a_1}{5 + 2b + b^2}\ ,\ A_2 = \frac{C_b a_2}{1 + 2b + 5b^2}\ ,\ A_4 = \frac{C_b a_4}{5 + 6b + 5b^2}\ , &(7)\\
A_3 &= \frac{3b^2 + 2b + 3}{3}\cdot\left(\left(1 + 15b^2 + 4b\right)a_3 - 2(b+1)a_5\right)\ , &(8)\\
A_5 &= \frac{3b^2 + 2b + 3}{3}\cdot\left(-2b(b+1)a_3 + \left(4b + 15 + b^2\right)a_5\right)\ , &(9)
\end{aligned}$$

where $C_b = \left(5b^4 + 20b^3 + 78b^2 + 20b + 5\right)\left(3b^2 + 2b + 3\right)$.

3 Positivity of solutions for affine data functions on the ellipsoid

In this section we discuss the positivity preserving property (P) in case the data function f is affine.

Proposition 3.1 *Let $\alpha \in \mathbb{R}$ and $\gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{R}^n$. An affine non-zero function $f(x) = \alpha + \langle \gamma, x \rangle$ is positive on the ellipsoid $E_\psi := \left\{ a_1 x_1^2 + ... + a_n x_n^2 < 1 \right\}$ if and only if*

$$\alpha > 0 \quad \text{and} \sum_{j=1}^{n} \frac{\gamma_j^2}{a_j} \leq \alpha^2.$$

Proof. If $f \geq 0$ in E_ψ, then $f(0) = \alpha \geq 0$. If $\alpha = 0$ then $f(x) = \langle \gamma, x \rangle$ and since f is positive this implies that $\gamma = 0$, so $f = 0$. Thus we may assume that $\alpha > 0$. The case $\gamma = 0$ is trivial, so we may assume as well $\gamma \neq 0$.

An affine function is harmonic, so it attains its maximum on the boundary. We apply the method of Lagrange multipliers in order to determine the minimum of f on the boundary. Let $\lambda \in \mathbb{R}$ be such that $\nabla f(x) = \lambda \nabla \psi(x)$ for some $x \in \partial E_\psi$. Then

$$\frac{\partial f}{\partial x_j}(x) = \gamma_j = \lambda \frac{\partial \psi}{\partial x_j} = \lambda 2 a_j x_j.$$

Since $\gamma \neq 0$ we infer that $\lambda \neq 0$. Thus $x_j = \gamma_j / 2 a_j \lambda$ and since $x \in \partial E_\psi$

$$1 = \sum_{j=1}^{n} a_j x_j^2 = \frac{1}{\lambda^2} \sum_{j=1}^{n} \frac{\gamma_j^2}{4 a_j}, \quad \text{so } \lambda = \pm \sqrt{\sum_{j=1}^{n} \frac{\gamma_j^2}{4 a_j}}. \tag{10}$$

Thus the critical points x on the boundary are of the form $\pm(\gamma_1/2a_1\lambda, ..., \gamma_n/2a_n\lambda)$ and

$$f(x) = \alpha \pm \sum_{j=1}^{n} \gamma_j \frac{\gamma_j}{2 a_j |\lambda|} \geq \alpha - \sum_{j=1}^{n} \frac{\gamma_j^2}{2 a_j |\lambda|} = \min_{y \in \partial E_\psi} f(y).$$

We conclude that f is positive on E_ψ if and only if $\sum_{j=1}^{n} \frac{\gamma_j^2}{2 a_j |\lambda|} \leq \alpha$ which in view of (10) is equivalent to the statement of the proposition. □

Proposition 3.2 *Let $\psi(x) = \sum_{k=1}^{n} a_k x_k^2 - 1$ and define $A = \sum_{k=1}^{n} a_k$ and $B = \sum_{k=1}^{n} a_k^2$. Then*

$$\Delta^2 \left(\psi^2 \cdot (\alpha + \langle \gamma, x \rangle) \right) = \alpha \left(8A^2 + 16B \right) + \sum_{k=1}^{n} \left(8A^2 + 16B + 32 A a_k + 64 a_k^2 \right) x_k. \tag{11}$$

Proof. Recall that $\Delta(fg) = f \Delta g + 2 \langle \nabla f, \nabla g \rangle + \Delta f \cdot g$ where ∇f is the gradient of f. Define $q(x) = \alpha + \langle \gamma, x \rangle$. Since $\Delta q = 0$ we conclude that $\Delta \left(\psi^2 q \right) = \Delta \left(\psi^2 \right) \cdot q + 2 \sum_{j=1}^{n} \gamma_j \frac{\partial}{\partial x_j} \left(\psi^2 \right)$. We apply the Laplace operator to the last equation, and using again that $\Delta q = 0$, we obtain

$$\Delta^2 \left(\psi^2 q \right) = \Delta^2 \left(\psi^2 \right) \cdot q + 4 \sum_{j=1}^{n} \gamma_j \Delta \frac{\partial}{\partial x_j} \left(\psi^2 \right). \tag{12}$$

Clearly $\frac{\partial \psi}{\partial x_j} = 2 a_j x_j$ and $\frac{\partial^2 \psi}{\partial x_j^2} = 2 a_j$ and therefore $\Delta \psi = 2 \sum_{j=1}^{n} a_j =: 2A$. Furthermore, $\Delta \left(\psi^2 \right) = 2 \psi \Delta(\psi) + 2 \sum_{j=1}^{n} \left(\frac{\partial \psi}{\partial x_j} \right)^2 = 4 A \psi + 8 \sum_{j=1}^{n} a_j^2 x_j^2$, so $\Delta^2 \left(\psi^2 \right) = 8A^2 + 16 \sum_{j=1}^{n} a_j^2 = 8A^2 + 16B$ and $\frac{\partial}{\partial x_k} \Delta \left(\psi^2 \right) = 8A \cdot a_k x_k + 16 a_k^2 x_k = 8 \left(A a_k + 2 a_k^2 \right) x_k$. Now we infer from (12) that

$$\Delta^2 \left(\psi^2 q \right) = q \cdot \left(8A^2 + 16B \right) + 32 \sum_{k=1}^{n} \gamma_k \left(A a_k + 2 a_k^2 \right) x_k$$

which is equal to (11) using that $q(x) = \alpha + \langle \gamma, x \rangle$. □

Theorem 3.3 *If an affine function f is positive on the ellipsoid E_ψ then the solution q of $\Delta^2\left(\psi^2 q\right)=f$ is positive on E_ψ.*

Proof. For the affine function f there exists a unique affine function $q=\alpha+\langle\gamma,x\rangle$ such that $\Delta^2\left(\psi^2 u\right)=f$. Let us define $\delta_k=8A^2+16B+32Aa_k+64a_k^2$, $1\leq k\leq n$. By Proposition 3.2

$$f=\Delta^2\left(\psi^2 q\right)=8\alpha\left(A^2+2B\right)+\sum_{k=1}^{n}x_k\gamma_k\delta_k.$$

Since f is positive on the ellipsoid, Proposition 3.1 implies that $\sum_{k=1}^{n}\frac{\gamma_k^2\delta_k^2}{a_k}\leq\alpha^2 64\left(A^2+2B\right)^2$. From the definition of δ_k we see that $\delta_k\geq 8\left(A^2+2B\right)$, so

$$\sum_{k=1}^{n}\frac{\gamma_k^2}{a_k}\leq\sum_{k=1}^{n}\frac{\gamma_k^2}{a_k}\frac{\delta_k^2}{64\left(A^2+2B\right)^2}\leq\alpha^2.$$

Proposition 3.1 shows that q is positive on the ellipsoid E_ψ. □

4 Positivity for quadratic polynomials on the strip

In this section we consider the case $\psi(x,y)=x^2-1$, so that E_0 is a strip. Let $f(x,y)=a_0+a_1x+a_2y+a_3x^2+a_4xy+a_5y^2$ be a polynomial of degree ≤ 2 and consider the operator S defined in (4) and (5) by

$$S(f)(x,y)=360F_\psi^{-1}f(x,y)=A_0+A_1x+A_2y+A_3x^2+A_4xy+A_5y^2. \tag{13}$$

Using the formulae in Section 2 for $b=0$ we obtain

$$\begin{array}{lll} A_0=15a_0+2a_3+6a_5, & A_1=3a_1, & A_2=15a_2,\\ A_3=a_3-2a_5 & A_4=3a_4, & A_5=15a_5. \end{array}$$

Theorem 4.1 *Let $\psi(x,y)=x^2-1$ and let f be a polynomial of degree ≤ 2 such that $f\geq 0$ $[-1,1]\times\mathbb{R}$. Then the unique solution u of (1)-(2) satisfies*

$$u\geq\frac{1}{120}\psi^2 f\geq 0\quad\textit{in } [-1,1]\times\mathbb{R}. \tag{14}$$

Proof. According to Theorem 2.1, $u=\psi^2\cdot F_\psi^{-1}(f)$. Thus, it is enough to show that

$$120\cdot F_\psi^{-1}(f)\geq f\quad\text{ in } [-1,1]\times\mathbb{R}.$$

1. Suppose that $f(x,y)=a_0+a_1x+a_2y+a_3x^2+a_4xy+a_5y^2$ is positive on the strip. It is easy to see that this implies $a_5\geq 0$. In view of (13) and (14) it suffices to show that $D(x,y):=S(f)(x,y)-3f(x,y)\geq 0$ for all $x\in[-1,1]$ and $y\in\mathbb{R}$. Clearly

$$D(x,y)=12a_0+2a_3+6a_5+12a_2y+\left(-2a_3-2a_5\right)x^2+12a_5y^2.$$

2. Consider the case $a_5=0$. Suppose that there exists $x_0\in[-1,1]$ such that $a_2+a_4x_0\neq 0$. Since $f(x_0,y)=a_0+a_1x_0+a_3x_0^2+\left(a_2+a_4x_0\right)y\geq 0$ for all $y\in\mathbb{R}$, we obtain a contradiction. Hence $a_2+a_4x=0$ for all $x\in[-1,1]$ and this implies that $a_2=a_4=a_5=0$. It follows that $f(x,y)=a_0+a_1x+a_3x^2$ and

$$D(x,y)=12a_0+2a_3-2a_3x^2\geq 10a_0+2\left(a_0+a_3\right)\left(1-x^2\right)\geq 0$$

since $a_0=f(0,0)\geq 0$ and $0\leq f(\pm x,0)=a_0\pm a_1+a_3$, so $a_0+a_3\geq 0$.

3. Now assume $a_5>0$. We prove that the polynomial $y\longmapsto D(x,y)$ is positive on $\mathbb{R}$ for all $x\in[-1,1]$ which means that the discriminant

$$4\cdot 12a_5\left(12a_0+2a_3+6a_5-2a_3x^2-2a_5x^2\right)-12^2a_2^2\geq 0,$$

for each $x \in [-1, 1]$. Dividing by 48, this amounts to

$$p(x) := \left(-2a_5 a_3 - 2a_5^2\right) x^2 + 12a_5 a_0 + 2a_5 a_3 + 6a_5^2 - 3a_2^2 \geq 0,$$

for all $x \in [-1, 1]$. This is equivalent to

$$\begin{aligned} p(\pm 1) &= 4a_5^2 + 12a_5 a_0 - 3a_2^2 = 4a_5^2 + 3\left(4a_5 a_0 - a_2^2\right) \geq 0 \\ p(0) &= 12a_5 a_0 + 2a_5 a_3 + 6a_5^2 - 3a_2^2 \geq 0. \end{aligned}$$

To this aim, we shall use the fact that f is positive on the strip. Since $y \longmapsto f(x, y)$ is a positive polynomial, we conclude that

$$\Delta(x) := 4a_5(a_0 + a_1 x + a_3 x^2) - (a_2 + a_4 x)^2 \geq 0,$$

for all $x \in [-1, 1]$. For $x = 0$ we infer that $4a_5 a_0 - a_2^2 \geq 0$. It follows that $p(\pm 1) \geq 0$.

4. If a polynomial $g(x) = \alpha + \beta x + \gamma x^2$ is positive on $[-1, 1]$ then $g(\pm 1) = \alpha \pm \beta + \gamma \geq 0$, so $\alpha + \gamma \geq 0$. We apply this fact to $x \longmapsto \Delta(x)$ using that

$$\Delta(x) = \left(4a_5 a_3 - a_4^2\right) x^2 + \left(4a_5 a_1 - 2a_2 a_4\right) x + 4a_5 a_0 - a_2^2.$$

Hence $4a_5 a_3 - a_4^2 + 4a_5 a_0 - a_2^2 \geq 0$ and therefore $4a_5 a_3 \geq a_2^2 + a_4^2 - 4a_5 a_0$. This inequality provides the estimate

$$2p(0) \geq 24a_5 a_0 + a_2^2 + a_4^2 - 4a_5 a_0 + 12a_5^2 - 6a_2^2 = 5\left(4a_5 a_0 - a_2^2\right) + a_4^2 + 12a_5^2 \geq 0.$$

Hence the proof is accomplished. □

Remark 4.2 Positive polynomial data functions of higher degree do not lead in general to positive solutions. By calculating the inverse matrix of the Fischer operator on the space of all polynomials of degree ≤ 4 one obtains that

$$q(x, y) := 24F_\psi^{-1}\left(y^4\right) = \frac{41}{210} - \frac{11}{35}x^2 + \frac{12}{5}y^2 + \frac{3}{70}x^4 - \frac{4}{5}x^2 y^2 + y^4.$$

Since $q(1, 0) = -8/105 < 0$ it follows that $q < 0$ in a neighborhood of $(1, 0)$ so the solution $u = \psi^2 q$ is not positive.

5 Positivity for quadratic polynomials on the ellipse

Throughout this section $\psi(x, y) = x^2 + by^2 - 1$ and $E_b = \{x \in \mathbb{R}^2 : \psi(x, y) < 0\}$. We start this section with a result that gives a necessary condition for a quadratic polynomial to be positive on an ellipse.

Proposition 5.1 *If the polynomial $f(x, y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2$ is positive on the ellipsoid E_b then*

$$a_0 \geq 0\ ,\ a_0 + a_3 \geq |a_1| \text{ and } ba_0 + a_5 \geq |a_2|\,.$$

Proof. Clearly $f(0, 0) = a_0 \geq 0$. We know that $H(x, y) := a_0 + a_1 x + a_3 x^2 + a_5 y^2 \geq y(-a_2 - a_4 x)$ on E_b. Since $H(x, y) = H(x, -y)$ we conclude that $H(x, y) \geq |y(a_2 + a_4 x)| \geq 0$. Since $(\pm 1, 0) \in E_b$ and $H(\pm 1, 0) \geq 0$ we infer $a_0 + a_3 \geq -a_1 \cdot (\pm 1)$ which implies that $a_0 + a_3 \geq |a_1|$. Similiarly $a_0 + a_2 y + a_3 x^2 + a_5 y^2 \geq |a_1 x + a_4 xy|$ and since $\left(0, \pm 1/\sqrt{b}\right) \in E_b$ we obtain $a_0 + a_5 \frac{1}{b} \geq |a_2| / \sqrt{b}$. □

Now we state our main result:

Theorem 5.2 *Let $b_0 \approx 18.94281916$ be the unique positive solution of the equation*

$$-11b^6 + 178b^5 + 531b^4 + 796b^3 + 811b^2 + 210b + 45 = 0. \tag{15}$$

Let $f(x,y) = a_0 + a_1 x + a_3 x^2$ be a positive polynomial on the ellipse E_b. If one of the following conditions hold

(i) $a_3 \leq 0$, (ii) $|a_1|^2 > 4a_0 a_3$, (iii) $0 \leq b \leq b_0$,

then the unique solution u of (1)-(2) is positive on E_b. Furthermore, for each $b > b_0$ there exists a_b in $\left[\frac{1}{5}, 1\right)$ such that if $f = (x - a_b)^2$ then the unique solution u of (1)-(2) is not positive on E_b.

Proof. 1. We know that $S(f) = A_0 + A_1 x + A_3 x^2 + A_5 y^2$, cf. (5) and using $a_4 = 0$. In the first case assume that $a_3 \leq 0$. From (8) and (9) and the fact that $a_5 = 0$ we infer that $A_3 \leq 0$ and $A_5 \geq 0$. Then $A_5 y^2 \geq 0$ for all y and it suffices to show that $G_b(x) := A_0 + A_1 x + A_3 x^2 \geq 0$ for all $x \in [-1,1]$. The polynomial G_b is concave since $A_3 \leq 0$ and it suffices to show that $G_b(\pm 1) \geq A_0 - |A_1| + A_3 \geq 0$.

So let us consider the expression $A_0 - |A_1| + A_3$ (without the assumption $a_3 \leq 0$). Since f is positive, Proposition 5.1 implies that $a_0 \pm a_1 + a_3 \geq 0$. Further

$$A_1 = c_1 a_1 \text{ with } c_1 = \frac{3b^2 + 2b + 3}{5 + 2b + b^2}\left(5b^4 + 20b^3 + 78b^2 + 20b + 5\right). \tag{16}$$

Hence $A_0 - |A_1| + A_3 = A_0 + A_3 - c_1 a_0 - c_1 a_3 + c_1 (a_0 - |a_1| + a_3)$. It suffices to show that

$$M := \left(5 + 2b + b^2\right)(A_0 - c_1 a_0 + A_3 - c_1 a_3) \geq 0.$$

Since $a_5 = 0$ we obtain

$$\begin{aligned} A_0 &= 3\left(5b^4 + 20b^3 + 78b^2 + 20b + 5\right) a_0 + \frac{2}{3}\left(9b^3 + 41b^2 + 11b + 3\right) a_3 \\ A_3 &= \frac{3b^2 + 2b + 3}{3}\left(1 + 15b^2 + 4b\right) a_3. \end{aligned}$$

A calculation show that

$$M = 4(b+3)\left(5b^4 + 20b^3 + 78b^2 + 20b + 5\right) a_0 - \left(72b^3 + 32b^2 + 4b + 128b^4 + 20b^5\right) a_3$$

Since $a_0 \geq 0$ and $a_3 \leq 0$ (by the first case) we infer that $M \geq 0$ and the positivity of $S(f)$ is established for the case (i).

2. We shall need later the following statement: if $a_3 > 0$ and $a_0 \geq a_3$ then $M \geq 0$. Indeed, the inequality $a_0 \geq a_3$ is used to estimate M and a direct calculation shows that the remaining sum is positive.

3. Now assume that $a_3 > 0$. Then $A_3 > 0$ and $A_5 < 0$ (if $b \neq 0$) and $S(f)(x,y) = A_0 + A_1 x + A_3 x^2 + A_5 y^2$ does not have a local minimum. Thus $S(f)$ takes its minimum on its boundary. It suffices to prove that

$$G_b(x) := A_0 + \frac{1}{b} A_5 + A_1 x + \left(A_3 - \frac{1}{b} A_5\right) x^2 \geq 0$$

for all $x \in [-1,1]$. From (8) and (9) it is easy to see that

$$A_3 - \frac{1}{b} A_5 = \left(3b^2 + 2b + 3\right)\left(5b^2 + 2b + 1\right) a_3 > 0. \tag{17}$$

Hence, the leading coefficient of $G_b(x)$ is positive. Further we compute

$$A_0 + \frac{1}{b} A_5 = 3\left(5b^4 + 20b^3 + 78b^2 + 20b + 5\right) a_0 + 4b\left(b^2 + 6b + 1\right) a_3. \tag{18}$$

In order to prove that G_b is positive on $[-1,1]$, we first consider the case that G_b has a critical point $x_0 \in [-1,1]$, i.e. that $x_0 := -A_1/2\left(A_3 - \frac{1}{b} A_5\right) \in [-1,1]$. Then

$$|A_1| \leq 2\left(A_3 - \frac{1}{b} A_5\right) = 2\left(5b^2 + 2b + 1\right)\left(3b^2 + 2b + 3\right) a_3.$$

Using (16) this is equivalent to

$$|a_1| \le \frac{2\left(5b^2+2b+1\right)\left(5+2b+b^2\right)}{\left(5b^4+20b^3+78b^2+20b+5\right)} a_3. \tag{19}$$

On the other hand, a straightforward calculation yields

$$\frac{1}{2} \le \frac{\left(5b^2+2b+1\right)\left(5+2b+b^2\right)}{\left(5b^4+20b^3+78b^2+20b+5\right)} \le 1,$$

which combined with (19) leads us to $|a_1| \le 2a_3$. Hence, the assumption that G_b has a critical point in $[-1,1]$ implies that the critical point of f is in the interval $[-1,1]$ and it follows that $a_1^2 \le 4a_0a_3$.

4. Now assume case (ii), so $a_3 > 0$ and $a_1^2 > 4a_0a_3$. Then the critical point of f, namely $x_0 = -a_1/2a_3$, can not be in the interval $[-1,1]$ since f is positive on $[-1,1]$ and

$$f(x_0) = a_0 - \frac{1}{4}\frac{a_1^2}{a_3} = \frac{1}{4a_3}\left(4a_0a_3 - a_1^2\right).$$

Thus $|a_1| \ge 2a_3$. Now $a_0 + a_3 \ge |a_1| \ge 2a_3$ implies $a_0 \ge a_3$. Since $a_1^2 > 4a_0a_3$ we know that $G_b(x)$ does not possess a critical point and G_b takes its extremum at the point 1 or -1. Thus it suffices to show that $G_b(\pm 1) = A_0 - |A_1| + A_3$, i.e. $M \ge 0$. This follows now from part 2 of the proof.

5. Now assume case (iii), so $0 \le b \le b_0$, $a_3 > 0$ and $a_1^2 \le 4a_0a_3$. Since

$$f(x,y) = a_0 + a_1x + a_3x^2 = \left(\sqrt{a_3}x - \frac{a_1}{2\sqrt{a_3}}\right)^2 + \frac{4a_0a_3 - a_1^2}{4a_3}$$

is a sum of a positive polynomial with discriminant 0 and a positive constant it suffices to show the result for polynomials with $4a_0a_3 = a_1^2$.

6. We shall show that $G_b(\pm 1) = A_0 - |A_1| + A_0$ is positive for $b \le b_0$. A straightforward computation shows that $A_0 + A_3 = \gamma_0 a_0 + \gamma_3 a_3$ where

$$\gamma_0 = 3\left(5b^4 + 20b^3 + 78b^2 + 20b + 5\right) \text{ and } \gamma_3 = 46b^2 + 15b^4 + 20b^3 + 12b + 3.$$

Using the notation $A_1 = c_1a_1$ from (16) and $4a_0a_3 = a_1^2$ we can write

$$G_b(\pm 1) = \gamma_0 a_0 - c_1|a_1| + \gamma_3 a_3 = \gamma_0 a_0 - c_1 2\sqrt{a_0a_3} + \gamma_3 a_3 = a_3\left(\gamma_0 z^2 - 2c_1 z + \gamma_3\right)$$

where $z := \sqrt{a_0/a_3}$. It is clear that $G_b(\pm 1) \ge 0$ for all choices of z if and only if $4\gamma_0\gamma_3 \ge 4c_1^2$. A further calculation shows that

$$\frac{\left(5+2b+b^2\right)^2\left(\gamma_0\gamma_3 - c_1^2\right)}{4\left(5b^4+20b^3+78b^2+20b+5\right)} = -11b^6 + 178b^5 + 531b^4 + 796b^3 + 811b^2 + 210b + 45.$$

Note that the right-hand side in the above equality is exactly the polynomial in (15) whose unique positive rooth is b_0. Thus for $b \le b_0$ we have proved that $G_b(\pm 1) \ge 0$.

If $b > b_0$ we consider the polynomial

$$f(x,y) = a_0 + a_1x + a_3x^2 := \frac{c_1^2}{\gamma_0} - 2\frac{c_1}{\gamma_0}x + x^2 = \left(x - \frac{c_1}{\gamma_0}\right)^2.$$

Then $G_b(\pm 1) < 0$ since with $z = \sqrt{a_0/a_3} = c_1/\gamma_0$ we obtain that

$$G_b(\pm 1) = a_3\left(\gamma_0 z^2 - 2c_1 z + \gamma_3\right) = a_3\left(-\frac{c_1^2}{\gamma_0} + \gamma_3\right) < 0.$$

Further we see that

$$\frac{c_1}{\gamma_0} = \frac{1}{3}\frac{3b^2+2b+3}{5+2b+b^2} \in \left[\frac{1}{5}, 1\right).$$

So for $b = 19$ we obtain the polynomial $f(x,y) = \left(x - \frac{281}{303}\right)^2$ which was presented in the introduction.

7. We still have to prove that $G_b(x) \geq 0$ for all $x \in [-1,1]$ in the case $a_3 > 0$, $a_1^2 = 4a_0a_3$ and $0 \leq b \leq b_0$.

If G_b has no critical points in $[-1,1]$, then G_b attains its minimum at ± 1 and since $G_b(\pm 1) \geq 0$ for all $b \leq b_0$ it follows that $G_b \geq 0$ on $[-1,1]$ for all $b \leq b_0$.

Assume next that G_b has a critical point in the interval $[-1,1]$ which is actually the local and global minimum of G_b. It is easy to see that $G_b \geq 0$ in $[-1,1]$ holds if

$$4\left(A_0 + \frac{1}{b}A_5\right)\left(A_3 - \frac{1}{b}A_5\right) - A_1^2 \geq 0.$$

From (17) and (18) we infer that

$$D := 4\left(A_0 + \frac{1}{b}A_5\right)\left(A_3 - \frac{1}{b}A_5\right) = \left(3b^2+2b+3\right)\left(5b^2+2b+1\right)\cdot D_0$$

where $D_0 := 3\left(5b^4+20b^3+78b^2+20b+5\right)4a_3a_0 + 4b\left(b^2+6b+1\right)4a_3^2$. Thus, $D - A_1^2$ is positive if and only if

$$D_1 := \frac{\left(5+2b+b^2\right)^2\left(D - A_1^2\right)}{\left(5b^4+20b^3+78b^2+20b+5\right)} \geq 0.$$

Since $4a_3a_0 = a_1^2$ and $A_1 = c_1a_1$ (cf. (16)), we arrive at

$$\begin{aligned} D_1 &= \left(5+2b+b^2\right)^2\left(3b^2+2b+3\right)\left(5b^2+2b+1\right)3a_1^2 \\ &\quad + \frac{\left(5+2b+b^2\right)^2 4b\left(b^2+6b+1\right)}{\left(5b^4+20b^3+78b^2+20b+5\right)}\left(3b^2+2b+3\right)\left(5b^2+2b+1\right)4a_3^2 \\ &\quad - \left(3b^2+2b+3\right)^2\left(5b^4+20b^3+78b^2+20b+5\right)a_1^2, \end{aligned}$$

which is equal to

$$\begin{aligned} D_1 &= 4(b+1)\left(3b^2+2b+3\right)\left(-b^4-12b^3+42b^2+20b+15\right)a_1^2 \\ &\quad + \frac{\left(5+2b+b^2\right)^2 4b\left(b^2+6b+1\right)}{\left(5b^4+20b^3+78b^2+20b+5\right)}\left(3b^2+2b+3\right)\left(5b^2+2b+1\right)4a_3^2. \end{aligned}$$

Using (19) we obtain

$$\begin{aligned} D_1 &\geq 4(b+1)\left(3b^2+2b+3\right)\left(-b^4-12b^3+42b^2+20b+15\right)a_1^2 \\ &\quad + \frac{4b\left(b^2+6b+1\right)}{\left(5b^2+2b+1\right)}\left(3b^2+2b+3\right)\left(5b^4+20b^3+78b^2+20b+5\right)a_1^2. \end{aligned}$$

We see that $D_1 \geq 0$ for $b \leq 21.601$, since the right-hand side of the above inequality multiplied by $(5b^2+2b+1)/a_1^2$ equals

$$180 + 960b + 17\,216b^5 + 3824b^2 + 9280b^3 + 17720b^4 + 3776b^7 + 12\,784b^6 - 204b^8$$

which is positive for all $b \leq 21.601$. The proof is accomplished. □

It seems to be difficult to characterize those parameters b for which all positive quadratic polynomials f have positive solutions u. In Theorem 5.2 we investigated polynomials f which depend only on the variable x. Next we prove positivity for data function $f(x,y)$ which have no linear terms. In that case the following is true:

Theorem 5.3 *Suppose that $f(x,y) = a_0 + a_3x^2 + a_4xy + a_5y^2$ is positive on the ellipse E_b. Then, the unique solution u of (1)-(2) satisfies*

$$u \geq \frac{24}{5+6b+5b^2} \cdot \psi^2 f \geq 0 \quad \text{in } E_b.$$

P r o o f. Using the definition of the Fischer operator and the fact that $S = 24C_bF_\psi^{-1}$, it suffices to show that $(5+6b+5b^2) \cdot S(f) \geq C_b f$ in E_b. For this purpose, let

$$D(x,y) := \left(5+6b+5b^2\right) \cdot S(f) - C_b f.$$

We have to show that $D(x,y)$ is positive on E_b. By a straightforward computation, $D(x,y) = B_0 + B_3x^2 + B_5y^2$, where

$$\begin{aligned}
B_0 &= \left(5b^4+20b^3+78b^2+20b+5\right)\left(12+16b+12b^2\right)a_0 \\
&\quad + \frac{2}{3}\left(5+6b+5b^2\right)\left(9b^3+41b^2+11b+3\right)a_3 \\
&\quad + \frac{2}{3}\left(5+6b+5b^2\right)\left(3b^3+11b^2+41b+9\right)a_5, \\
\frac{3B_3}{3b^2+2b+3} &= \left(-10-130b^2-34b+50b^3+60b^4\right)a_3 - 2(b+1)\left(5+6b+5b^2\right)a_5, \\
\frac{3B_5}{3b^2+2b+3} &= -2b(b+1)\left(5+6b+5b^2\right)a_3 + \left(50b+60-130b^2-34b^3-10b^4\right)a_5.
\end{aligned}$$

We write $D(x,y)$ in the trivial way:

$$D(x,y) = B_0\psi(x,y) + (B_0+B_3)x^2 + (bB_0+B_5)y^2. \tag{20}$$

If we know that $B_0 + B_3 \geq 0$ and $bB_0 + B_5 \geq 0$ then $D(x,y)$ is a sum of positive polynomials on the ellipse and the positivity of $D(x,y)$ is evident. The positivity of $B_0 + B_3$ follows from the identity

$$\begin{aligned}
B_0 + B_3 &= 4\left(5b^2+6b+5\right)\left(9b^3+41b^2+11b+3\right)a_0 \\
&\quad + 4b\left(34b^3+30b^4+40b^2+7b+2+15b^5\right)(a_0+a_3) \\
&\quad + 4\left(5b^2+6b+5\right)\left(b^2+6b+1\right)(a_5+ba_0),
\end{aligned}$$

and the positivity of the coefficients $a_0 + a_3$ and $ba_0 + a_5$, cf. Proposition 5.1. Similarly, $bB_0 + B_5$ is positive since it is equal to

$$\begin{aligned}
bB_0 + B_5 &= 4b^2\left(5+6b+5b^2\right)\left(3b^3+11b^2+41b+9\right)a_0 \\
&\quad + 4b^2\left(5+6b+5b^2\right)\left(b^2+6b+1\right)(a_0+a_3) \\
&\quad + \left(8b^5+28b^4+160b^3+136b^2+120b+60\right)(a_5+ba_0) \geq 0.
\end{aligned}$$

The proof is now complete. □

References

[1] T. Boggio, *Sulle funzioni di Green d'ordine m*, Rend. Circ. Mat. Palermo **20** (1905), 97-135.

[2] P. G. Garabedian, *A partial differential equation arising in conformal mapping,* Pacific J. Math. **1** (1951), 485-524.

[3] F. Gazzola, H.-C. Grunau, G. Sweers, *Polyharmonic boundary value problems*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2010.

[4] H.-C. Grunau, *Positivity, change of sign and buckling eigenvalues in a one-dimensional fourth order problem,* Adv. Differ. Equations **7** (2002), 177-196.

[5] H.-C. Grunau, F. Robert, *Positivity issues of biharmonic Green's functions under Dirichlet boundary conditions,* Arch. Ration. Mech. Anal. **195** (2010), 865-898.

[6] H.-C. Grunau, G. Sweers, *The maximum principle and positive eigenfunctions for polyharmonic equations,* Lecture Notes in Pure and Appl. Math., 194, Dekker, New York, 1998.
[7] H.-C. Grunau, G. Sweers, *The role of positive boundary data in generalized clamped plate equations,* Z. Angew. Math. Phys. **49** (1998), no. 3, 420-435.
[8] H.-C. Grunau, G. Sweers, *Positivity for perturbations of polyharmonic operators with Dirichlet boundary conditions in two dimensions,* Math. Nachr. **179** (1996), 89–102.
[9] S.A. Lurie, V.V. Vasiliev, *The biharmonic problem in the theory of elasticity*, Gordon Breach, Luxembourg 1995.
[10] V.V. Meleshko, *Bending of an Elastic Rectangular Clamped Plate: Exact Versus 'Engineering' Solutions,* J. Elasticity **48** (1997), 1–50.
[11] H. Render, *Real Bargmann spaces, Fischer decompositions and sets of uniqueness for polyharmonic functions,* Duke Math. J. **142** (2008), 313-352.
[12] H. Render, *Cauchy-Goursat and Dirichlet problems for holomorphic PDE's*, Comput. Methods Function Theory 10 (2010) 519-554.
[13] H.S. Shapiro, *An algebraic theorem of E. Fischer and the Holomorphic Goursat Problem,* Bull. London Math. Soc. **21** (1989), 513–537.
[14] H.S. Shapiro, M. Tegmark, *An elementary proof that the biharmonic Green function of an eccentric ellipse changes sign,* Siam Review **36** (1994), 99–101.