

Boundary behaviour of functions which possess universal Taylor series

Stephen J. Gardiner

Abstract

It is known that, for any simply connected proper subdomain Ω of the complex plane and any point ζ in Ω , there are holomorphic functions on Ω that possess “universal” Taylor series expansions about ζ ; that is, partial sums of the Taylor series approximate arbitrary polynomials on arbitrary compacta in $\mathbb{C}\setminus\Omega$ that have connected complement. This paper establishes a strong unboundedness property for such functions near every boundary point. The result is new even in the case of the disc, where it strengthens work of several authors.

1 Introduction

Throughout this paper Ω denotes a simply connected proper subdomain of the complex plane \mathbb{C} and ζ is a point in Ω . A holomorphic function f on Ω is said to belong to the collection $\mathcal{U}(\Omega, \zeta)$, of functions with universal Taylor series expansions about ζ , if the partial sums

$$S_N(f, \zeta) = \sum_{n=0}^N \frac{f^{(n)}(\zeta)}{n!} (z - \zeta)^n$$

of the power series expansion of f about ζ have the following property:

for every compact set $K \subset \mathbb{C}\setminus\Omega$ with connected complement and every function g which is continuous on K and holomorphic on K° , there is a subsequence $(S_{N_k}(f, \zeta))$ that converges to g uniformly on K .

Nestoridis [16], [17] has shown that possession of such universal Taylor series expansions is a generic property of holomorphic functions on simply connected domains; that is, $\mathcal{U}(\Omega, \zeta)$ is a dense G_δ subset of the space of all holomorphic functions on Ω endowed with the topology of local uniform

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convergence. Further, Müller, Vlachou and Yavrian [14] (see also Theorem 9.1 in [12]) have shown that the collection $\mathcal{U}(\Omega, \zeta)$ is independent of the choice of the centre of expansion ζ .

A significant avenue of investigation concerns the boundary behaviour of functions in $\mathcal{U}(\Omega, \zeta)$: see [17], [13], [6], [10], [12], [5], [14], [1], [3], [4]. For example, in the case where Ω is the unit disc \mathbb{D} , Nestoridis [17] showed that $\mathcal{U}(\mathbb{D}, 0)$ does not intersect the Hardy space $H^1(\mathbb{D})$. This was subsequently strengthened by Melas, Nestoridis and Papadoperakis [13], as follows.

Theorem A *The collection $\mathcal{U}(\mathbb{D}, 0)$ does not intersect the Nevanlinna class; that is, for every $f \in \mathcal{U}(\mathbb{D}, 0)$, we have*

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \rightarrow \infty \quad (r \rightarrow 1-). \quad (1)$$

Bayart [3] considered the local boundary behaviour of functions in $\mathcal{U}(\mathbb{D}, 0)$ and answered a question raised by Armitage and Costakis [1] by establishing the next result. We recall that a subset Z of the unit circle $\partial\mathbb{D}$ is called *residual* if $\partial\mathbb{D} \setminus Z$ is of first Baire category relative to $\partial\mathbb{D}$.

Theorem B *If $f \in \mathcal{U}(\mathbb{D}, 0)$, then there is a residual subset Z of $\partial\mathbb{D}$ such that $\{f(rw) : 0 < r < 1\}$ is unbounded for every $w \in Z$.*

A complementary result of Armitage and Costakis [1] tells us that, for any set $Y \subset \partial\mathbb{D}$ which is of first category relative to $\partial\mathbb{D}$, there are functions f in $\mathcal{U}(\mathbb{D}, 0)$ such that $\{f(rw) : 0 < r < 1\}$ is bounded for every $w \in Y$.

There is no implication in either direction between the conclusions of Theorems A and B.

Much less is known about boundary behaviour in the case of general simply connected domains Ω . Müller, Vlachou and Yavrian [14] (see also [12]) have shown that no member of $\mathcal{U}(\Omega, \zeta)$ is holomorphically extendable beyond Ω . However, it has remained an open question whether such functions are necessarily unbounded. (I am grateful to Vassili Nestoridis for drawing this problem to my attention.) The purpose of this paper is to show that functions with universal Taylor series expansions have a strong unboundedness property near every boundary point. We denote by $D(w, r)$ the open disc of centre w and radius r .

Theorem 1 *Let $f \in \mathcal{U}(\Omega, \zeta)$. Then, for any $w \in \partial\Omega$, any $r > 0$ and any component U of $D(w, r) \cap \Omega$, the function $\log^+ |f|$ does not have a harmonic majorant on U , and thus $\mathbb{C} \setminus f(U)$ is polar.*

Corollary 2 *If $f \in \mathcal{U}(\Omega, \zeta)$, then f is unbounded near every point of $\partial\Omega$.*

Theorem 1 is new even in the case of the disc. It contains Theorem A because condition (1) is equivalent to saying that the subharmonic function $\log^+ |f|$ does not have a harmonic majorant on the whole of \mathbb{D} . The special case of the corollary where $\Omega = \mathbb{D}$ is equivalent to Theorem B because, by the Collingwood Maximality Theorem (Proposition 2.1 in [18]), the radial cluster set of f agrees with the full cluster set at a residual subset of $\partial\mathbb{D}$. Theorem 1 fails if we drop our overall assumption that Ω is simply connected, as is clear from Corollary 1 in Melas [11].

The proof of Theorem 1 presents challenges because no special assumptions are made concerning the boundary of Ω . In particular, since we have not assumed that $\partial\Omega$ is locally connected, we cannot rely on having a conformal mapping $g : \mathbb{D} \rightarrow \Omega$ with continuous extension to $\overline{\mathbb{D}}$. We will make extensive use of potential theoretic tools to overcome such difficulties, and refer to the book [2] for the background theory.

2 A preparatory lemma

Let G_U denote the Green function of a non-empty open set $U \subset \mathbb{C} \cup \{\infty\}$ when it exists, that is, when the complement of U is non-polar. Under the same assumption we denote by H_ϕ^U the (Perron-Wiener-Brelot) solution to the Dirichlet problem on U with resolutive boundary function ϕ . We also write χ_A for the characteristic function valued 1 on a set A and 0 elsewhere.

Lemma 3 *Let U be a simply connected proper subdomain of \mathbb{C} and $\zeta_0 \in U$, and let ω be a non-empty open subset of U . Suppose that (v_k) is a decreasing sequence of harmonic functions on U such that $v_1/G_U(\zeta_0, \cdot)$ is bounded above on ω and $\lim v_k < 0$ on U . If $H_{\chi_U}^\omega \not\equiv 1$, then there exists $k' \in \mathbb{N}$ such that the open set $\omega_1 = \omega \cap \{v_{k'} < 0\}$ is non-empty and $H_{\chi_U}^{\omega_1} \not\equiv 1$.*

If $A \subset U$ and u is a positive superharmonic function on U , we define the reduced function

$${}^U R_u^A = \inf\{w : w \text{ is positive and superharmonic on } U \text{ and } w \geq u \text{ on } A\}.$$

In the context of the above lemma, since ${}^U R_1^{U \setminus \omega} = H_{\chi_U}^\omega$ on ω and ${}^U R_1^{U \setminus \omega} = 1$ on $U \setminus \omega$, the hypothesis that $H_{\chi_U}^\omega \not\equiv 1$ is equivalent to saying that ${}^U R_1^{U \setminus \omega} \not\equiv 1$. Now let \mathbb{H} denote the upper halfplane and $g : \mathbb{H} \rightarrow U$ be a conformal mapping such that $g(i) = \zeta_0$. Since superharmonicity is preserved by conformal mappings,

$${}^U R_1^{U \setminus \omega} \circ g = {}^{\mathbb{H}} R_1^{\mathbb{H} \setminus g^{-1}(\omega)}.$$

Thus ${}^U R_1^{U \setminus \omega} \not\equiv 1$ if and only if ${}^{\mathbb{H}} R_1^{\mathbb{H} \setminus g^{-1}(\omega)} \not\equiv 1$. Also, $G_U(\zeta_0, g(z)) = G_{\mathbb{H}}(i, z)$. It is therefore enough to prove the lemma in the case where $U = \mathbb{H}$ and

$\zeta_0 = i$. Starting with the assumption that ${}^{\mathbb{H}}R_1^{\mathbb{H}\setminus\omega} \neq 1$, we will show that ${}^{\mathbb{H}}R_1^{\mathbb{H}\setminus\omega_1} \neq 1$ for a suitable choice of k' .

Let $Z = \mathbb{R} \times \{0\}^3$. For any function $\psi : \mathbb{H} \rightarrow [-\infty, +\infty]$ we define

$$\psi^*(x_1, \dots, x_4) = \frac{\psi(x_1, y(x_2, x_3, x_4))}{y(x_2, x_3, x_4)} \quad ((x_1, \dots, x_4) \in \mathbb{R}^4 \setminus Z),$$

where

$$y(x_2, x_3, x_4) = \sqrt{x_2^2 + x_3^2 + x_4^2}.$$

If $\psi \in C^2(\mathbb{H})$, then a straightforward calculation shows that

$$(\Delta\psi^*)(x_1, \dots, x_4) = \frac{(\Delta\psi)(x_1, y(x_2, x_3, x_4))}{y(x_2, x_3, x_4)}. \quad (2)$$

Thus the transformation $\psi \mapsto \psi^*$ preserves harmonicity, and even (by a smoothing argument) superharmonicity. In fact, it establishes a one-to-one correspondence between superharmonic functions on \mathbb{H} and Z -axially symmetric superharmonic functions Ψ on $\mathbb{R}^4 \setminus Z$, the inverse transformation being given by $\Psi \mapsto \Psi^\dagger$, where

$$\Psi^\dagger(x_1, x_2) = x_2\Psi(x_1, x_2, 0, 0)$$

(cf. [9]).

The function u defined by

$$u(x_1, \dots, x_4) = \frac{1}{y(x_2, x_3, x_4)}$$

is superharmonic on \mathbb{R}^4 and harmonic on $\mathbb{R}^4 \setminus Z$. Further, by the above transformation, our hypothesis that ${}^{\mathbb{H}}R_1^{\mathbb{H}\setminus\omega} \neq 1$ implies that

$$\mathbb{R}^4 \setminus Z R_u^{\mathbb{R}^4 \setminus (W \cup Z)} \neq u,$$

where

$$W = \{(x_1, \dots, x_4) : (x_1, y(x_2, x_3, x_4)) \in \omega\}.$$

Indeed, since the polar set Z is removable for positive superharmonic functions on $\mathbb{R}^4 \setminus Z$, we have

$$\mathbb{R}^4 R_u^{\mathbb{R}^4 \setminus W} \neq u. \quad (3)$$

The function u can be expressed as a Newtonian potential,

$$u = \frac{1}{\pi} \int_{\mathbb{R}} u_t dt, \quad (4)$$

where

$$u_t(x_1, \dots, x_4) = \frac{1}{(x_1 - t)^2 + x_2^2 + x_3^2 + x_4^2} \quad (t \in \mathbb{R}).$$

Hence

$$\mathbb{R}^4 R_u^{\mathbb{R}^4 \setminus W} = \frac{1}{\pi} \int_{\mathbb{R}} \mathbb{R}^4 R_{u_t}^{\mathbb{R}^4 \setminus W} dt. \quad (5)$$

(The interchange of the integration and the reduction is justified by Tonelli's theorem because, as we have previously noted, in the set W the reduction can be expressed as a Dirichlet solution and so can be viewed as an integral against harmonic measure for W .)

Let λ denote Lebesgue measure on \mathbb{R} and let A be the Borel set defined by

$$A = \left\{ t \in \mathbb{R} : \mathbb{R}^4 R_{u_t}^{\mathbb{R}^4 \setminus W} \not\equiv u_t \right\}.$$

Then (3), (4) and (5) together show that $\lambda(A) > 0$. Further, $A \times \{0\}^3$ is precisely the set of points in Z at which $\mathbb{R}^4 \setminus W$ is thin (see Theorem 7.3.2 in [2]). We now recall that the *fine topology* of classical potential theory is the coarsest topology on \mathbb{R}^n that makes all superharmonic functions continuous (see Chapter 7 of [2]). Further, a set $V \subset \mathbb{R}^n$ is a fine neighbourhood of a point $x \in V$ if and only if $\mathbb{R}^n \setminus V$ is thin at x . In our context the set $W \cup A$ is thus a finely open set in \mathbb{R}^4 .

Now let (v_k) be as in the statement of the lemma. The transformed function v_k^* is certainly harmonic on $\mathbb{R}^4 \setminus Z$. Further, since $v_1/G_{\mathbb{H}}(i, \cdot)$ is bounded above on ω , and the function

$$x + iy \mapsto G_{\mathbb{H}}(i, x + iy)/y$$

has a continuous extension to $\overline{\mathbb{H}} \cup \{\infty\}$, we see that each v_k^* is bounded above on W . If $W \cup A$ were a Euclidean neighbourhood of points in A , we would be able to conclude that v_k^* has a subharmonic extension to $W \cup A$ by a standard removable singularity theorem for polar sets (Corollary 5.2.2 in [2]). However, we know only that $W \cup A$ is a fine neighbourhood of such points, so we instead appeal to the theory of fine subharmonicity on finely open sets (see Fuglede [7]). This tells us that v_k^* has a finely subharmonic extension to the finely open set $W \cup A$ (see Theorem 9.14 in [7]). It follows that $\lim v_k^*$ is either finely subharmonic or identically valued $-\infty$ on $W \cup A$, according to whether $\lim v_k$ is harmonic or identically $-\infty$ on \mathbb{H} . In either case, since $\lim v_k^* < 0$ on W , we must have $\lim v_k^* < 0$ on A also.

For each $k \in \mathbb{N}$ let A_k be the Borel set defined by

$$\begin{aligned} A_k &= \{t \in A : v_k^*(t, 0, 0, 0) < 0\} \\ &= \{t \in A : \{x \in W : v_k^*(x) \geq 0\} \text{ is thin at } (t, 0, 0, 0)\}. \end{aligned}$$

Since $A_k \uparrow A$ and $\lambda(A) > 0$, we can choose $k' \in \mathbb{N}$ such that $\lambda(A_{k'}) > 0$. Recalling that $\mathbb{R}^4 \setminus W$ is thin at each point of $A \times \{0\}^3$, we now see that $\mathbb{R}^4 \setminus W_1$ is thin at each point of $A_{k'} \times \{0\}^3$, where $W_1 = W \cap \{v_{k'}^* < 0\}$. In view of (5) the function $\mathbb{R}^4 R_u^{\mathbb{R}^4 \setminus W_1}$, and hence also its lower semicontinuous

regularization v , differs from u . Further, the set W_1 is obviously invariant under rotation about the axis Z , so v is Z -axially symmetric. The associated function v^\dagger is thus superharmonic on \mathbb{H} and majorizes 1 on $\mathbb{H} \setminus \omega_1$, where $\omega_1 = \omega \cap \{v_{k'} < 0\}$, apart possibly from a polar set. (Clearly Z -axially symmetric polar sets in $\mathbb{R}^4 \setminus Z$ correspond to polar sets in \mathbb{H} .) Since v^\dagger takes values less than 1 somewhere in the set ω_1 , it follows that $\mathbb{H}R_1^{\mathbb{H} \setminus \omega_1} \neq 1$, as required.

3 Proof of Theorem 1

Before giving the formal proof of Theorem 1 we briefly outline our strategy in the special case where $f \in \mathcal{U}(\mathbb{D}, 0)$ and $w = 1$, say. Suppose that $\log^+ |f|$ has a harmonic majorant h on the set $U = D(1, r) \cap \mathbb{D}$, where $r < 1$, and let $K = \overline{D}(1, r) \setminus \mathbb{D}$ and $\Omega_1 = (\mathbb{C} \cup \{\infty\}) \setminus K$. By universality we can find an increasing sequence (N_k) of natural numbers such that $|S_{N_k}| < k^{-1}$ on K , where $S_{N_k} = S_{N_k}(f, 0)$. It follows from Bernstein's lemma (Theorem 5.5.7 in [19]) and standard estimates for the Green function that

$$N_k^{-1} \log |S_{N_k}| \leq G_{\Omega_1}(\infty, \cdot) \leq cG_U(0, \cdot) \quad \text{on } \omega,$$

where $\omega = D(1, r/2) \cap \mathbb{D}$ and $c > 1$ is a suitable constant. It is also easy to see that $\limsup_{k \rightarrow \infty} u_k < 0$ on \mathbb{D} , where $u_k = N_k^{-1} \log |S_{N_k} - f|$. We then construct a sequence of functions (v_k) as in Lemma 3 such that $N_k u_k - h - N_k v_k$ is bounded above on U , and deduce that the subharmonic functions $\log |S_{N_k}| - h$ ($k \in \mathbb{N}$) have a uniform upper bound near 1. This, in turn, leads to a contradiction. The proof in the case of general simply connected domains Ω involves quite delicate potential theoretic arguments, as we will now see.

Let $f \in \mathcal{U}(\Omega, \zeta)$. We suppose, for the sake of contradiction, that there exist $w_0 \in \partial\Omega$, $r_0 > 0$, a component U of $D(w_0, r_0) \cap \Omega$, and a positive harmonic function h on U such that

$$\log^+ |f| \leq h \quad \text{on } U. \tag{6}$$

Clearly U is a simply connected domain. Without loss of generality we may assume that $\zeta \in \Omega \setminus \overline{D}(w_0, r_0)$ and that h is positive and harmonic on an open set containing $\overline{U} \cap \Omega$.

We define a domain

$$\Omega_1 = \Omega \cup (\mathbb{C} \setminus \overline{D}(w_0, r_0)) \cup \{\infty\}$$

in the extended complex plane, and a compact set

$$K = \mathbb{C} \setminus \Omega_1 = \overline{D}(w_0, r_0) \setminus \Omega,$$

whence $K \subset \mathbb{C} \setminus \Omega$ and $\mathbb{C} \setminus K$ is connected. (See Figure 1.) If Ω is unbounded,

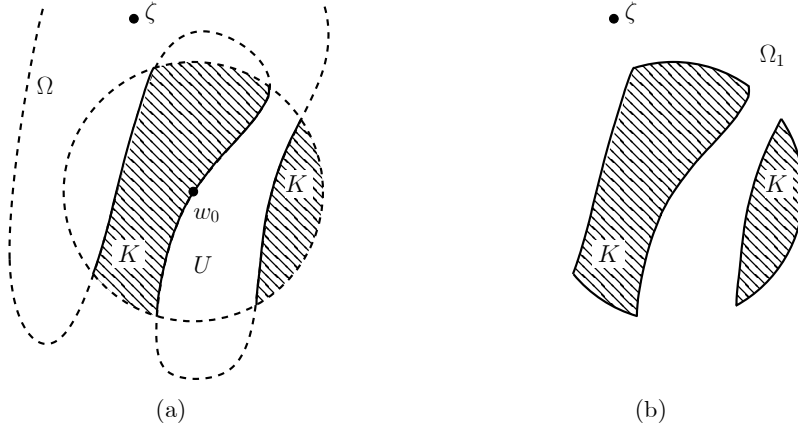


Figure 1.

we define $F = \mathbb{C} \setminus \Omega$. Otherwise, we choose F to be a ray going to infinity in $\mathbb{C} \setminus (\overline{\Omega} \cup K)$.

Since $f \in \mathcal{U}(\Omega, \zeta)$, we can choose an increasing sequence (N_k) of natural numbers such that

$$|S_{N_k}| < k^{-1} \quad \text{on } (\overline{D}(w_0, r_0 + k) \cap F) \cup K, \quad (7)$$

where $S_{N_k} = S_{N_k}(f, \zeta)$. We define the subharmonic functions

$$u_k(z) = \frac{1}{N_k} \log |S_{N_k}(z) - f(z)| \quad (z \in \Omega; k \in \mathbb{N}). \quad (8)$$

Since F is non-thin at infinity we can apply a result of Müller and Yavrian [15] to see that $\limsup_{k \rightarrow \infty} u_k < 0$ on Ω . Further (cf. [8], page 250), since $u_k = \log |\zeta - \cdot| + g_k$ for some subharmonic function g_k on Ω , and since $G_\Omega(\cdot, \zeta)$ is the least non-negative superharmonic function on Ω of the form $-\log |\zeta - \cdot| - g$, where g is subharmonic on Ω , we see that

$$\limsup_{k \rightarrow \infty} u_k \leq -G_\Omega(\zeta, \cdot) \quad \text{on } \Omega. \quad (9)$$

Also, by the triangle inequality,

$$u_k(z) \leq \frac{1}{N_k} \log (2 \max \{|S_{N_k}(z)|, |f(z)|\}). \quad (10)$$

In view of (6) we obtain an upper-bounded function s_k on \overline{U} by writing

$$s_k(z) = \begin{cases} u_k(z) - h(z)/N_k & (z \in \overline{U} \cap \Omega) \\ \limsup_{w \rightarrow z} \{u_k(w) - h(w)/N_k\} & (z \in \partial U \cap \partial \Omega) \end{cases}. \quad (11)$$

Bernstein's lemma tells us that

$$\frac{1}{N_k} \log |S_{N_k}(z)| \leq \frac{1}{N_k} \log \max_K |S_{N_k}| + G_{\Omega_1}(\infty, z) \quad (z \in \Omega_1).$$

Thus, by (7), and Harnack's inequalities applied to functions of the form $G_{\Omega_1}(\cdot, z)$, there is a positive constant C such that

$$\frac{1}{N_k} \log |S_{N_k}(z)| \leq CG_{\Omega_1}(\zeta, z) \quad (z \in \bar{U} \cap \Omega; k \in \mathbb{N}). \quad (12)$$

We will make use of an ideal boundary for Ω_1 , known as the Martin boundary (see Chapter 8 of [2]) and denoted by $\partial^M \Omega_1$. (In the case of a simply connected domain the Martin compactification is homeomorphic to the prime end compactification, but our domain Ω_1 is, in general, multiply connected.) There is an associated Martin kernel, analogous to the Poisson kernel for \mathbb{D} , given by

$$M(z, w) = \lim_{\xi \rightarrow w} \frac{G_{\Omega_1}(z, \xi)}{G_{\Omega_1}(\zeta, \xi)} \quad (z \in \Omega_1, w \in \partial^M \Omega_1).$$

(The Martin boundary is constructed in such a way that these limits exist.) The functions $M(\cdot, w)$ are positive and harmonic on Ω_1 . We denote by $\partial_1^M \Omega_1$ the set of those points $w \in \partial^M \Omega_1$ for which the function $M(\cdot, w)$ is minimal; that is, for which the only positive harmonic minorants of $M(\cdot, w)$ are constant multiples of $M(\cdot, w)$. Then, for any positive harmonic function v on Ω_1 , there is a unique measure μ_v on $\partial^M \Omega_1$ such that

$$v(z) = \int_{\partial^M \Omega_1} M(z, w) d\mu_v(w) \quad (z \in \Omega_1) \quad (13)$$

and

$$\mu_v(\partial^M \Omega_1 \setminus \partial_1^M \Omega_1) = 0. \quad (14)$$

A set $E \subset \Omega_1$ is said to be *minimally thin* at a point $w \in \partial_1^M \Omega_1$ if $\Omega_1 R_{M(\cdot, w)}^E \not\equiv M(\cdot, w)$. Further, a function $\phi : \Omega_1 \rightarrow [-\infty, +\infty]$ is said to have a *minimally fine limit* (denoted by mf lim) l at such a point w if there is a set E , minimally thin at w , such that $\lim_{z \rightarrow w, z \in \Omega_1 \setminus E} \phi(z) = l$. In fact, this definition can be relaxed to require only that the function ϕ is defined on a set $\Omega_2 \subset \Omega_1$ such that $\Omega_1 \setminus \Omega_2$ is minimally thin at w .

We can now resume the proof of Theorem 1. Since $D(w_0, r_0)$ intersects $\partial\Omega$, we know that $H_{\chi_\Omega}^U \not\equiv 1$, whence $\Omega_1 R_1^{\Omega_1 \setminus U} \not\equiv 1$. The constant function 1 has the representation

$$1 = \int_{\partial^M \Omega_1} M(z, w) d\mu_1(w) \quad (z \in \Omega_1) \quad (15)$$

(see (13)), so

$$\Omega_1 R_1^{\Omega_1 \setminus U}(z) = \int_{\partial^M \Omega_1} \Omega_1 R_{M(\cdot, w)}^{\Omega_1 \setminus U}(z) d\mu_1(w) \quad (z \in \Omega_1). \quad (16)$$

Since the left hand side of (16) is not identically valued 1, and (14) holds in particular when $v \equiv 1$, the set

$$A = \{w \in \partial_1^M \Omega_1 : \Omega_1 \setminus U \text{ is minimally thin at } w\}$$

must satisfy $\mu_1(A) > 0$. (For the Borel measurability of A we refer to Lemma 9.3.6 in [2].)

Let $\zeta_0 \in U$. We know from Theorem 9.6.2 of [2] that the quotient

$$G_U(\zeta_0, \cdot) / G_{\Omega_1}(\zeta, \cdot)$$

has a positive minimal fine limit at each point of A . Hence $A = \cup_m A_m$, where

$$A_m = \left\{ w \in A : \text{mf} \lim_{z \rightarrow w} \frac{G_U(\zeta_0, z)}{G_{\Omega_1}(\zeta, z)} \geq \frac{1}{m} \right\} \quad (m \in \mathbb{N}).$$

We can thus choose $m' \in \mathbb{N}$ such that $\mu_1(A_{m'}) > 0$. Let

$$\omega = \left\{ z \in U : G_U(\zeta_0, z) > \frac{1}{m'+1} G_{\Omega_1}(\zeta, z) \right\}. \quad (17)$$

Then ω is a domain containing ζ_0 , by the maximum principle,

$$\partial\omega \cap \partial U \subset \partial\Omega_1 \subset K, \quad (18)$$

and $\Omega_1 \setminus \omega$ is minimally thin at each point of $A_{m'}$. It now follows from the choice of m' that

$$\int_{A_{m'}} \Omega_1 R_{M(\cdot, w)}^{\Omega_1 \setminus \omega}(z) d\mu_1(w) < \int_{A_{m'}} M(z, w) d\mu_1(w) \quad (z \in \omega).$$

Hence $\Omega_1 R_1^{\Omega_1 \setminus \omega} \not\equiv 1$ (cf. (15) and (16)), so $H_{\chi_{\Omega_1}}^\omega \not\equiv 1$, and thus

$$H_{\chi_U}^\omega \not\equiv 1. \quad (19)$$

By choosing a suitable subsequence of (S_{N_k}) , if necessary, we may assume, in view of (11) and (9), that

$$s_k \leq -\frac{k}{k+1} G_\Omega(\zeta, \cdot) \quad \text{on } Y_k, \quad (20)$$

where

$$Y_k = \{z \in \bar{U} \cap \Omega : \text{dist}(z, \partial\Omega) \geq k^{-1}\}$$

(see Corollary 5.7.2 in [2]). Now, for each $k \in \mathbb{N}$, we have

$$s_k \leq H_{s_k}^U = w_k + h_k \quad \text{on } U, \quad (21)$$

where

$$w_k = H_{s_k \chi_{\Omega \cap \partial U}}^U \quad \text{and} \quad h_k = H_{s_k \chi_{\partial \Omega \cap \partial U}}^U. \quad (22)$$

Further, we see from (11), (10) and (6) that

$$w_k \leq \frac{1}{N_k} H_{(\log 2 + \log^+ |S_{N_k}|) \chi_{\partial U \cap (\Omega \setminus Y_k)}}^U + H_{s_k \chi_{\partial U \cap Y_k}}^U,$$

and then from (12) and (20) that

$$w_k \leq v_k + (\log 2)/N_k \quad \text{on } U, \quad (23)$$

where

$$v_k = H_{\phi_k}^U$$

and

$$\phi_k(\xi) = \begin{cases} CG_{\Omega_1}(\zeta, \xi) & \text{when } \xi \in \partial U \cap (\Omega \setminus Y_k) \\ -\frac{k}{k+1} G_{\Omega}(\zeta, \xi) & \text{when } \xi \in \partial U \cap Y_k \\ 0 & \text{when } \xi \in \partial U \cap \partial \Omega \end{cases}.$$

It follows from (17) that

$$v_1 \leq CG_{\Omega_1}(\zeta, \cdot) \leq C(m' + 1)G_U(\zeta_0, \cdot) \quad \text{on } \omega.$$

In the light of the above formulae and (19), we see that U , ω and (v_k) satisfy the hypotheses of Lemma 3. Thus we can find a domain $\omega_1 \subset \omega$ and a number $k' \in \mathbb{N}$ such that

$$v_k < 0 \quad \text{on } \omega_1 \quad (k \geq k') \quad \text{and} \quad H_{\chi_{\partial \omega_1 \cap \partial U}}^{\omega_1} > 0. \quad (24)$$

It follows, in view of (21) and (23), that

$$s_k \leq h_k + (\log 2)/N_k \quad \text{on } \omega_1 \quad \text{when } k \geq k'.$$

However, $s_k \leq (\log 2)/N_k$ on $\partial U \cap \partial \Omega$, by (11), (10), (6) and (7), so $h_k \leq (\log 2)/N_k$ on U , by (22). Thus $s_k \leq (\log 4)/N_k$ on ω_1 . It follows from (11) and (8) that

$$\log |S_{N_k} - f| \leq h + \log 4 \quad \text{on } \omega_1$$

and so, by (6),

$$|S_{N_k}| \leq |f| + 4e^h \leq 5e^h \quad \text{on } \omega_1.$$

Since $\log |S_{N_k}| - h$ is subharmonic and bounded above on ω_1 we deduce that

$$\log |S_{N_k}| - h \leq H_{(\log 5) \chi_{\partial \omega_1 \cap U}}^{\omega_1} + H_{(\log |S_{N_k}|) \chi_{\partial \omega_1 \cap \partial U}}^{\omega_1} \quad \text{on } \omega_1,$$

and so

$$\log |S_{N_k}| \leq h + \log 5 + H_{(\log |S_{N_k}|)_{\chi_{\partial\omega_1 \cap \partial U}}}^{\omega_1} \quad \text{on } \omega_1.$$

We know that $S_{N_k} \rightarrow 0$ uniformly on K , and that $\partial\omega_1 \cap \partial U \subset \partial\omega \cap \partial U \subset K$, by (18). Since $S_{N_k} \rightarrow f$ on Ω (see the sentence following (8)), we can use Fatou's lemma to see that

$$\log |f| \leq h + \log 5 + (-\infty)H_{\chi_{\partial\omega_1 \cap \partial U}}^{\omega_1} = -\infty \quad \text{on } \omega_1,$$

by (24). We have now arrived at the contradictory conclusion that $f \equiv 0$, so $\log^+ |f|$ cannot have a harmonic majorant on U . It follows that $\mathbb{C} \setminus f(U)$ is polar, by Myrberg's theorem (Theorem 5.3.8 of [2]).

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School of Mathematical Sciences,
 University College Dublin,
 Belfield, Dublin 4, Ireland.
 e-mail: stephen.gardiner@ucd.ie