Cauchy, Goursat and Dirichlet problems for holomorphic partial differential equations

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Abstract. In this paper we survey recent results about Fischer decompositions of polynomials or entire functions and their applications to holomorphic partial differential equations. We discuss Cauchy and Goursat problems for the polyharmonic operator. Special emphasis is given to the Khavinson-Shapiro conjecture concerning polynomial solvability of the Dirichlet problem.

Keywords. Cauchy problem, Goursat problem, Dirichlet problem, holomorphic PDE, polyharmonic function, Fischer decomposition, Jacobi polynomial, Khavinson-Shapiro conjecture.

2000 MSC. Primary: 35A10. Secondary: 35A20, 31B30.

1. Introduction

The theory of partial differential equations is a very active area of research with a variety of methods and techniques. Classical methods such as the power series method and the Fourier analysis method have as point of departure explicit exact solutions. Some more recent developments, e.g. pseudo-differential and Fourier integral operator methods, depend on explicit approximate solutions. Other approaches in PDE's, e.g. variational methods and distribution theory, have lead to new branches in mathematics. Last but not least, methods of complex analysis are considered as indispensable for a deeper understanding of the subject, see e.g. [51] or [13].

In this survey paper we shall take a rather elementary approach to questions in the theory of linear partial differential equations with analytic coefficients. Our methods are based on so-called Fischer decompositions of polynomials or entire functions in the spirit of the work of D.J. Newman and H. S. Shapiro, see [80], [81] and [90]. Although this approach is elementary we shall derive various interesting consequences concerning general Goursat problems and the polynomial solvability of the Dirichlet problem. For another application, uniqueness of polyharmonic functions vanishing on prescribed hyper surfaces, we refer the reader to [86] and [55].

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In order to give a first intuition for these methods let us consider the classical Dirichlet problem for a domain Ω in $\mathbb{R}^d$: given a real or complex valued continuous function f defined on the boundary $\partial\Omega$, a *solution u_f of the Dirichlet problem* is a function u_f defined and continuous on the closure $\overline{\Omega}$ of Ω which is harmonic in Ω and satisfies the boundary value condition

$$u_f(\xi) = f(\xi) \text{ for } \xi \in \partial\Omega.$$

The question of existence of solutions to the Dirichlet problem had an important impact on the development of mathematics in the early 20th century and it is connected with prominent names like H. Poincaré, C. Neumann, D. Hilbert, I. Fredholm and O. Perron, see e.g the exposition [68]. Explicit solutions for the Dirichlet problem can be constructed only for domains of special geometry. For a ball the Poisson formula provides an explicit formula, see [8]. If Ω is an ellipsoid, i.e. if Ω is, up to a rotation and a translation, given by

$$(1) \qquad \Omega_e = \{x \in \mathbb{R}^d : \sum_{j=1}^{d} a_j x_j^2 < 1\} \text{ with } a_1, ..., a_d > 0$$

it is already very cumbersome to provide an explicit formula, see e.g. [88]. On the other hand it was already known in the 19th century that for a *polynomial data function f*, restricted to the boundary $\partial\Omega$ of the ellipsoid Ω, the solution u_f of the Dirichlet problem is a harmonic *polynomial*. This result can be proved by means of elliptical coordinates and separation of variables (see [98]) and is associated with the names E. Heine, G. Lamé and M. Ferrers. A much simpler proof based on techniques of Fischer operators was given in [70], see also [60] and [10]. In the second Section of this paper we shall present this elementary approach to the Dirichlet problem for an ellipsoid since it illustrates in a nice way the basic ideas and the powerful tool of Fischer decompositions.

The aim of this survey is to show that Fischer decomposition techniques can be extended to a much more general setting addressing as well problems for higher order differential operators like the polyharmonic operator Δ^k which is defined iteratively by $\Delta^k := \Delta\left(\Delta^{k-1}\right)$ where k is a natural number and

$$\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_d^2}$$

is the Laplace operator. In particular we shall discuss later the Goursat problem for perturbations of the polyharmonic operator Δ^k.

In the third Section we shall discuss systematically the concept of a Fischer operator: if $Q(x)$ is a polynomial, $Q(D)$ the associated differential operator and $P(x)$ a polynomial then we define the *Fischer operator $F_{Q,P}$* acting on the space of all polynomials by

$$F_{Q,P}(q) := Q(D)(Pq).$$

If Q and P have the same degree k and Q is *homogeneous* then $F_{Q,P}$ maps the space of all polynomials of degree $\leq m$ into itself. This important property has the consequence that the linear operator $F_{Q,P}$ is bijective if and only if it is injective. It is not very difficult to prove that the surjectivity of $F_{Q,P}$ implies the following property: For each polynomial f of degree $\leq m$ there exist polynomials q, r of degree $\leq m$ such that

$$(2) \qquad\qquad f = Pq + r \text{ and } Q(D)r = 0.$$

The injectivity of $F_{Q,P}$ implies that the representation is unique. Equation (2) is called the *Fischer decomposition* of the polynomial f with respect to the polynomials P and Q (provided that the Fischer operator is bijective). *Equation* (2) *provides a solution for the following generalized Dirichlet problem:* for any polynomial data function f of degree m there exists a polynomial r of degree $\leq m$ such that

$$f(\xi) = r(\xi) \text{ for all } \xi \in P^{-1}(0) := \left\{ x \in \mathbb{R}^d : P(x) = 0 \right\}$$

and r is a solution of the partial differential equation $Q(D)r = 0$. For example, if we take $P(x) = \sum_{j=1}^{d} a_j x_j^2 - 1$ and $Q(x) = |x|^2 := x_1^2 + ... + x_d^2$, then $Q(D)$ is the Laplace operator Δ and we rediscover the Dirichlet problem for the ellipsoid.

An old theorem due to E. Fischer [45] in 1917 says that for any *homogeneous* polynomial P the operator

$$(3) \qquad\qquad F_P(q) := P^*(D)(Pq)$$

is bijective where P^* is the polynomial arising from P by conjugating the coefficients of P. In [86] we have been able to identify a new class of bijective Fischer operators: if P is a polynomial of degree $2k$ whose leading part is non-negative on $\mathbb{R}^d$ then the operator

$$(4) \qquad\qquad F_{\Delta^k, P}(q) := \Delta^k(Pq)$$

is bijective. Even for $k = 1$ this was proved only recently in [9].

In complex analysis we are dealing with power series or limits of polynomials. For this reason it is desirable to extend the Fischer decomposition to a larger class of functions than polynomials. Let us denote the ball with center 0 and radius $R > 0$ by

$$B_R := \left\{ x \in \mathbb{R}^d : |x| < R \right\}$$

where it is allowed that R takes the value ∞. Assume that f is infinitely differentiable in a neighborhood of 0 and consider the Taylor polynomial of f of order m, defined as $f_0 + ... + f_m$, where the *homogeneous* polynomials f_m are given by

$$f_m = \sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(0) \ x^\alpha.$$

We define $A(B_R)$ as the space of all infinitely differentiable complex-valued functions f on B_R such that for any compact set $K \subset B_R$ the series $\sum_{m=0}^{\infty} f_m(x)$

converges absolutely and uniformly to f on K. The second main result in Section 3 generalizes the existence of Fischer decompositions of polynomials to the class $A(B_R)$: for an elliptic polynomial $P(x)$ of degree $2k$ there exists $R > 0$ such that for each $f \in A(B_R)$ there exist unique functions $q, r \in A(B_R)$ with

$$f = Pq + r \text{ and } \Delta^k r = 0.$$

In the fourth section we shall discuss Cauchy and Goursat problems. Our approach is motivated by the work of P. Ebenfelt and H.S. Shapiro in [39] and [40] who used Fischer operators of the type (3). Using the new type of Fischer decompositions, P. Ebenfelt and the present author established in [37] the following result: Let $R > 0$ and consider the differential operator

$$L = \Delta^k + \sum_{|\alpha| \leq k_0} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}, \text{ where } a_\alpha \in A(B_R).$$

Let $P(x)$ be an elliptic, homogeneous polynomial of degree $2k$. If $k_0 < k$ then for any $f \in A(B_R)$ there exists $v \in A(B_R)$ such that

$$L(P \cdot v) = f.$$

If $k_0 = k$ then there exists $r > 0$ such for any $f \in A(B_R)$ there exists $v \in A(B_r)$ such that $L(P \cdot v) = f$.

We illustrate the power of the Fischer decomposition method for a Goursat problem with respect to the Helmholtz operator $\Delta + c$ (see [38] where also the polyharmonic operator is discussed): Let Γ_1, Γ_2 be two distinct lines through the origin in $\mathbb{R}^2$, and denote by $\theta = 2\pi\alpha$ the acute angle between them. Suppose that α satisfies the Diophantine condition

$$\left| \alpha - \frac{n}{m} \right| \geq \frac{C}{m^2}, \quad \text{for all } n, m \in \mathbb{N}_0, m \neq 0$$

for some constant $C > 0$. Let $c \in A(B_R)$. Then there exists $0 < r \leq R$ such that the Goursat problem

$$(\Delta + c) u = f \text{ and } u = g \text{ on } \Gamma_1 \cup \Gamma_2$$

has a unique solution $u \in A(B_r)$ for every $f, g \in A(B_R)$. In the last subsection of Section 4 we discuss some old and new results about the Dirichlet problem for general differential operators and connections to dynamical systems.

In the fifth Section we shall return to the classical Dirichlet problem. We discuss the following property introduced by D. Khavinson and H.S. Shapiro for a domain Ω in $\mathbb{R}^d$ for which the Dirichlet problem is solvable:

(KS) For any polynomial f the solution u_f of the Dirichlet problem for $f \mid \partial\Omega$ is a polynomial.

The conjecture of D. Khavinson and H.S. Shapiro [70] says that property (KS) for a bounded domain Ω implies that Ω is an ellipsoid. In other words: if Ω is bounded but not an ellipsoid then there must exist a polynomial such that

the solution u_f of the Dirichlet problem is not a polynomial. Using our results about Fischer decompositions for polynomials we shall establish a large class of domains such that the polynomial

$$|x|^2 = x_1^2 + ... + x_d^2$$

does not have a polynomial solution. Unfortunately there are examples of domains for which the test function $|x|^2$ has a polynomial solution to the Dirichlet problem but nonetheless (KS) does not hold.

The central motivation behind the Khavinson-Shapiro conjecture is the following question: is it possible to describe the singularities of the solution u_f of the Dirichlet problem for a given data function f in terms of the singularities of f and characteristics of the domain Ω? If we assume that f is a polynomial (so f does not have singularities) and Ω is not an ellipsoid, is it true that u_f develops somewhere a singularity, say in $\mathbb{R}^d$ or in $\mathbb{C}^d$? This question leads to the following condition:

(KSe) For any polynomial f the solution u_f of the Dirichlet problem for $f \mid \partial\Omega$ has an extension to a holomorphic function on $\mathbb{C}^d$.

The *second* conjecture of Khavinson and Shapiro states that for a bounded domain Ω condition (KSe) implies that Ω is an ellipsoid. Using our results about Fischer decompositions for the algebra $A(B_R)$ we can establish this conjecture for a large class of domains. Roughly speaking, we shall assume that the boundary of the domain Ω is given by the zero set of an *elliptic* polynomial. On the other hand, it should be emphasized that the conjecture of Khavinson and Shapiro is still open in its full generality, and we shall address recent developments in this area at the end of Section 5.

Section 6 is devoted to a short introduction to the Schwarz potential in $\mathbb{R}^d$ which generalizes the Schwarz function known from the two-dimensional case, see [29]. The interested reader is referred to the excellent expositions [64], [65], [69] and [91] for a deeper analysis.

Finally let us fix some notations and definitions. Throughout the paper we shall use multi-index notation: for $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$ we define $|\alpha| = \alpha_1 + ... + \alpha_d$. The monomial x^α is defined by $x_1^{\alpha_1}...x_d^{\alpha_d}$ for $x = (x_1, ..., x_d)$. A polynomial $P(x)$ of degree $\leq k$ is an expression of the the form

$$P(x) = \sum_{|\alpha| \leq k} c_\alpha x^\alpha$$

where c_α are complex numbers. By $\mathbb{C}[x]$ we denote the set of all polynomials with complex coefficients, and by $\mathbb{R}[x]$ those with real coefficients. For $P(x) = \sum_{|\alpha| \leq k} c_\alpha x^\alpha$ we define $P^*(x) = \sum_{|\alpha| \leq k} \overline{c_\alpha} x^\alpha$ where $\overline{c_a}$ is the complex conjugate of c_α. The differential operator $P(D)$ associated to a polynomial $P(x)$ is defined

by

$$P(D) = \sum_{|\alpha| \le k} c_\alpha \frac{\partial^\alpha}{\partial x^\alpha}.$$

For an open set $\Omega \subset \mathbb{R}^d$ we denote by $C^k(\Omega)$ the set of all functions $f : \Omega \to \mathbb{C}$ which are differentiable up to order k. A function $f \in C^{2k}(\Omega)$ is called *poly-harmonic of order k* if $\Delta^k f(x) = 0$ for all $x \in \Omega$. For $k = 1$ one obtains the definition of a harmonic function. For a treatise about polyharmonic functions we refer the reader to [7] and for applications see e.g. [72].

A function $f : \Omega \to \mathbb{C}$ is real-analytic at $x_0 \in \Omega$ if there exists $c_\alpha \in \mathbb{C}$ and a neighborhood U of x_0 such that $f(x) = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha (x - x_0)^\alpha$ where the sum converges locally uniformly in U.

2. An elementary approach to the Dirichlet problem for the ellipsoid and quadratic surfaces

In the first part of this Section the main ideas are taken from [70], see also [11].

Theorem 1. *Let P be a polynomial of degree ≤ 2. If the Fischer operator $F_{\Delta,P} : \mathbb{C}[x] \to \mathbb{C}[x]$ defined by*

$$F_{\Delta,P}(q) := \Delta(Pq) \ \text{for all } q \in \mathbb{C}[x]$$

is injective then for each polynomial $f(x)$ of degree $\le m$ there exists a harmonic polynomial u of degree $\le m$ such that

(5) $\qquad u(\xi) = f(\xi) \ \text{for all } \xi \in P^{-1}(0) = \left\{ x \in \mathbb{R}^d : P(x) = 0 \right\}.$

Proof. Let $\mathcal{P}_m(\mathbb{R}^d)$ be the space of all polynomials of degree $\le m$. Since P has degree ≤ 2 the operator $F_{\Delta,P}$ maps $\mathcal{P}_m(\mathbb{R}^d)$ into itself. Thus injectivity implies bijectivity of $F_{\Delta,P}$. The solution u in (5) is then defined by

$$u = f - P \cdot F_{\Delta,P}^{-1}(\Delta f),$$

where $F_{\Delta,P}^{-1}$ is the inverse of the bijective operator $F_{\Delta,P}$ defined on $\mathcal{P}_m(\mathbb{R}^d)$. Then u obviously satisfies (5) and u is harmonic since

$$\Delta u = \Delta f - F_{\Delta,P} \circ F_{\Delta,P}^{-1}(\Delta f) = 0.$$

Hence the polynomial u is a solution to the generalized Dirichlet problem stated in the theorem. ∎

Theorem 2. *Let E be an ellipsoid. Then for any polynomial f of degree $\le m$ the solution u of the Dirichlet problem for $f \mid \partial E$ is a polynomial of degree $\le m$.*

Proof. By Theorem 1 it suffices to prove the injectivity of $F_{\Delta,P}$: if $\Delta(Pq) = 0$ then $u := Pq$ is a harmonic function vanishing on the boundary $\partial E = P^{-1}(0)$, hence it is zero by the maximum principle for harmonic functions. ∎

Theorem 3. *The Dirichlet problem is solvable for the ellipsoid E.*

Proof. Let f be a continuous function on the boundary ∂E. By the Stone-Weierstraß theorem we can approximate f by a sequence of polynomials $p_n, n \in \mathbb{N}$. For each p_n there exists a harmonic polynomial u_n with $p_n(\xi) = u_n(\xi)$ for $\xi \in \partial E$. Using the maximum principle we infer that

$$\max_{x \in \overline{E}} |u_n(x)| = \max_{\xi \in \partial E} |u_n(\xi)| = \max_{\xi \in \partial E} |p_n(\xi)|.$$

Thus u_n is a Cauchy sequence in the space $C\left(\overline{E}\right)$. By completeness of $C\left(\overline{E}\right)$ there exists a continuous function u on $\overline{E}$ such that u_n converges uniformly to u. Then the function u is harmonic in E and it is easy to see that $u(\xi) = f(\xi)$ for $\xi \in \partial E$. $\blacksquare$

An elementary treatment of the Dirichlet problem for the ellipse can also be found in [60]. For a nice account about the potential theory on ellipsoids we refer the reader to [65]. Interesting remarks about the history of the Dirichlet problem are contained in [30, pp. 568–573], for a survey of potential methods in classical mechanics we refer to [52]. Finally we mention that S.M. Nikol'skiǐ has generalized Theorem 2 to the case of elliptic self adjoint operators of degree $2l$ and the ellipsoid with appropriate boundary conditions, see [82].

The reader who is interested in further algebraic aspects of solutions to the Dirichlet problem is referred to the work [14], [15], [36] and [42].

Finally we mention that Theorem 2 has been generalized in the following way by D. Khavinson and H.S. Shapiro in [70], see also [6] where growth conditions of entire functions are considered:

Theorem 4. *Let Ω be an ellipsoid. If $f : \mathbb{R}^d \to \mathbb{C}$ has an holomorphic extension to $\mathbb{C}^d$ then the solution u of the Dirichlet problem for $f \mid \partial\Omega$ extends to an holomorphic harmonic function on $\mathbb{C}^d$.*

We want to illustrate the usefulness of Fischer decompositions by another example: Consider for $\gamma \in (0,1)$ the quadratic homogeneous polynomial

$$P_\gamma(x_1, ..., x_d) = \gamma^2 \left(x_1^2 + ... + x_{d-1}^2\right) + \left(\gamma^2 - 1\right) x_d^2.$$

Then for $\gamma \in (0,1)$ the zero set of P_γ is a cone passing through 0 containing all $(x_1, ..., x_{d-1}, x_d)$ such that

$$x_1^2 + ... + x_{d-1}^2 = \frac{1 - \gamma^2}{\gamma^2} x_d^2.$$

We shall consider the Dirichlet problem for the cone

$$\Omega := \left\{ x \in \mathbb{R}^d : x_d > 0 \text{ and } P_\gamma(x) < 0 \right\} \text{ for } \gamma \in (0,1).$$

Note that the boundary $\partial\Omega$ is contained properly in the algebraic set $P_\gamma^{-1}(0)$.

Let us recall that the Pochhammer symbol $(\alpha)_k$ for a complex number α and a natural number $k \geq 0$ is defined by

$$(\alpha)_k = \alpha \, (\alpha + 1) \dots (\alpha + k - 1)$$

with the convention that $(\alpha)_0 = 1$. The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ of degree n for complex parameters α and β is defined by

$$P_n^{(\alpha,\beta)}(x) = (-1)^n \, \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{(n+\alpha+\beta+1)_k}{(\alpha+1)_k} \left(\frac{1-x}{2} \right)^k,$$

see [4, p. 99]. The following result was proved by D. Armitage [5].

Theorem 5. *Let $\gamma \in (0,1)$ and $d \geq 3$. Then the Fischer operator F_{Δ, P_γ} for the polynomial $P_\gamma(x)$ is injective if and only if*

(6) $$P_n^{(k+(d-3)/2, \, k+(d-3)/2)}(\gamma) \neq 0 \text{ for all } k, n \in \mathbb{N}_0,$$

where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials of degree n.

It is well known that the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ have their zeros in the interval $(-1,1)$, and they are clearly algebraic numbers. Thus for given *rational* number $\gamma \in (0,1)$ it is not obvious whether condition (6) is satisfied. In [87] we proved the following:

Theorem 6. *Let b and c be relatively prime natural numbers and $d \geq 3$. If n is even and $b \neq 1$, or if n is odd and $b \neq 1, 3$, then*

$$P_n^{(k+(d-3)/2, \, k+(d-3)/2)} \left(\sqrt{\frac{b}{c}} \right) \neq 0 \text{ for all } k \in \mathbb{N}_0.$$

Combining Theorem 1 and 5 one obtains:

Theorem 7. *Let $d \geq 3$ and $\gamma = \sqrt{b/c}$ with relatively prime natural numbers $b, c \neq 0$ with $b \neq 1, 3$. Then for each polynomial f of degree $\leq m$ there exists a harmonic polynomial u of degree $\leq m$ such that $f(\xi) = u(\xi)$ for all $\xi \in P_\gamma^{-1}(0) = \{x \in \mathbb{R}^d : P_\gamma(x) = 0\}$.*

For a different approach to the Dirichlet problem for a cone we refer the reader to [73, p. 210].

Legendre polynomials are by definition the Jacobi polynomials $P_n^{(0,0)}(x)$. It is still an unsolved question whether the Legendre polynomials are irreducible over the rationals, see [57], [77], [96] and [97]. H. Ille has shown in [58] that $P_n^{(0,0)}(x)$ has no quadratic factor which implies that $P_n^{(0,0)}\left(\sqrt{b/c}\right) \neq 0$ for all $n, b, c \in \mathbb{N}$ (even for the case $b = 1, 3$).

A result related to Theorem 5 was proved in 1988 by V.P. Burskiĭ for the Dirichlet problem for the unit ball and the wave equation

$$(7) \qquad u_{x_1 x_1} + u_{x_2 x_2} - a^2 u_{x_3 x_3} = 0.$$

It is shown in [21] that the set of all solutions of (7) satisfying $u(\xi) = 0$ for all $|\xi| = 1$ is trivial if and only if $P_n^{(k,k)}\left(1/\sqrt{1+a^2}\right) \neq 0$ for all $k, n \in \mathbb{N}_0$. A generalization to n variables can be found in [25].

3. Fischer decompositions

A polynomial $P(x)$ is called *homogeneous* of degree k if $P(rx) = r^k P(x)$ for all $x \in \mathbb{R}^d$ and $r > 0$. Let $P(x) = \sum_{|\alpha| \leq k} c_\alpha x^\alpha$ be a polynomial. By defining

$$P_m(x) := \sum_{|\alpha|=m} c_\alpha x^\alpha$$

we see that each polynomial can be written as a sum of homogeneous polynomials P_m for $m = 0, ..., k$, i.e. that

$$P = P_0 + ... + P_k.$$

The polynomial $P_k \neq 0$ is called the *leading part* or *principal part* of $P(x)$. A polynomial $P(x)$ of degree $2k$ is called *elliptic* if there there exists $C > 0$ such that the *leading part* P_{2k} satisfies

$$P_{2k}(x) \geq C \cdot |x|^{2k} = C \cdot \left(x_1^2 + ... + x_d^2\right)^k \text{ for all } x \in \mathbb{R}^d.$$

As before, the Fischer operator for polynomials P and Q is defined by

$$F_{Q,P}(q) := Q(D)(Pq).$$

At first we recall the well known fact that surjectivity of the Fischer operator corresponds to a polynomial decomposition property (see e.g. [90], [78]):

Theorem 8. *Let Q be a homogeneous polynomial. Then the Fischer operator $F_{Q,P}$ is surjective if and only if for each polynomial f there exist polynomial q and r such that*

$$(8) \qquad f = Pq + r \text{ and } Q(D)r = 0.$$

Proof. Assume that $F_{Q,P}$ is surjective and let f be a polynomial. By surjectivity, we can find a polynomial q with $F_{Q,P}(q) = Q(D)f$. We define $r := f - Pq$. Then

$$Q(D)r = Q(D)f - Q(D)(Pq) = Q(D)f - F_{Q,P}(q) = 0.$$

For the converse we shall use without proof the well known fact that for every polynomial f there exists a polynomial g with $Q(D)g = f$. By assumption we can write $g = Pq + r$ with $Q(D)r = 0$. Then $f = Q(D)g = Q(D)(Pq) = F_{Q,P}(q)$, so $F_{Q,P}$ is surjective. ∎

Similarly it is easy to see that injectivity of the Fischer operator corresponds to the uniqueness of the decomposition (8).

3.1. Fischer decompositions for polynomials. The following result is proved in the same manner as Theorem 1, see also Theorem 8.

Theorem 9. *Let P be polynomial of degree $\leq k$ and Q be a homogeneous polynomial of degree $\leq k$. If the Fischer operator $F_{Q,P} : \mathbb{C}[x] \to \mathbb{C}[x]$ defined by*

$$F_{Q,P}(q) = Q(D)(Pq) \text{ for all } q \in \mathbb{C}[x],$$

is injective then for each polynomial $f(x)$ of degree $\leq m$ there exist unique polynomials q and u of degree $\leq m$ such that

(9) $f = Pq + r \text{ and } Q(D)r = 0.$

In general, it is very difficult to decide whether a given Fischer operator is injective. A simple example of a non-injective Fischer operator is the following: take a *harmonic* polynomial $P(x)$ of exact degree 2 and define

$$F_{\Delta,P}(q) = \Delta(Pq).$$

If q is the constant function 1 then $F_{\Delta,P}(1) = 0$ and the Fischer operator is not injective.

In the following we want to develop criteria which ensure the injectivity of the Fischer operator. It is amazing that elementary Hilbert space techniques are very useful in this context. One key tool is the following scalar product defined for polynomials $f = \sum_{|\alpha| \leq N} c_\alpha x^\alpha$ and $g = \sum_{|\alpha| \leq N} d_\alpha x^\alpha$ by the simple formula

(10) $\langle f, g \rangle_F = \sum_{|\alpha| \leq N} \alpha! c_\alpha \overline{d_\alpha}.$

This scalar product is often called the *Fischer inner product* or the *apolar inner product* and its origin goes back to classical invariant theory. We note that it is often used in the treatment of spherical harmonics, see e.g. [31]. The apolar inner product has the following basic property:

(11) $\langle f, Q \cdot g \rangle_F = \langle Q^*(D) f, g \rangle_F$

for all $f, g \in \mathbb{C}[x]$. Thus the adjoint of the multiplication operator $g \longmapsto Q \cdot g$ is the differential operator $Q^*(D)$. The identity (11) is easily checked for monomials $f(x) = x^\alpha$ and $g(x) = x^\beta$, and by bilinearity the result follows.

Theorem 10. *(Fischer 1917) Let $P(x)$ be a homogeneous polynomial. Then the Fischer operator $F_P : \mathbb{C}[x] \to \mathbb{C}[x]$ defined by $F_P(q) = P^*(D)(Pq)$ is a bijection.*

Proof. It suffices to show that F_P is injective. Suppose that $F_P(q) = 0$. Then

$$0 = \langle q, F_P(q) \rangle_F = \langle q, P^*(D)(Pq) \rangle_F = \langle Pq, Pq \rangle_F = \|Pq\|_F^2.$$

This implies $Pq = 0$ and $q = 0$.   ∎

The apolar inner product possesses an integral representation. Indeed, let us define the *Bargmann space* $\mathcal{F}_d$ (or *Fock space* or *Fischer space,* see [12] and [90]) as the space of all entire functions $f : \mathbb{C}^d \to \mathbb{C}$ satisfying

$$(12) \qquad \|f\|_F^2 := \frac{1}{\pi^d} \int_{\mathbb{C}^d} |f(z)|^2 e^{-|z|^2} dz < \infty.$$

Clearly the norm $\|f\|_F$ is induced by the scalar product

$$(13) \qquad \frac{1}{\pi^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x+iy) \overline{g(x+iy)} e^{-|x|^2 - |y|^2} dx dy < \infty$$

where $dxdy$ is the Lebesgue measure over $\mathbb{R}^{2d}$. By a direct computation one may prove that for polynomials f, g the apolar inner product $\langle f, g \rangle_F$ defined in (10) is identical to the expression (13); moreover,

$$x^\alpha / \sqrt{\alpha!}, \qquad \alpha \in \mathbb{N}_0^d,$$

are orthonormal polynomials.

One disadvantage of the apolar inner product is the fact the integration in (13) has to be taken over all complex arguments. Thus an assumption like ellipticity of a polynomial $P(x)$ can not easily be used. In analogy to (12) we have defined in [86] the *real Bargmann space* $\mathcal{RF}_n$ as the space of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$(14) \qquad \|f\|^2 := \int_{\mathbb{R}^n} |f(x)|^2 e^{-|x|^2} dx < \infty,$$

endowed with the scalar product

$$\langle f, g \rangle_{rF} := \int_{\mathbb{R}^d} f(x) \overline{g(x)} e^{-|x|^2} dx.$$

The following result is crucial and for a proof we refer to [86].

Theorem 11. *Let f be a homogeneous polynomial of degree m, and $k \in \mathbb{N}$ with $2(k-1) \le m$. Then $\Delta^k f = 0$ if and only if*

$$\langle f, g \rangle_{rF} = 0 \quad \textit{for all polynomials } g \textit{ with} \quad \deg g + 2(k-1) < m.$$

As a consequence we obtain an important theorem due to Brelot-Choquet [20]:

Corollary 12. *(Brelot-Choquet) Let f be a homogeneous harmonic polynomial of degree m. Then f does not have a non-negative non-constant factor.*

Proof. Since f is harmonic and homogeneous we infer from Theorem 11 for $k = 1$ that $\langle f, g \rangle_{rF} = 0$ for all polynomials g of degree $< m$. Suppose that $f = f_1 f_2$ where f_1 is non-negative and has degree ≥ 1. Then we conclude that

$$0 = \langle f, f_2 \rangle_{rF} = \langle f_1 f_2, f_2 \rangle_{rF} = \int_{\mathbb{R}^d} f_1(x) |f_2(x)|^2 e^{-|x|^2} dx.$$

Since f_1 is non-negative we infer that $f_1 f_2^2 = 0$, a contradiction. ∎

Theorem 13. *Let $P(x)$ be a polynomial of degree $2k$ whose leading part is non-negative. Then the Fischer operator $F_{\Delta^k,P}$ defined by*

$$F_{\Delta^k,P}(q) := \Delta^k(Pq)$$

is a bijection on $\mathbb{C}[x]$ and for each polynomial f of degree $\leq m$ there exist polynomials q and r of degree $\leq m$ such that

$$f = Pq + r \ \text{ and } \ \Delta^k r = 0.$$

Proof. It suffices to prove the injectivity of $F_{\Delta^k,P}$. Suppose that $F_{\Delta^k,P}(q) = 0$ for a polynomial $q \neq 0$ of degree m. By expanding q and P into sums of homogeneous polynomials with leading parts $q_m \neq 0$ and $P_{2k} \neq 0$ and comparing the homogeneous summands one arrives at the equation $\Delta^k(P_{2k}q_m) = 0$. By Theorem 11 applied to the polynomial $f := P_{2k}q_m$ we see that $\langle P_{2k}q_m, g\rangle_{rF} = 0$ for all polynomials g with $\deg g + 2(k-1) < 2k + m$. Thus we may take $g = q_m$ and obtain that

$$0 = \langle P_{2k}q_m, q_m\rangle_{rF} = \int_{\mathbb{R}^d} P_{2k}(x)\,|q_m(x)|^2\,e^{-|x|^2}dx.$$

Since P_{2k} is non-negative we infer that $q_m = 0$. This contradiction finishes the proof. ∎

Let us illustrate Theorem 13 by two examples:

(i) At first consider the one-dimensional case $d = 1$: then $\Delta^k = \frac{d^{2k}}{dx^{2k}}$ and the condition $\Delta^k r = 0$ means that $\deg r < 2k$. Thus the decomposition $f = Pq + r$ just leads to the Euclidean algorithm.

(ii) Consider the domain

$$\Omega := \left\{x \in \mathbb{R}^d : x_1^{2k} + ... + x_d^{2k} < 1\right\},$$

sometimes called the TV-screen. Theorem 13 shows that for any polynomial f there exists a unique *polynomial* u such that $\Delta^k u = 0$ and $u(x) = f(x)$ for all $x \in \partial\Omega$.

3.2. Fischer decompositions for analytic functions. In the last Section we investigated the Fischer decomposition for a polynomial f. Now we want to extend this result to a special class of analytic functions. Let us recall from the introduction that $B_R := \left\{x \in \mathbb{R}^d : |x| < R\right\}$ is the ball with center 0 and radius $0 < R \leq \infty$, and $A(B_R)$ is the space of all $f \in C^\infty(B_R)$ such that for any compact set $K \subset B_R$ the series $\sum_{m=0}^\infty f_m(x)$ converges absolutely and uniformly to f on K where f_m

$$(15) \qquad\qquad f_m = \sum_{|\alpha|=m} \frac{1}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}(0)\ x^\alpha.$$

It is easy to see that $f \in A(B_R)$ is real-analytic in B_R. The converse is not true as the simple example $f(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$ shows. A characterization of the

class $A(B_R)$ in terms of holomorphy is given in the next theorem. Since we shall not use this result we omit the details, see [86].

Theorem 14. *Each $f \in A(B_R)$ has an holomorphic extension to the Lie ball, also called the classical domain R_{IV} of E. Cartan, defined by*

$$\widehat{B_R} := \left\{ z \in \mathbb{C}^d : |z|^2 + \sqrt{|z|^4 - |q(z)|^2} < R^2 \right\}$$

where $q(z) = z_1^2 + ... + z_d^2$ and $|z|^2 = |z_1|^2 + ... + |z_d|^2$. In particular, each $f \in A(B_R)$ for $R = \infty$ has an holomorphic extension to $\mathbb{C}^d$.

The next theorem says that for functions $f \in A(B_R)$ there exists a Fischer decomposition provided that the polynomial $P(x)$ is homogeneous and elliptic:

Theorem 15. *Let $P(x)$ be a homogeneous elliptic polynomial of degree $2k$. Then for each $f \in A(B_R)$ there exist unique functions $q, r \in A(B_R)$ such that*

$$f = Pq + r \ \text{ and } \ \Delta^k r = 0.$$

For applications it is important to consider *non-homogeneous* elliptic polynomials $P(x)$. However, in order that Theorem 15 holds for non-homogeneous polynomials one has to require that the radius R for defining the class $A(B_R)$ is large enough (see [86]):

Theorem 16. *Let $P(x)$ a be polynomial of degree $2k$ and $P = P_{2k} + ... + P_0$ be its homogeneous decomposition, and assume that $CP_{2k}(x) \geq |x|^{2k}$ for all $x \in \mathbb{R}^d$. Let l_P be the cardinality of $E_P := \{s \in \{0, ..., 2k-1\} : P_s \neq 0\}$ and let α denote the smallest and β the largest element in E_P. Define*

$$D := \max_{s=0,...,2k-1} \max_{\theta \in \mathbb{S}^{n-1}} |P_s(\theta)|.$$

Assume that R is so large such that

(16) $$l_P CD < R^\gamma \text{ for all } \gamma \text{ with } 2k - \beta \leq \gamma \leq 2k - \alpha.$$

Then for each $f \in A(B_R)$ there exist unique functions $q, r \in A(B_R)$ such that

$$f = Pq + r \ \text{ and } \ \Delta^k r = 0.$$

At a first glance one may be surprised that one needs the requirement of a large radius. But consider the following example: define $P := |x|^{2k} - d$ with $d > 0$. Suppose that the radius R is small, e.g. suppose that $R^{2k} < d$. Then $P := |x|^{2k} - d$ has no zeros in B_R and it is invertible in $A(B_R)$. Let $u \in A(B_R)$ be an arbitrary harmonic function. Then we can write for any $f \in A(B_R)$ the following trivial and useless decomposition

$$f = u + Pq \text{ with } q := (f - u) P^{-1} \in A(B_R).$$

Thus uniqueness of the representation fails.

A proof of Theorem 16 runs as follows. Since $f \in A(B_R)$ we can write $f = \sum_{m=0}^{\infty} f_m$ with homogeneous polynomials f_m. For each f_m there exists a Fischer decomposition

$$(17) \qquad\qquad\qquad f_m = Pq_m + r_m$$

with polynomials q_m and r_m of degree $\leq m$ and $Q(D) r_m = 0$. Then we define $q = \sum_{m=0}^{\infty} q_m$ and $r = \sum_{m=0}^{\infty} r_m$. The only difficulty is to establish the convergence of the last two sums, and this is the place where one needs the assumption that the radius R is large enough. Basic ingredients of the proof are estimates of the norms of q_m and r_m in the decomposition (17) in dependence of the norm of f_m. For details we refer the interested reader to [86].

4. Cauchy and Goursat problems

The original proof of the Cauchy-Kovalevski theorem goes back to 1874 but it was preceded by the work of A. Cauchy in 1842 who proved an existence theorem for analytic differential equations of second order. For a general introduction to the subject we refer the reader to the excellent books of J. Rauch [85], or F. John [62], or D. Khavinson [65].

As explained in the introduction we want to generalize results from [39] and [40] to the framework of the new classes of Fischer operators presented in Section 3. The first central result is Theorem 21 below, due to P. Ebenfelt and the author. Since the reader might be not very familiar with some extensions of the Cauchy-Kovalevski theorem, like the Goursat theorem, we shall use the opportunity to provide background material in order to facilitate the comparison of Theorem 21 with related theorems in the literature.

4.1. The Cauchy-Kovalevski theorem for hyperplanes. Let us recall the Cauchy-Kovalevski theorem in a form which is surely known to every mathematician:

Theorem 17. *Let $x_0 = (t_0, y_0) \in \mathbb{R} \times \mathbb{R}^{d-1}$, and V and U open sets with $t_0 \in V$ and $y_0 \in U$. Assume that $a_{(j,\beta)} : V \times U \to \mathbb{C}$ are real-analytic functions and consider the partial differential operator*

$$L = \frac{\partial^m}{\partial t^m} + \sum_{j=0}^{m-1} \sum_{|\beta| \leq m-j} a_{(j,\beta)}(t,y) \frac{\partial^j}{\partial t^j} \frac{\partial^\beta}{\partial y^\beta}.$$

If $f(t,y)$ and $w_0(y), ..., w_{m-1}(y)$ are real-analytic on $V \times U$ and U resp., then there exists a unique real-analytic function u defined on a neighborhood $V_0 \times U_0$ of (t_0, y_0) such that

$$Lu \quad (t,y) = f(t,y) \ \text{for all} \ (t,y) \in V_0 \times U_0$$

$$\frac{\partial^j u}{\partial t^j}(t_0, y) = w_j(y) \ \text{for all} \ y \in U_0 \ \text{and for all} \ j = 0, ..., m-1.$$

In many applications the function $f(t, y)$ will be the zero function while the real-analytic functions $w_0(y), ..., w_{m-1}(y)$ express the initial conditions. However, from a proof-theoretic point of view the following well known observation is very useful:

Remark 1. It is sufficient to prove the theorem for arbitrary real-analytic functions $f(t, y)$ and initial condition $w_0 = ... = w_{m-1} = 0$.

Proof. Let $f(t, y)$ and $w_1(y), ..., w_{m-1}(y)$ be real-analytic functions. Define $w(t, y) = \sum_{j=0}^{m-1} \frac{1}{j!} (t - t_0)^j w_j(y)$ and $g = f - L(w)$. Then g is real-analytic. By assumption there exists a solution u_0 of the equation $L(u_0) = f - L(w)$ and zero boundary conditions $\frac{\partial^j u_0}{\partial t^j}(t_0, y) = 0$ for $j = 0, ..., m$. Then $u := u_0 + w$ is a solution of $L(u) = f$ with boundary conditions $\frac{\partial^j u}{\partial t^j}(t_0, y) = w_j(y)$ for $j = 0, ..., m - 1$. ∎

The Cauchy-Kovalevski theorem is a *local result*: the solution u is defined only on a neighborhood of the point x_0 and it does not give the maximal domain of regularity of the solution of the partial differential equation in terms of the regularity of the data. This is a severe limitation for applications, and for this reason the Cauchy-Kovalevski theorem is usually considered as a theoretical result. However, under certain additional assumptions it is possible to derive a *global result* and we cite from [83].

Theorem 18. *(Persson) Assume that $a_{(j,\beta)}$ and f and $w_0, ..., w_{m-1}$ are entire functions. Then the solution of the Cauchy-Kovalevski problem has an entire solution u provided that for those indexes (j, β) with $j + |\beta| = m$ the coefficients $a_{(j,\beta)}$ are polynomials in y of degree $\leq |\beta| = m - j$, i.e. of the form*

$$a_{j,\beta}(t, y) = \sum_{|\gamma| \leq |\beta|} a_{j,\beta,\gamma}(t) \cdot y^\gamma.$$

S. Kovalevski proved her result for *systems of non-linear* partial differential equations with analytic coefficients. In case of one equation this is usually expressed in the form

$$(18) \qquad \frac{\partial^m}{\partial t^m} u(t, y) = F\left(t, y, D_t^r D_y^\alpha\right)$$

where $F(t, y, z)$ is an analytic function and the number $r \in \mathbb{N}_0$ of derivatives $D_t := \frac{\partial}{\partial t}$ in (18) is smaller than m, and $\alpha \in \mathbb{N}_0^{d-1}$ is a multi-index with $r + |\alpha| \leq m$, so the expression on the right hand side only contains derivatives $D_t^r D_y^\alpha u$ of order at most m.

The Cauchy-Kovalevski theorem has been generalized in many respects. For example, A. Friedman [48] allows infinitely differentiable functions $F(t, y, z)$ which are real-analytic in t and z, while for the variable y weaker estimates for the derivatives are required, and the solution $u(t, y)$ is of the same type. There are many contributions to the Cauchy-Kovalevski problem (non-linear case, characteristic case, uniqueness questions beyond analytic functions, Gevrey classes)

which can be connected to the following *incomplete* list of names given in alphabetical order: M.S. Baouendi, A. Bergamasco, P. Ebenfelt, A. Friedman, L. Gårding, C. Goulaouic, T. Gramchev, Y. Hamada, L. Hörmander, K. Igari, F. John, T. Kotake, J. Leray, M. Miyake, J. Persson, H.S. Shapiro, F. Treves.

Note that an equation of the type

$$(19) \qquad a\left(t,y\right)\frac{\partial^m}{\partial t^m}u\left(t,y\right)=F\left(t,y,D_t^r D_y^\alpha\right)$$

can be reduced to the equation (18) if we assume that $a\left(t_0,y_0\right)\neq 0$, simply by dividing (19) by $a\left(t,y\right)$ and restricting the values (t,y) to a suitable small neighborhood of (t_0,y_0). The case $a\left(t_0,y_0\right)=0$ leads to many difficulties and new phenomena and it is called the *characteristic Cauchy problem* for the differential operator L and the hyperplane.

4.2. The Cauchy-Kovalevski theorem for hyper surfaces. It is straightforward to generalize the result to the important case that data are given on a hyper surface: assume that φ is a real-analytic function defined on an open set Ω and define

$$\Sigma=\left\{x\in\Omega:\varphi\left(x\right)=0\right\}.$$

If Σ is non-empty and $\nabla\varphi\left(x\right)\neq 0$ for all $x\in\Sigma$ then Σ is a hyper surface. Consider a linear differential operator of the form

$$Lu=\sum_{|\alpha|\leq m}a_\alpha\left(x\right)\frac{\partial^\alpha}{\partial x^\alpha}u.$$

where a_α are real-analytic functions on $\Omega\subset\mathbb{R}^d$. Assume that $x_0\in\Sigma$ is a given point. By a suitable change of coordinates the differential operator can be transformed to an equation of type (19), transforming the point x_0 to (t_0,y_0). In order to apply the Cauchy-Kovalevski theorem one has to ensure that $a\left(t_0,y_0\right)$ is not equal to zero. This condition can be formulated for the original equation and x_0 in the following form:

$$(20) \qquad L_m\left(x_0,\nabla\varphi\left(x_0\right)\right)\neq 0 \text{ for } L_m\left(x,\xi\right):=\sum_{|\alpha|=m}a_\alpha\left(x\right)\cdot\xi^\alpha.$$

Here L_m is called the *principal symbol of* L and we say that x_0 is *non-characteristic for* (L,Σ) if (20) holds.

Theorem 19. *Let $x_0\in\Sigma$ and L as above. Assume that f and w are real-analytic data on Ω and that x_0 is non-characteristic for (L,Σ). Then there exists a unique real-analytic function u defined on a neighborhood U of x_0 such that*

$$(21) \qquad Lu\quad\left(x\right)=f\left(x\right)\ \text{for all } x\in U,$$

$$(22) \qquad \frac{\partial^\alpha Lu}{\partial x^\alpha}\left(\xi\right)=\frac{\partial^\alpha}{\partial x^\alpha}w\left(\xi\right)\ \text{for all } \xi\in\Sigma\cap U\ \text{and }|\alpha|\leq m-1.$$

4.3. Goursat problems.

Goursat considered an initial boundary problem for the operator

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}u$$

where the initial data set Σ is defined in accordance to the differential operator:

$$\Sigma := \left\{ (x,y) \in \mathbb{R}^2 : xy = 0 \right\} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}.$$

Clearly Σ is not a hyper surface since it has a singularity at $(0,0)$, so the Cauchy-Kovalevski theorem can not be used for solving this problem. However, Goursat proved that for a given real-analytic function $f : \mathbb{R}^2 \to \mathbb{R}$ one can find a real-analytic solution u of the problem

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}u(x,y) = f(x,y) \text{ and } u(x,0) = 0 \text{ and } u(0,y) = 0 \text{ for all } x,y \in \mathbb{R}.$$

This type of problem is called the *Goursat problem*. L. Hörmander has generalized this result in his classical treatment [56]:

Theorem 20. *Let* $\gamma \in \mathbb{N}_0^d$ *be a fixed multi-index and* a_α *be real-analytic in a neighborhood of* x_0. *Then the equation*

$$L(u) := \frac{\partial^\gamma}{\partial x^\gamma}u + \sum_{|\alpha|\leq|\gamma|,\alpha\neq\gamma} a_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}u = f$$

has a unique real-analytic solution u *defined in a neighborhood of* x_0 *satisfying for* $j = 1, ..., d$:

(23) $$\frac{\partial^{m_j}}{\partial x_j^{m_j}}u \ (x_1, ..x_{j-1}, 0, x_{j+1}, ..., x_d) = 0 \text{ for all } \leq m_j < \gamma_j.$$

4.4. Mixed Cauchy problems with data on singular conics.

The boundary conditions (23) for the solution u can be expressed in the following way: the solution u of the equation $L(u) = f$ is of the form $u(x) = x^\gamma v(x)$ for some real-analytic function v. Hence Theorem 20 is equivalent to the statement that there exists a real-analytic function $v(x)$ with

$$L(x^\gamma v(x)) = f.$$

Now we replace the monomial x^γ be a general homogeneous polynomial $P(x)$ of degree $2k$. Instead of the partial differential operator $\frac{\partial^\gamma}{\partial x^\gamma}$ we consider the polyharmonic operator Δ^k. Using refined results about Fischer operators P. Ebenfelt and the author have been able to provide a proof of the following result (see [37]):

Theorem 21. *Let* $P(x)$ *be an elliptic, homogeneous polynomial of degree* $2k$. *Let* $R > 0$ *be a positive number and* $k_0 \leq k$ *a natural number and define*

$$L = \Delta^k + \sum_{|\alpha|\leq k_0} a_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}, \text{ where } a_\alpha \in A(B_R).$$

If $k_0 < k$ then for any $f \in A(B_R)$ there exists $v \in A(B_R)$ such that

$$L(P \cdot v) = f.$$

If $k_0 = k$ then there exists $r > 0$ such for any $f \in A(B_R)$ there exists $v \in A(B_r)$ such that $L(P \cdot v) = f$.

4.5. Goursat problems for the Helmholtz operator. In this Section we are dealing only with the two-dimensional case. In Theorem 21 we considered perturbations of the polyharmonic operator Δ^k and the data have been related to an *elliptic* homogeneous polynomial $P(x)$ of degree $2k$. Now we are turning to another extreme: the homogeneous polynomial $P(x)$ of degree $2k$ is a product of $2k$ lines, so it is highly non-elliptic. For simplicity let us discuss in the following only the case $k = 1$: then $P(x)$ is a product of two lines Γ_1, Γ_2. In general, the problem

(24) $$\Delta u = f \text{ and } u = g \text{ on } \Gamma_1 \cup \Gamma_2$$

does not allow unique solutions: Denote by $\theta = 2\pi\alpha$ the acute angle between Γ_1 and Γ_2. If α is rational then there exists infinitely many solutions to the problem (24) for $g = 0$ (e.g. for the case $\Gamma_1 = \mathbb{R} \times \{0\}$ and $\Gamma_2 = \{0\} \times \mathbb{R}$ we see that 0 and the function xy are solutions of the problem for $f = g = 0$). Thus for rational α the Dirichlet-type problem in (24) does not have unique solutions. However, for α irrational there exists for every polynomial data function f and g a unique polynomial u solving (24) since one might prove that the Fischer operator $q \mapsto \Delta(Pq)$ is injective, and therefore bijective. For data functions $f, g \in A(B_R)$ the question of existence of solutions $u \in A(B_R)$ is much more subtle. In recent joint work with P. Ebenfelt the following result below was obtained; the interested reader may find in [38] as well a discussion of the more difficult case of the polyharmonic operator Δ^k.

Theorem 22. *Let Γ_1, Γ_2 be two distinct lines through the origin in $\mathbb{R}^2$, and denote by $\theta = 2\pi\alpha$ the acute angle between them. Suppose that α is irrational and satisfies the condition*

(25) $$\tau := \liminf_{m \to \infty} \left(\inf_{n \in \mathbb{Z}} \left| \alpha - \frac{n}{m} \right| \right)^{1/m} > 0.$$

Then, the homogeneous Goursat problem

$$\Delta u = f \text{ and } u = g \text{ on } \Gamma_1 \cup \Gamma_2$$

has a unique solution $u \in A(B_{\tau R})$ for every $f, g \in A(B_R)$.

For the Helmholtz operator we have the following result:

Corollary 23. *Let Γ_1, Γ_2 be two distinct lines through the origin in $\mathbb{R}^2$, and denote by $\theta = 2\pi\alpha$ the acute angle between them. Suppose that α satisfies the*

Diophantine condition

$$(26) \qquad \left| \alpha - \frac{n}{m} \right| \geq \frac{C}{m^2}, \quad \textit{for all } n, m \in \mathbb{N}_0, m \neq 0$$

for some constant $C > 0$. Then, for any $c \in A(B_R)$, there exists $0 < r \leq R$ such that the Goursat problem

$$(\Delta + c)\, u = f \ \textit{and } u = g \ \textit{on } \Gamma_1 \cup \Gamma_2$$

has a unique solution $u \in A(B_r)$ for every $f, g \in A(B_R)$.

In the following we want to show that Theorem 22 is equivalent to a result of Leray in [74] who considered the homogeneous Goursat problem

$$(27) \qquad \left(\Delta + \lambda \frac{\partial^2}{\partial x \partial y} \right) u = f \text{ and } u = g \text{ on } \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$$

where λ is a real constant. The general theory of Goursat (or mixed Cauchy) problems shows that (27) has a unique real-analytic solution near 0, for all f and g, if $|\lambda| > 2$ (see Gårding [50]; see also Theorem 9.4.2 in Hörmander [56]). The case where $\lambda \in [-2, 2]$ is much more subtle, and was analyzed by Leray in [74] (see also the work of Yoshino [99], [100] for extensions to complex parameters and higher dimensions). For $\lambda \in [-2, 2]$, let $\beta \in [-1/4, 1/4]$ denote the angle such that $\lambda = 2\sin(2\pi\beta)$. Leray showed that the unique solvability of (27) depends on Diophantine properties of β. For instance, there is a unique formal power series solution u for every f and g if and only if β is irrational. Leray also gave a necessary and sufficient Diophantine condition on irrational β guaranteeing that this formal solution u converges for all convergent f and g,

$$(28) \qquad \liminf_{\mathbb{Z} \ni m \to \infty} \left(\inf_{n \in \mathbb{Z}} \left| \beta - \frac{n}{m} \right|^{1/m} \right) > 0.$$

In order to show that the result for $\lambda \in (-2, 2)$ is equivalent to Theorem 22 we consider the linear change of variables

$$(29) \qquad x \to -\sqrt{1 - \frac{\lambda^2}{4}}\, x + \frac{\lambda}{2} y.$$

This leads to the following transformation for the principal symbol of the operator

$$\lambda \frac{\partial^2}{\partial x \partial y} + \Delta \to \Delta.$$

Hence, the Goursat problem (27) is transformed into the following problem:

$$(30) \qquad \Delta u = f \text{ and } u = g \text{ on the set } x\,(x - ay) = 0$$

where

$$(31) \qquad a := \frac{\lambda/2}{\sqrt{1 - (\lambda/2)^2}}.$$

If we let $\theta = 2\pi\alpha$ denote the acute angle between the two lines $L_1 := \{y = 0\}$ and $L_2 := \{x = by\}$ and β the angle such that $\lambda := 2\sin(2\pi\beta)$, then we have

$$\alpha = \frac{1 - 2\beta}{4}.$$

Clearly, we have

$$\liminf_{\mathbb{Z} \ni m \to \infty} \left(\inf_{n \in \mathbb{Z}} \left| \beta - \frac{n}{m} \right| \right)^{1/m} = \liminf_{\mathbb{Z} \ni m \to \infty} \left(\inf_{n \in \mathbb{Z}} \left| \alpha - \frac{n}{m} \right| \right)^{1/m}.$$

This shows that Leray's result, with $\lambda \in (-2, 2)$, is equivalent to Theorem 22.

As mentioned above, we discussed in [38] the polyharmonic operator Δ^k for data given on a homogeneous polynomial consisting of $2k$ linear factors. The invertibility of the Fischer operator $q \longmapsto \Delta^k (Pq)$ on the space $\mathcal{P}_{\leq m} \left(\mathbb{R}^d \right)$ of all polynomials of degree $\leq m$ can be expressed by the requirement that certain determinants M_m do not vanish; similar results can be found in [25], see also [22]. The solvability of the equation for data functions $f, g \in A(B_R)$ depends on the asymptotic behavior of $\sqrt[m]{|M_m|}$ for $m \to \infty$.

4.6. The Dirichlet problem for general differential operators and dynamical systems.
In this Section we present some results about Dirichlet problems for a domain Ω in $\mathbb{R}^2$ for a *general differential operator of second order* and continuous data on the boundary $\partial\Omega$. We include these results in this survey because there are some fascinating similarities with the results in the last section. It should be noted that these Dirichlet problems do not represent natural problems of mathematical physics and they have a completely different character from the classical (elliptic) Dirichlet problem.

The Dirichlet problem for the vibrating string equation

$$(32) \qquad \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) u(x, y) = 0$$

is the problem to find for a continuous function $f : \partial\Omega \to \mathbb{C}$ a solution u of (32) such that $u(\xi) = f(\xi)$ for all $\xi \in \partial G$. The first systematic results are due to A. Huber in 1932 who considered the case of an rectangle

$$\Omega_{a,b} = [0, a] \times [0, b].$$

In 1939 D. G. Bourgin and R. Duffin [18] showed that the qualitative behavior of the Dirichlet problem depends on number-theoretic properties of the quotient b/a : if b/a is an irrational number then solutions for the Dirichlet problem are unique: if $u \in C^2 \left(\overline{\Omega}_{a,b} \right)$ is a solution of (32) and u vanishes on ∂G then u is identical zero. If b/a is rational then many solutions exist: the function u_n defined by

$$u_n(x, t) = \sin\left(\frac{1}{a} n\pi x \right) \sin\left(\frac{1}{a} n\pi y \right)$$

satisfies (32) and clearly vanishes if $x = 0$, $x = a$, or $y = 0$. If b/a is rational we can find infinitely many n such that nb/a is a natural number, so u_n vanishes as well for $y = b$. Thus the question of uniqueness of solutions is completely solved.

The problem of existence of solutions u for data functions f is much more subtle. A *sufficient condition* is that the number $\alpha := b/a$ has the following Diophantine property: there exist a positive constant A, a natural number K such that for all natural numbers m, n such that $m \leq 2\alpha n$ the inequality

$$\left| \alpha - \frac{m}{n} \right| > \frac{A}{n^{K+1}}$$

holds. Under this assumption there exists for any smooth data function f a solution $u \in C^2\left(\overline{\Omega}_{a,b}\right)$ of the Dirichlet problem. We refer the interested reader to [33] for a generalization to the case of n variables, and to [19], [101] for the Dirichlet problem for more general hyperbolic operators.

The above-mentioned results are proved for the rectangle and depend on classical methods from Fourier analysis. In [61] F. John introduced a completely different method which reveals a connection of this problem to *dynamical systems*. It is assumed that the boundary of $\Omega \subset \mathbb{R}^2$ is a Jordan curve and that Ω is convex in the x- and y-direction in the following sense: if L is a line parallel to the x-axis or to the y-axis then the intersection of the line with the boundary $\partial\Omega$ has at most two points. Using this property one may define a map

$$T : \partial\Omega \to \partial\Omega$$

in the following way: given a point $P \in \partial\Omega$ there exists by our assumption a unique point $AP \in \partial\Omega$ which has the same abscissa as P. For AP we can find a unique point $Q \in \partial\Omega$ which the same ordinate as AP, and we define finally $T(P) = Q$. The uniqueness for the Dirichlet problem is now connected to properties of the transformation T. We recall that P is a periodic point of T if there exists a natural number n such that $T^n P = P$.

Theorem 24. *Let $\Omega \subset \mathbb{R}^2$ be convex in the x- and y-direction and $\partial\Omega$ a Jordan curve. Then the solution for the Dirichlet problem*

$$(33) \qquad \frac{\partial^2}{\partial x \partial y} u(x, y) = 0 \text{ and } u \mid \partial\Omega = 0$$

is uniquely determined in the space $C^2\left(\overline{\Omega}\right)$ if the set of all periodic points of the transformation T is either finite or denumerable.

In the interesting paper of V.P. Burskiĭ and A.S. Zhedanov [27] (see also [26]), the transformation T is called the *John mapping*. The interested reader can find there a deep analysis of the Dirichlet problem for the hyperbolic operator (33) for domains Ω whose boundary $\partial\Omega$ is given by a biquadratic algebraic curve

$$F(x, y) = \sum_{k,j=0}^{2} a_{k,j} x^k y^j.$$

For a discussion of the Dirichlet problem for non-linear wave equations we refer to [16] and the references given there.

5. The conjecture of Khavinson and Shapiro

The reader can find an excellent survey about the Khavinson-Shapiro conjecture in [67] which is illustrated by many heuristic motivations and illuminating examples. Our presentation emphasizes the connection to Fischer decomposition methods. Let us recall that the Khavinson-Shapiro conjecture says that for a bounded domain Ω condition (KS) implies that Ω is an ellipsoid where

(KS) For any polynomial f the solution u_f of the Dirichlet problem for $f \mid \partial\Omega$ is a polynomial.

It is not difficult to see that it suffices to show that the boundary $\partial\Omega$ is contained in the zero-set of a polynomial $P(x)$ of degree 2 using the classification of conical sections.

It is a well-known fact that condition (KS) implies that the boundary $\partial\Omega$ is contained in an algebraic set. We include the short proof:

Lemma 25. *Suppose that the data function $|x|^2$ for the Dirichlet problem of domain Ω has a polynomial harmonic solution $u(x)$. Then $\partial\Omega$ is contained in the zero set of $Q(x) := |x|^2 - u(x)$.*

Proof. By assumption, there exists a harmonic polynomial u such that $u(\xi) = |\xi|^2$ for all $\xi \in \partial\Omega$. Then $Q(\xi) := |\xi|^2 - u(\xi) = 0$ for $\xi \in \partial\Omega$, and

(34) $$\partial\Omega \subset Q^{-1}(\{0\}).$$

This completes the proof. ■

It is important to note that the inclusion (34) is in general proper: consider for example the rectangle $R := [0,1] \times [0,1]$. Then the boundary ∂R is properly contained in the zero set of

$$Q(x,y) = x(x-1)y(y-1).$$

This examples shows as well that the set $\mathbb{R}^d \setminus P^{-1}(0)$ decomposes into several connected components, so one can associate *different* domains to *one* polynomial $P(x)$. In contrast to complex algebraic geometry, the zero set $P^{-1}(0) := \{x \in \mathbb{R}^d : P(x) = 0\}$ of an *irreducible* polynomial $P(x)$ is in general not connected, for examples see [17] or [75].

5.1. The Khavinson-Shapiro conjecture and polynomial decompositions. Lemma 25 tells us that we may assume that the boundary $\partial\Omega$ of the domain Ω is contained in the zero set of a non-constant polynomial P with real coefficients. Hence there exist irreducible polynomials $\psi_1, ..., \psi_r$ in $\mathbb{R}[x]$ and natural numbers $m_1, ..., m_r$ and a constant $C \neq 0$ such that

$$P(x) = C\psi_1^{m_1}....\psi_r^{m_r}.$$

For the inclusion (34) we may assume that $m_1 = ... = m_r = 1$, so one may assume that ψ_j is not a scalar multiple of ψ_k for $k \neq j$. But we have also to guarantee that each factor ψ_j really contributes to the description of the boundary $\partial\Omega$, so we have to disregard those factors which have non-empty intersection with the boundary. We can achieve this by requiring that there exists open balls U_j such that

$$(35) \qquad \emptyset \neq \left\{x \in \mathbb{R}^d : \psi_j(x) = 0\right\} \cap U_j \subset \partial\Omega.$$

Still it might happen that the intersection in (35) is just one point. In order to guarantee that the intersection has many points we shall require that ψ_j *changes sign* over U_j which means that there exist

$$x \neq y \in U_j \text{ such that } \psi_j(x) < 0 < \psi_j(y).$$

Now are ready to connect the conjecture of Khavinson-Shapiro with a question which is purely formulated in terms of polynomial decompositions:

Theorem 26. *Let Ω be a domain in $\mathbb{R}^d$. Let $P \in \mathbb{R}[x]$ be of the form $P = \psi_1....\psi_r$ such that $\psi_1, ..., \psi_r$ are irreducible and $\psi_j \neq c\psi_k$ for $j \neq k$. Suppose that for every $j = 1, ..., r$ there exists an open ball U_j such that*

$$(36) \qquad \left\{y \in \mathbb{R}^d : \psi_j(y) = 0\right\} \cap U_j \subset \partial\Omega$$

and ψ_j changes sign over U_j for $j = 1, ..., r$. Then condition (KS) implies that for any polynomial f there exist polynomials q_f and u_f with

$$(37) \qquad f = Pq_f + u_f \text{ and } \Delta u_f = 0.$$

Proof. Let f be a polynomial. By assumption there exists a harmonic polynomial u_f such that $u_f(\xi) = |\xi|^2$ for $\xi \in \partial\Omega$. Then $Q(x) := f(x) - u_f(x)$ is zero over $\{x \in U_j : \psi_j(x) = 0\}$ for each $j = 1, ..., r$. A theorem in real algebraic geometry [17, Theorem 4.5.1] (using the assumption that ψ_j is irreducible and changes sign) tells us that $Q = \psi_j \cdot f_j$ some polynomial f_j. Hence there exists a polynomial q such that

$$Q = \psi_1....\psi_r q = Pq.$$

Since $Q = f - u_f$ we infer $f = Pq + u_f$.   ■

Now we confirm the Khavinson-Shapiro conjecture for a large class of domains:

Theorem 27. *Let Ω be a domain in $\mathbb{R}^d$ and let $P \in \mathbb{R}[x]$ be of the form $P = \psi_1....\psi_r$ such that $\psi_1, ..., \psi_r$ are irreducible and $\psi_j \neq c\psi_k$ for $j \neq k$. Suppose that for every $j = 1, ..., r$ there exists an open ball U_j such that*

$$\left\{y \in \mathbb{R}^d : \psi_j(y) = 0\right\} \cap U_j \subset \partial\Omega$$

and ψ_j changes sign over U_j for $j = 1, ..., r$. Assume that $\deg \psi > 2$ and that the leading term of ψ contains a non-negative non-constant factor. Then the data function $|x|^2$ does not have a polynomial solution for the Dirichlet problem.

Proof. Suppose that the function $|x|^2$ has a polynomial solution. By Theorem 26 there exist polynomials q and u such that $|x|^2 = Pq + u$ where u is harmonic. Since $|x|^2$ is not harmonic it follows that $q \neq 0$, and clearly $2d = \Delta(Pq)$. Expand q and P into sums of homogeneous polynomials with leading parts $q_m \neq 0$ and $P_s \neq 0$. Since $\deg P \geq 3$ and $2d = \Delta(Pq)$ it follows that $\Delta(P_s q_m) = 0$. Thus $P_s q_m$ is harmonic. By Corollary 12 we infer that $P_s q_m$ must be zero. This contradiction completes the proof. ∎

As an example, consider the square $\Omega := (-1, 1) \times (-1, 1)$ in $\mathbb{R}^2$ and $P(x, y) = (x - 1)(x + 1)(y - 1)(y + 1)$. Clearly the leading part of P is non-negative. Since $\deg P = 4$ it follows from Theorem 27 that the solution of the Dirichlet problem for the data function $x^2 + y^2$ can not be a polynomial. Similarly, for $k > 1$,

$$\Omega_k := \left\{(x_1, ..., x_d) \in \mathbb{R}^d : x_1^{2k} + ... + x_d^{2k} < 1\right\}$$

is a domain for which the data function $|x|^2$ does not have a polynomial solution. For $d = 2$ this result was proved by P. Ebenfelt in [35] by different methods.

M.L. Agranovsky and Y. Krasnov introduced in [1] the concept of a harmonic divisor which arises naturally in the investigation of stationary sets for the wave and heat equation [2], and the injectivity of the spherical Radon transform [3]. We say that a polynomial P is a *harmonic divisor* if there exists a polynomial $q \neq 0$ such that Pq is harmonic. Analyzing again the proof of Theorem 27 we see that it is sufficient in Theorem 27 to assume that the leading term P_s is *not* a harmonic divisor. We used the assumption of a non-negative non-constant factor of the leading term in order to conclude via Corollary 12 that the leading term is not a harmonic divisor.

Theorem 27 was proved in [86] and generalizes a result of Chamberland and Siegel in [28] for the two dimensional case.

E. Volkov has shown in [94] that for a domain in $\mathbb{R}^2$ whose boundary is a polygon with more than 3 edges the function $x_1^2 + x_2^2$ does not have a polynomial solution. In [95] the author discusses the case of a polygonal domain whose edges consists of algebraic curves.

5.2. Degree preserving polynomial decompositions. It is an interesting fact in Theorem 27 that we can specify an explicit function f, namely

$$f(x) = |x|^2,$$

for which the solution u of the Dirichlet is not a polynomial. It is tempting to conjecture that the condition (KS) is satisfied if we know that the function $|x|^2$ has a polynomial solution for the Dirichlet problem. Unfortunately this is not true as the following example of L.J. Hansen and H.S. Shapiro [54, p. 125] shows (it also shows that Theorem 27 does not hold if we omit the assumption that the leading term is not a harmonic divisor):

Example 5.2: Let $\varphi \in \mathbb{R}[x]$ be a harmonic polynomial on $\mathbb{R}^d$. For $\varepsilon > 0$ define

(38) $$P_\varepsilon(x) := |x|^2 - 1 + \varepsilon\varphi(x).$$

If $\varepsilon > 0$ is small enough then $P_\varepsilon(0) < 0$ and positive on $|x| = 2$. Then the connected component Ω_ε of the open set $\{P_\varepsilon < 0\}$ containing the point 0 is a bounded domain in $\mathbb{R}^d$. The Dirichlet problem for the data function $|x|^2$ has the harmonic polynomial solution $u_f(x) = 1 - \varepsilon\varphi(x)$ since

$$|x|^2 = P_\varepsilon(x) \cdot 1 + 1 - \varepsilon\varphi(x).$$

Note that the degree of the solution $u_f = 1 - \varepsilon\varphi(x)$ is indeed larger than the degree of the data function $|x|^2$. On the other hand we shall see that condition (KS) is not satisfied for certain polynomials φ, see the arguments at the end of this Section.

In view of Theorem 26 it is natural to consider the following conjecture, see [76]:

Conjecture A: Let $P \in \mathbb{R}[x]$ be a polynomial with real coefficients such that for any polynomial $f \in \mathbb{R}[x]$ there exist polynomials q_f and u_f in $\in \mathbb{R}[x]$ with $f = Pq_f + u_f$ and $\Delta u_f = 0$. Then $\deg P \le 2$.

In joint work with E. Lundberg [76] we have been able to prove the conjecture A if we add a degree condition on the involved polynomials:

Theorem 28. *Let P be a polynomial of degree ≥ 2. Suppose that there exists a constant $C > 0$ such that for any polynomial $f \in \mathbb{R}[x]$ there exists a decomposition $f = Pq_f + u_f$ with $\Delta u_f = 0$ and*

(39) $$\deg u_f \le \deg f + C.$$

Then $\deg(P) = 2$.

In [76] we gave a criterion such that the degree condition (39) is automatically satisfied:

Theorem 29. *Suppose that P is a polynomial of degree $k > 2$ such that the decomposition into homogeneous polynomials $P = P_k + P_s + P_{s-1} + ... + P_0$ with $P_k \ne 0$ has the property that the second non-zero summand P_s of degree s contains*

a non-negative non-constant factor. Let f be a polynomial and assume that there exists a decomposition

$$f = Pq + u$$

where h is harmonic and q is a polynomial. Then $\deg u \leq \deg f + (k - s + 2)$.

Let us return to Example 5.2: we see that the leading term of P_ε is equal to $P_k = \varepsilon\varphi$, and the second non-zero summand is $P_s(x) = |x|^2$, so it is non-negative. Assume further that φ is a homogeneous harmonic polynomial of degree 3 such that P_ε is irreducible. By Theorem 26, Theorem 29 and Theorem 28 it follows that property (KS) is not satisfied, although we can find for $|x|^2$ a polynomial solution of the Dirichlet problem.

5.3. The conjecture of Khavinson-Shapiro with entire solutions. In the introduction we have explained the significance of the following condition:

(KSe) For any polynomial f the solution u_f of the Dirichlet problem for $f \mid \partial\Omega$ has an extension to a holomorphic function on $\mathbb{C}^d$.

The second conjecture of Khavinson-Shapiro states that for a bounded domain Ω, (KSe) implies that Ω is an ellipsoid (or contained in the zero set of a polynomial of degree 2).

In case of condition (KS) we concluded easily that the boundary of the domain must be contained in an algebraic set. In case of (KSe) we can only infer that the boundary is contained in an real-analytic set.

The proof of Theorem 27 uses in an essential way the fact that we were dealing with polynomials. However, the proof of Theorem 26 can be extended to the setting of entire functions, or what is equivalent, to the setting of the algebra $A(B_R)$ for $R = \infty$. Roughly speaking, the fact that the Fischer decomposition is unique in $A(B_\infty)$ will be the essential argument. However, this fact, proven in [86], is much more difficult to prove than the corresponding result in the polynomial case: it requires a series of technical estimates and the assumption of an *elliptic* polynomial $P(x)$.

Theorem 30. *Let Ω be a domain in $\mathbb{R}^d$ and let $P \in \mathbb{R}[x_1, ..., x_d]$ be of the form $P = \psi_1....\psi_r$ such that $\psi_1, ..., \psi_r$ are irreducible and $\psi_j \neq c\psi_l$ for $j \neq l$. Suppose that for every $j = 1, ..., r$ there exists an open ball U_j such that*

$$\{y \in \mathbb{R}^d : \psi_j(y) = 0\} \cap U_j \subset \partial\Omega$$

and ψ_j changes sign over U_j for $j = 1, ..., r$. If $\deg P > 2$ and $P = \psi_1...\psi_r$ is elliptic then there is no entire solution of the Dirichlet problem for the data function $|x|^2$ restricted to $\partial\Omega$.

Proof. Suppose that the function $|x|^2$ has an entire solution. Then there exists a harmonic function $u \in A(B_R)$ for $R = \infty$ such $Q(x) := |x|^2 - u(x)$ vanishes

on $\partial\Omega$. Using the assumptions about $\psi_1, ..., \psi_r$ we infer as in the proof Theorem 26 that there exist q such that $Q = Pq$; the critical reader may observe that Q is not a polynomial but a function in $A(B_R)$, and that q will be an element in $A(B_R)$; however, the necessary modifications for the proof are sort of mathematical folklore (see [86]) and we conclude that

$$Pq - |x|^2 + u(x) = 0.$$

Hence for $k := \frac{1}{2}\deg P > 1$ and $r(x) = -|x|^2 + u(x)$ we obtain $\Delta^k r = 0$ and $0 = Pq + r$. The uniqueness property in Theorem 16 implies that $r = 0$ and $q = 0$. Thus $|x|^2 = u(x)$, a contradiction to the harmonicity of u. ∎

Now let us summarize further results in the literature concerning the *second* Khavinson-Shapiro conjecture. To the best knowledge of the author, results have been achieved only for the *two-dimensional* case. Methods based on the Schwarz function have been used by P. Ebenfelt [35] to discuss the behavior of singularities of the Dirichlet problem for quadrature domains or domains which are bounded by k-th root of an ellipse. In contrast to our Theorem 30 he obtains in his deep work an explicit description of the singularities.

Although the Khavinson-Shapiro conjecture is a statement in *real analysis* there is a close connection to *complex analysis*, at least for the two-dimensional case. We illustrate this by a result proven by L. Hansen and H.S. Shapiro in [54]: assume that γ is a curve in the plane $\mathbb{R}^2$ defined by

$$P(x, y) = 0,$$

where P is an irreducible polynomial. We turn P into a polynomial in the variables z and $\overline{z}$ replacing x by $(z + \overline{z})/2$ and y by $(z - \overline{z})/2i$, so that

$$P(x, y) = \widetilde{P}(z, \overline{z})$$

for a suitable polynomial $\widetilde{P}$. Now substitute the variable $\overline{z}$ by w. We say that the curve γ *contains a rectangle* if there exists four *distinct* points $(z_j, w_k) \in \mathbb{C}^2$, $j, k \in \{1, 2\}$, so that $\widetilde{P}(z_j, w_k) = 0$ for $j, k \in \{1, 2\}$. In [54] the following is proved:

Theorem 31. *If a curve γ (defined by an irreducible polynomial P) contains a rectangle, and is the boundary of the bounded region Ω, then the solution of the Dirichlet problem on Ω with boundary data $|x|^2$ cannot be extended to be harmonic on all of $\mathbb{R}^2$.*

The last result can be extended to the framework of so-called *complex lightning bolts* and we refer the interested reader to the work of E. Lundberg [75] for more details and instructive examples of domains which are not covered by Theorem 30. In passing we mention that lightning bolts were used by Kolmogorov and Arnold to solve Hilbert's 13th problem regarding the solution of 7th degree equations using functions of two parameters, see [75].

We mention the following result in [54] which relates conjecture (KS) to conjecture (KSe):

Theorem 32. *Suppose that the domain $\Omega \subset \mathbb{R}^2$ is bounded by a curve Γ with the property that there exists a non constant entire function $F : \mathbb{C} \to \mathbb{C}$ mapping the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ into Γ. Suppose that the polynomial $f(x, y)$ of degree m has a solution u_f on Ω to the Dirichlet problem with boundary data f which extends to harmonic function on the entire space $\mathbb{R}^2$. Then $u_f(x, y)$ is a polynomial of degree $\leq m$.*

The paper [54] discovers also a connection between certain functional equations for entire functions and the property (KSe), see also [46].

5.4. Recurrence relations for planar orthogonal polynomials and the Khavinson-Shapiro conjecture. Recently M. Putinar and N.S. Stylianopoulos found a nice link between the Khavinson-Shapiro conjecture and recurrence relations of Bergman orthogonal polynomials in the complex plane. Let us introduce some notations and definitions:

Let Ω be a bounded domain in the complex plane $\mathbb{C}$ which will be identified with $\mathbb{R}^2$ (so we can speak about the Dirichlet problem for Ω). Define for polynomials $f(z)$ and $g(z)$ of a complex variable z the inner product

$$(40) \qquad \langle f, g \rangle_\Omega := \int_\Omega f(z) \overline{g(z)} dA(z)$$

where dA stands for the area measure. The *Bergman orthogonal polynomials* $p_n(z)$ of degree n are defined as the polynomials

$$p_n(z) = \gamma_n z^n + \gamma_{n-1} z^{n-1} + ... + \gamma_1 z + \gamma_0$$

with $\gamma_n > 0$ which are orthonormal with respect to the inner product (40). We say that the orthogonal polynomials p_n satisfy *a recurrence relation of order $N + 1$* if for each $n \in \mathbb{N}$, $n \geq N + 1$, there exists real numbers $a_{n+1,n}, ..., a_{n-N+1,n}$ such that

$$z \cdot p_n(z) = a_{n+1,n} p_{n+1}(z) + a_{n,n} p_n(z) + ... + a_{n-N+1,n} p_{n-N+1}(z).$$

For example, if $N = 2$, then the orthogonal polynomials p_n satisfies a recurrence relation of order 3 if for each n there exist $a_{n+1,n}, a_{n,n}$ and $a_{n-1,n}$ such that

$$(41) \qquad z \cdot p_n(z) = a_{n+1,n} p_{n+1}(z) + a_{n,n} p_n(z) + a_{n-1,n} p_{n-1}(z).$$

We say that a sequence of Bergman orthogonal polynomials satisfy a *finite recurrence relation* if for each fixed $k \geq 0$ there exists $N(k) \geq 0$ such that

$$\langle z \cdot p_n, \ p_k \rangle = 0 \text{ for all } n \geq N(k).$$

Let us denote by $L_a^2(\Omega)$ the space of all analytic functions $f : \Omega \to \mathbb{C}$ which are square integrable with respect to area measure. In the following it is always

assumed that polynomials are dense in $L_a^2(\Omega)$. This is e.g satisfied if the interior of $\overline{\Omega}$ is contained in Ω.

M. Putinar and N.S. Stylianopoulos established in [84] the following result:

Theorem 33. *Let Ω be a simply connected bounded domain in $\mathbb{C}$ such that the polynomials are dense in $L_a^2(\Omega)$. Let $N_0 \geq 2$ be a natural number such that*

$$(42) \qquad \langle z \cdot p_n(z), 1 \rangle_\Omega = 0 \text{ for all } n \geq N_0.$$

Then for the polynomial $|x|^2 = x_1^2 + x_2^2$ there exists a harmonic polynomial $u(x)$ of degree $\leq N_0$ such that

$$(43) \qquad |\xi|^2 = u(\xi) \text{ for all } \xi \in \partial\Omega.$$

If the orthogonal polynomials satisfy a recurrence relation of order 3, see (41), then (42) is satisfied for all $n \geq 2$; thus we conclude from Theorem 33 with $N_0 := 2$ that $\partial\Omega$ is contained in the zero-set of a polynomial of degree ≤ 2. Since Ω is bounded we conclude that Ω is an ellipse. Thus we obtain:

Theorem 34. *Let Ω be a simply connected bounded domain in $\mathbb{C}$ such that the polynomials are dense in $L_a^2(\Omega)$. If the Bergman orthogonal polynomials satisfy a recurrence relation of order 3 then Ω is an ellipse.*

With the same methods one obtains the following interesting result:

Theorem 35. *Let Ω be a simply connected bounded domain in $\mathbb{C}$ such that the polynomials are dense in $L_a^2(\Omega)$. Then the Bergman orthogonal polynomials satisfy a finite-term recurrence relation if and only if the condition (KS) holds for Ω.*

Thus one may use results about the Khavinson-Shapiro conjecture for proving the non-existence of finite-term recurrence relations of Bergman orthogonal polynomials or vice versa. Recurrence relations of order $N+1$ can be characterized in a similar way:

Theorem 36. *Let Ω be a simply connected bounded domain in $\mathbb{C}$ such that the polynomials are dense in $L_a^2(\Omega)$. The the following statements are equivalent for a given natural number $N \geq 2$:*

1. *The Bergman orthogonal polynomials $p_n(z)$ satisfy a recurrence relation of order $N+1$.*
2. *For all $m, n \in \mathbb{N}_0$ the Dirichlet problem with data function $\overline{z}^m z^n$ has a polynomial solution of degree $\leq m(N-1) + n$ in z and of degree $\leq n(N-1) + m$ in $\overline{z}$.*

Using results about the asymptotic of orthogonal polynomials $p_n(z)$ and some results about quadrature domains it is proved in [71] that condition 1 in Theorem 36 implies that Ω is an ellipse and $N = 2$ under the additional assumption that Ω

has a C^2-smooth Jordan boundary $\partial\Omega$. Thus the Khavinson-Shapiro conjecture is true in $\mathbb{R}^2$ if the domain has a C^2-smooth Jordan boundary $\partial\Omega$ (without cusps) and the degree of the polynomial solution u_f of the Dirichlet problem depends in a nice way on the degree of f.

Similar results hold for Hardy spaces and for Szegö orthogonal polynomials and we refer the reader to [84], [71], [67] and [32] for more information and for a description on the history of the subject refering to the work of P. Duren [34] in 1965.

6. The Schwarz potential conjecture

Let f be a real analytic function of two variables x, y and define

(44) $$\Gamma = \left\{ (x, y) \in \mathbb{R}^2 : f(x, y) = 0 \right\}.$$

If the gradient of f does not vanish on Γ we call Γ a real-analytic hyper surface. Setting $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$ where $z = x + iy$ we can rewrite (44) as

$$\Gamma = \{ z \in \mathbb{C} : F(z, \overline{z}) = 0 \}$$

where F is a suitable real analytic function of z and $\overline{z}$. Then $\partial F/\partial\overline{z}$ does not vanish on Γ, and by the implicit function theorem we can solve the equation $F(z, \overline{z}) = 0$, obtaining an analytic function $S(z)$ in a neighborhood of Γ such that

$$S(z) = \overline{z} \text{ for } z \in \Gamma.$$

In [29] the function S is called the *Schwarz function* and the reader will find there a detailed account, many examples and applications to various areas in complex function theory.

In [69] and [65] D. Khavinson and H.S. Shapiro introduced a Schwarz function in the context of several real variables and with respect to a real analytic hyper surface Γ in $\mathbb{R}^d$ and they showed that many features of the classical theory can be extended to this setting.

Definition 1. Let Ω be a domain in $\mathbb{R}^d$ such that the boundary $\partial\Omega$ is a real analytic hyper surface. The Schwarz potential is defined as the solution u (defined in a neighborhood of the boundary $\partial\Omega$) of the Cauchy problem

(45) $$\Delta u = 0$$

(46) $$u(\xi) = |\xi|^2 \text{ for all } \xi \in \partial\Omega,$$

(47) $$\frac{\partial u}{\partial x_j}(\xi) = 2\xi_j \text{ for all } \xi = (\xi_1, ..., \xi_d) \in \partial\Omega \text{ and } j = 1, ..., d.$$

One outstanding conjecture in this area is the Schwarz potential conjecture:

- Any solution u of the Laplace equation $\Delta u = 0$ with entire Cauchy data function f and initial conditions $u(\xi) = f(\xi)$ and $\frac{\partial u}{\partial x_j}(\xi) = \frac{\partial f}{\partial x_j}(\xi)$ for $\xi \in \partial\Omega$ and $j = 1, ..., d$ can be analytically continued as far as the Schwarz potential can be continued.

In [69] and [65] one can find explicit computations for the Schwarz potential of basic surfaces as planes, spheres and cylinders for which the Schwarz potential conjecture can be confirmed directly. For example, Khavinson has proved in an elementary way the following result in [66]:

Theorem 37. *Let* $\Omega = \{x \in \mathbb{R}^d : \sum_{j=1}^d x_j^2 < 1\}$ *be the unit ball. If* $f : \mathbb{R}^d \to \mathbb{C}$ *is entire then there exists a harmonic function* $u : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ *such that*

$$u(\xi) = f(\xi) \ \ for \ all \ \xi \in \partial\Omega,$$

$$\frac{\partial u}{\partial x_j}(\xi) = \frac{\partial f}{\partial x_j}(\xi) \ \ for \ all \ \xi \in \partial\Omega \ and \ j = 1, ..., d.$$

The theorem tells us that the Schwarz potential of the ball can be extended to the space $\mathbb{R}^d \setminus \{0\}$, and and it tells us that this is also true for every solution of the Cauchy problem for any entire data function; in particular the Schwarz potential conjecture holds for the ball. More generally, G. Johnsson has confirmed the conjecture for any surface given by a quadratic polynomials, see [63].

The general idea behind the Schwarz potential conjecture is that one needs to test only one particular data function, namely $|x|^2$, in order to understand the location of the singularities of the solutions u of the Cauchy problem for arbitrary entire data, or at least for all polynomial data.

The technique of Fischer decomposition gives only a slight reduction in the case that the boundary of the domain Ω in $\mathbb{R}^d$ is algebraic, i.e. that there exists a polynomial ψ of degree k with

$$(48) \qquad\qquad\qquad \partial\Omega \subset \psi^{-1}\{0\}.$$

If f is a polynomial data function we can use Theorem 13 for the polynomial $P := \psi^2$ which has clearly non-negative leading part. Thus we can write

$$(49) \qquad\qquad\qquad f = Pq + r = \psi^2 q + r$$

with $\Delta^{2k} r = 0$ where q and r are polynomials. From (48) and (49) it follows that $f(\xi) = r(\xi)$ for all $x \in \partial\Omega$ and $\frac{\partial f}{\partial x_j}(\xi) = \frac{\partial r}{\partial x_j}(\xi)$ for all $\xi \in \Omega$ and $j = 1, ..., d$. Thus it suffices to solve the Cauchy problem for the data function r instead of f. In other words, we may assume that the data function f already satisfies the polyharmonic equation $\Delta^{2k} r = 0$.

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