Modelling the slight compressibility of anisotropic soft tissue

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Abstract

In order to avoid the numerical difficulties in globally enforcing the incompressibility constraint using the displacement formulation of the Finite Element Method, slight compressibility is typically assumed when simulating transversely isotropic, soft tissue. The current standard method of accounting for slight compressibility assumes an additive decomposition of the strain-energy function into a volumetric and a deviatoric part. This has been shown, however, to be inconsistent with the linear theory. It is further shown here that, under hydrostatic tension or compression, a transversely isotropic cube modelled using this additive split is simply deformed into another cube, in contravention of the physics of the problem. A remedy for these defects is proposed here: the trace of the Cauchy stress is assumed linear in both volume change and fibre stretch. The general model consistent with this model is obtained and is shown to be a generalisation of the current standard method. A specific example is used to clearly demonstrate the differences in behaviour between the two models in hydrostatic tension and compression.

Keywords: anisotropic materials, soft tissue, slight compressibility.

1 Introduction

The assumption of perfect incompressibility has been central to the successful application of the theory of non-linear elasticity to practical problems. First formulated and incorporated into the theory by Rivlin [1], incompressibility has enabled a number of boundary values problems that have a direct physical relevance for non-linear elastic materials to be solved exactly. These solutions have played an essential role in the modelling of elastomeric materials, in particular. Rivlin's theory has also recently been widely applied to the mechanical response of biological, soft tissue, the materials of most interest here. The assumption of incompressibility for soft tissue is motivated primarily by the high water content of soft tissue (Vito and Dixon [2]). Experimental data supporting this hypothesis is limited, with only the work of Carew *et al.* [3] suggesting that, for most practical purposes, arterial tissue can be considered to be incompressible.

These solutions are of only limited value when simulating the mechanical response of biological soft tissue *in situ* because of the typical complexity of the geometry of the problem and the corresponding boundary conditions. Therefore in order to obtain realistic predictions of this mechanical response, numerical simulations are a necessity, with the Finite Element Method (FEM) being the standard numerical method of choice. There is a practical difficulty, however, with globally enforcing the incompressibility constraint using the usual displacement formulation of FEM. A common solution to this problem is to assume instead that soft tissue is slightly compressible. This, however, only has the effect of transferring a problem with the numerical implementation of FEM into constitutive problem as there is no generally accepted method of modelling slight compressibility. The most popular method is to additively decompose the strain-energy function, assuming hyerelasticity, into a volumetric and a deviatoric part. This decomposition is usually introduced without motivation, although exceptionally Weiss *et al.* [4] do acknowledge that the chief motivation is mathematical convenience rather than physics and certainly there is no experimental evidence ever advanced to support this key assumption.

There, however, is some theoretical support for this decomposition [5, 6, 7]: the additive decomposition of a strain-energy function into volumetric and deviatoric parts is equivalent to assuming that the hydrostatic Cauchy stress, defined as the sum of the principal stresses, is a function only of the volume change of the material. But even this argument only seems convincing for *isotropic* materials; certainly one would expect that the hydrostatic Cauchy stress to depend also on the orientation of the fibres if anisotropic soft tissue was being considered. It seems that the additive decomposition that seems reasonable for isotropic materials was simply adopted for anisotropic materials without proper consideration of the consequences. Some seemingly unreasonable consequences of this assumption have been noted recently. Helfenstein *et al.* [8] noted an unphysical response of growing lateral stretches in simulations of uniaxial stress experiments based on the additive split. More fundamentally, Vergori *et al.* [9] have showed that the linear theory is not fully recoverable from the non-linear theory based on the usual decomposition, with the result, for example, that transversely isotropic cubes remain cubic under hydrostatic tension for infinitesimal strains. It is further shown here that this unphysical property of the standard model is characteristic of these materials for *all* strains.

To remedy these deficiencies for the current standard model of slight compressibility, it is proposed here that the hydrostatic Cauchy stress is a natural quantity about which constitutive assumptions should be made when modelling slight compressibility due to the primary importance of the experiment in which the faces cuboid specimens are subjected to the same traction. There is some limited experimental evidence due to Penn [10] for vulcanised natural rubbers to suggest that the hydrostatic Cauchy stress is linear in the volume change; this is essentially the assumption that is usually made implicitly in the standard model of slight compressibility. In the absence of experimental data to suggest otherwise, it will also be assumed that the hydrostatic Cauchy stress is also linear in the fibre stretch, an assumption based on analogy with the volume change assumption and on the basis that in typical experiments the fibre change is likely to be infinitesimal and therefore well approximated by the linearity assumption. Assuming an additive dependence on both volume change and fibre stretch for the hydrostatic Cauchy stress leads to a model that recovers the standard model of compressibility as a special case. This is a key advantage of the method proposed here as it allows many of the previously developed models to be directly incorporated within the new approach. An additional essential feature of the method proposed here is that the linear theory can be fully recovered on restriction to infinitesimal deformations.

There is a serious and compelling need for the accurate representation of small volume changes in anisotropic soft tissue because numerical simulations of soft tissue are now an essential tool in the solution of some serious medical problems. One pressing problem, for example, is the design and development of protective systems to reduce diffuse axonal injury due to shear strains produced at the moment of injury to the head (Johnson *et al.* [11]); a mathematical model of slight compressibility that's fit for purpose is an essential requirement for such an endeavour. It is hoped that the analysis presented here will be a contribution towards the establishment of a rational model of slight compressibility that has universal acceptance.

2 Preliminaries

Let $\boldsymbol{F} \equiv \partial \boldsymbol{x}/\partial \boldsymbol{X}$ denote the deformation gradient tensor, with $J \equiv det\, \boldsymbol{F}$ and $\boldsymbol{B}, \boldsymbol{C}$ the left and right Cauchy-Green strain tensors respectively, which are therefore positive definite and symmetric. The corresponding principal strain invariants are defined by

$$I_1 = tr(\boldsymbol{B}), \quad I_2 = \frac{1}{2}\left[I_1^2 - tr\left(\boldsymbol{B}^2\right)\right], \quad I_3 = det(\boldsymbol{B}) = J^2,$$

and are therefore positive. Consider now a transversely anisotropic, non-linearly elastic material with a preferred direction $\boldsymbol{M}$ in the undeformed configuration. The so-called pseudo-invariants are defined by

$$I_4 = \boldsymbol{M}.\boldsymbol{C}\boldsymbol{M}, \quad I_5 = \boldsymbol{M}.\boldsymbol{C}^2\boldsymbol{M}. \tag{2.1}$$

The constitutive law for compressible, homogeneous, transversely anisotropic, non-linearly hyperelastic materials is given by (Ogden [12])

$$\begin{aligned} J\boldsymbol{\sigma} = 2W_1\boldsymbol{B} + 2W_2\left(I_1\boldsymbol{B} - \boldsymbol{B}^2\right) + 2I_3W_3\boldsymbol{I} + 2W_4\boldsymbol{F}\boldsymbol{M} \otimes \boldsymbol{F}\boldsymbol{M} + \\ 2W_5\left(\boldsymbol{F}\boldsymbol{M} \otimes \boldsymbol{B}\boldsymbol{F}\boldsymbol{M} + \boldsymbol{B}\boldsymbol{F}\boldsymbol{M} \otimes \boldsymbol{F}\boldsymbol{M}\right), \end{aligned} \tag{2.2}$$

where $\boldsymbol{\sigma}$ denotes the Cauchy stress and $W = W\left(I_1, I_2, I_3, I_4, I_5\right)$ is the strain-energy function per unit undeformed volume with attached subscripts denoting partial differentiation with respect to the appropriate principal strain invariant or pseudo-invariant. To ensure that the stress is identically zero in the undeformed configuration, it will be required that

$$W_1^0 + 2W_2^0 + W_3^0 = 0, \quad W_4^0 + 2W_5^0 = 0, \tag{2.3}$$

where the 0 superscript denotes evaluation at $I_1 = I_2 = 3$, $I_j = 1, j = 3, 4, 5$. It will also be assumed that the strain-energy vanishes in the undeformed configuration, i.e., that

$$W^0 = 0. \tag{2.4}$$

The preferred direction is usually physically induced by the presence of fibres embedded in an elastic matrix and we will assume this here. The trace operator will play an important role in motivating the new models of slight compressibility introduced here. Taking the trace of both sides in (2.2) yields

$$I_3^{1/2} \operatorname{tr} \boldsymbol{\sigma} = 2I_1W_1 + 4I_2W_2 + 6I_3W_3 + 2I_4W_4 + 4I_5W_5. \tag{2.5}$$

The preferred direction $\boldsymbol{M}$ in the undeformed configuration is transformed into the vector $\boldsymbol{FM}$ in the deformed configuration. Since $I_4 = \boldsymbol{M.CM} = \boldsymbol{FM.FM}$, I_4 is therefore the squared stretch of line elements aligned in the original direction of anisotropy. Let $\boldsymbol{m}$ denote the new direction of anisotropy. Then

$$\boldsymbol{m} \equiv I_4^{-1/2} \boldsymbol{FM}.$$

Let $\boldsymbol{n} \equiv \boldsymbol{Bm}$. Then

$$I_5 = \boldsymbol{M.C}^2\boldsymbol{M} = \boldsymbol{CM.CM} = I_4\boldsymbol{Bm.m} = I_4\boldsymbol{n.m}, \tag{2.6}$$

and the constitutive relation (2.2) can be rewritten as

$$\begin{aligned} J\boldsymbol{\sigma} = 2W_1\boldsymbol{B} + 2W_2\left(I_1\boldsymbol{B} - \boldsymbol{B}^2\right) + 2I_3W_3\boldsymbol{I} + 2I_4W_4\boldsymbol{m}\otimes\boldsymbol{m} + \\ 2I_4W_5\left(\boldsymbol{m}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{m}\right). \end{aligned} \tag{2.7}$$

If the material is assumed perfectly incompressible, then $I_3 \equiv 1$ and the stress-strain relation can then be formally obtained from (2.2) by replacing the $2I_3W_3$ term by an arbitrary scalar field $-p$ to obtain

$$\begin{aligned} \boldsymbol{\sigma} = -p\boldsymbol{I} + 2W_1\boldsymbol{B} + 2W_2\left(I_1\boldsymbol{B} - \boldsymbol{B}^2\right) + 2I_4W_4\boldsymbol{m}\otimes\boldsymbol{m} + \\ 2I_4W_5\left(\boldsymbol{m}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{m}\right), \end{aligned} \tag{2.8}$$

where here $W = W(I_1, I_2, I_4, I_5)$. Again it will be required that both the strain energy and stress vanish in the reference configuration. Thus it will be required that (2.4) holds and that

$$2W_1^0 + 4W_2^0 = p^0, \quad W_4^0 + 2W_5^0 = 0,$$

where p^0 is the value of the arbitrary field in the reference configuration.

3 The linear theory

Let (x_1, x_2, x_3) denote the typical Cartesian coordinates of a typical point of a linear, transversely isotropic solid and assume without loss of generality, that the x_3 axis is aligned along the axis of symmetry. Then (see, for example, Lubarda and Chen [13])

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\left(S_{11} - S_{12}\right) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}, \tag{3.1}$$

where the compliances S_{ij} can be defined in terms of material constants as follows:

$$S_{11} = \frac{1}{E}, \quad S_{33} = \frac{1}{E_0}, \quad S_{12} = -\frac{\nu}{E}, \quad S_{13} = -\frac{\nu_0}{E_0}, \quad S_{44} = \frac{1}{\mu_0}, \tag{3.2}$$

where E, E_0 are the Young's moduli in the plane of isotropy and in the fibre direction respectively, ν is Poisson's ratio in the plane of isotropy when forces are applied in the orthogonal direction within the plane of isotropy, ν_0 is Poisson's ratio when forces are applied in the fibre direction and μ, μ_0 are the shear moduli in the plane of isotropy and in any plane perpendicular to the plane of isotropy respectively, noting that

$$\mu = \frac{E}{2\left(1+\nu\right)}.$$

In what follows, it will be assumed that $E, E_0, \mu, \mu_0, 1+\nu > 0$.

Inversion of the normal strain-stress relations in (3.1) yields

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{11} & c_{13} \\ c_{13} & c_{13} & c_{33} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \end{bmatrix}, \tag{3.3}$$

where the elastic constants c_{ij} are given by

$$\begin{aligned} c_{11} &= \frac{E^2\nu_0^2 - EE_0}{\left(1+\nu\right)\left(2E\nu_0^2 + E_0\left(\nu-1\right)\right)}, \\ c_{12} &= -\frac{E^2\nu_0^2 + EE_0\nu}{\left(1+\nu\right)\left(2E\nu_0^2 + E_0\left(\nu-1\right)\right)}, \\ c_{13} &= -\frac{EE_0\nu_0}{2E\nu_0^2 + E_0\left(\nu-1\right)}, \\ c_{33} &= \frac{E_0^2\left(\nu-1\right)}{2E\nu_0^2 + E_0\left(\nu-1\right)}, \end{aligned} \tag{3.4}$$

assuming that $2E\nu_0^2 + E_0\left(\nu-1\right) \neq 0$. The shear stress-strain law has the form

$$\begin{aligned} \sigma_{23} &= 2c_{44}\epsilon_{23}, \\ \sigma_{13} &= 2c_{44}\epsilon_{13}, \\ \sigma_{12} &= \left(c_{11} - c_{12}\right)\epsilon_{12}, \end{aligned} \tag{3.5}$$

where

$$c_{44} = \mu_0 \quad c_{11} - c_{12} = \frac{E}{1+\nu} = 2\mu. \tag{3.6}$$

Note that in terms of the usual physical constants of the linear theory, expressing strain as a function of stress seems a more natural representation that the classical stress-strain relation (3.3), (3.4), (3.5), (3.6). Such strain-stress relations have been promoted recently by Rajagopal and co-workers [14, 15, 16].

In addition to the initial conditions (2.3), (2.4) assumed earlier, to ensure compatibility of the stress-strain relation (2.2) with the linear theory, Merodio and Ogden [17] have shown that the following restrictions on the strain-energy function must also be imposed:

$$\begin{aligned} W_{11}^0 + 4W_{12}^0 + 4W_{22}^0 + 2W_{13}^0 + 4W_{23}^0 + W_{33}^0 &= c_{11}/4, \\ W_2^0 + W_3^0 = \left(c_{12} - c_{11}\right)/4 \quad W_1^0 + W_2^0 + W_5^0 &= c_{44}/2, \\ W_{14}^0 + 2W_{24}^0 + 2W_{15}^0 + W_{34}^0 + 4W_{25}^0 + 2W_{35}^0 &= \left(c_{13} - c_{12}\right)/4, \\ W_{44}^0 + 4W_{45}^0 + 4W_{55}^0 + 2W_5^0 &= \left(c_{33} - c_{11} + 2c_{12} - 2c_{13}\right)/4. \end{aligned} \tag{3.7}$$

Utilising the initial conditions (2.3) results in the following alternative form for the third of these:

$$W_5^0 = -\frac{1}{2}W_4^0 = \frac{1}{4}\left(c_{12} + 2c_{44} - c_{11}\right). \tag{3.8}$$

In terms of the physical constants considered here, these restrictions become

$$\begin{aligned} &W_{11}^0 + 4W_{12}^0 + 4W_{22}^0 + 2W_{13}^0 + 4W_{23}^0 + W_{33}^0 = \frac{E^2\nu_0^2 - EE_0}{4\left(1+\nu\right)\left(2E\nu_0^2 + E_0\left(\nu - 1\right)\right)}, \\ &W_2^0 + W_3^0 = -\frac{E}{4\left(1+\nu\right)}, \\ &W_5^0 = \frac{2\mu_0\left(1+\nu\right) - E}{4\left(1+\nu\right)}, \\ &W_{14}^0 + 2W_{24}^0 + 2W_{15}^0 + W_{34}^0 + 4W_{25}^0 + 2W_{35}^0 = \frac{EE_0\left(\nu - \nu_0 - \nu\nu_0\right) + E^2\nu_0^2}{4\left(1+\nu\right)\left(2E\nu_0^2 + E_0\left(\nu - 1\right)\right)}, \\ &W_{44}^0 + 4W_{45}^0 + 4W_{55}^0 + 2W_5^0 = \frac{EE_0\left(1 - 2\nu + 2\nu_0 + 2\nu\nu_0\right) - 3E^2\nu_0^2 + E_0^2\left(\nu^2 - 1\right)}{4\left(1+\nu\right)\left(2E\nu_0^2 + E_0\left(\nu - 1\right)\right)}. \end{aligned} \tag{3.9}$$

It is generally accepted as a modelling axiom that the corresponding linear theory should be fully recoverable from a non-linear model on restriction to infinitesimal deformations. Therefore no restrictions should be imposed on the physical constants by the form of the non-linear theory in the absence of experimental data to support such restrictions. It follows immediately therefore from the initial condition $(2.3)_2$ and $(3.9)_3$ that the strain-energy function must depend on *both* anisotropic invariants, a point developed more fully in Murphy [18], Destrade *et al.* [19] and Feng *et al.* [20].

It has been shown recently by Vergori *et al.* [9] that the standard method of accounting for slight compressibility discussed in Section 5 does not satisfy this modelling axiom. This violation results, for example, in a transversely isotropic sphere remaining spherical under hydrostatic tension, almost certainly a non-physical result. Alternatives therefore are needed. Before these alternatives are discussed, the theory of linear elasticity for slightly compressible transversely isotropic materials is considered to gain some insight into the forms of viable models of almost incompressible behaviour.

4 The linear theory and slight compressibility

If a material is suspected of being slightly compressible, then the natural first step in the modelling of this physical characteristic is consideration of the linear theory, even though the material might typically undergo non-linear deformations in applications. The material will be assumed homogenous and hyperelastic. The insights that can be gained from the linear theory will be explored in turn for isotropic and transversely isotropic materials. In what follows, the infinitesimal strain tensor will be denoted by $\boldsymbol{\epsilon}$ and the stress tensor by $\boldsymbol{\sigma}$, with an obvious notation used for the Cartesian components of the same. The quantity of particular interest here in each case is the measure of infinitesimal volume change, $\operatorname{tr}\boldsymbol{\epsilon}$.

4.1 Isotropic materials

The classical stress-strain relation for isotropic, linearly elastic materials is given by

$$\boldsymbol{\sigma} = \lambda \operatorname{tr}\boldsymbol{\epsilon}\,\boldsymbol{I} + 2\mu\boldsymbol{\epsilon},$$

where $\boldsymbol{I}$ denotes the identity tensor and λ, μ are the usual Lame constants. In terms of the Young's modulus E and Poisson's ration ν, this relation has the form

$$\boldsymbol{\sigma} = \frac{\nu E}{(1+\nu)(1-2\nu)}\operatorname{tr}\boldsymbol{\epsilon}\,\boldsymbol{I} + \frac{E}{1+\nu}\boldsymbol{\epsilon}, \tag{4.1}$$

Taking the trace of both sides then yields

$$\operatorname{tr}\boldsymbol{\epsilon} = \frac{1-2\nu}{E}\operatorname{tr}\boldsymbol{\sigma}. \tag{4.2}$$

It is rarely, if ever, explicitly stated that the applied stress is of the order of Young's modulus for isotropic materials, even in the nonlinear regime. Let $\hat{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma}/E$ denote the corresponding non-dimensional stress measure. Then (4.2) becomes

$$\operatorname{tr}\boldsymbol{\epsilon} = (1-2\nu)\operatorname{tr}\hat{\boldsymbol{\sigma}}.$$

Two assumptions typically used in the literature to characterise slight compressibility of elastomers: infinitesimal volume change and/or $\nu \approx 1/2$. The relationship between these is now clear: the volume change and $1-2\nu$ are of *the same order*. Formally, the classical theory of perfect incompressibility can therefore be obtained from (4.1) by letting $\operatorname{tr}\boldsymbol{\epsilon}$ and $1-2\nu$ simultaneously go to zero and writing the quotient $\operatorname{tr}\boldsymbol{\epsilon}/(1-2\nu)$ as $-p \times 1+\nu/\nu E$.

4.2 Transversely isotropic materials

Then

$$\operatorname{tr}\boldsymbol{\epsilon} = (\sigma_{11}+\sigma_{22})\left(\frac{1-\nu}{E} - \frac{\nu_0}{E_0}\right) + \sigma_{33}\frac{1-2\nu_0}{E_0}.$$

It will be assumed that the normal stresses in the plane of isotropy and along the fibres are of $\mathcal{O}(E)$ and $\mathcal{O}(E_0)$ respectively. Non-dimensionalising the stresses in the obvious way then yields

$$\operatorname{tr}\boldsymbol{\epsilon} = (\hat{\sigma}_{11}+\hat{\sigma}_{22})\left(1-\nu-\nu_0\frac{E}{E_0}\right) + \hat{\sigma}_{33}(1-2\nu_0).$$

It follows then that for an infinitesimal volume change tending towards zero,

$$\nu_0 \to 1/2, \quad \nu \to 1 - \frac{E}{2E_0}. \tag{4.3}$$

In practical applications of fibre-reinforced materials

$$\epsilon \equiv \frac{E}{E_0} << 1, \tag{4.4}$$

and therefore

$$\nu \to 1.$$

Therefore

$$\begin{aligned}
\operatorname{tr}\boldsymbol{\sigma} &= (\epsilon_{11}+\epsilon_{22})(c_{11}+c_{12}+c_{13}) + \epsilon_{33}(2c_{13}+c_{33}) \\
&= (\epsilon_{11}+\epsilon_{22}+\epsilon_{33})(c_{11}+c_{12}+c_{13}) + \epsilon_{33}(c_{13}+c_{33}-c_{11}-c_{12}) \\
&= (\epsilon_{11}+\epsilon_{22})\left(\frac{EE_0(1+\nu_0)}{E_0(1-\nu)-2E\nu_0^2}\right) + \epsilon_{33}\frac{2EE_0\nu_0 + E_0^2(1-\nu)}{E_0(1-\nu)-2E\nu_0^2} \\
&= \operatorname{tr}\boldsymbol{\epsilon}\frac{EE_0(1+\nu_0)}{E_0(1-\nu)-2E\nu_0^2} + \epsilon_{33}\frac{EE_0(\nu_0-1)+E_0^2(1-\nu)}{E_0(1-\nu)-2E\nu_0^2}.
\end{aligned} \tag{4.5}$$

It follows then that hydrostatic stress and strain are equivalent for transversely isotropic materials if, and only if, either

$$c_{13} + c_{33} = c_{11} + c_{12}, \tag{4.6}$$

a result first obtained by Musgrave [21] and Vergori *et al.* [9], or, in terms of the physical constants,

$$EE_0(\nu_0 - 1) + E_0^2(1-\nu) = 0. \tag{4.7}$$

If $\nu_0 = 1$ this condition is satisfied if $\nu = 1$ and if $\nu_0 \neq 1$, then

$$\frac{E}{E_0} = \frac{\nu - 1}{\nu_0 - 1}.$$

Satisfaction of this identity for any real material is highly unlikely and, almost certainly, for linear transversely isotropic materials hydrostatic stress does not result in hydrostatic strain and vice versa.

It follows then from (4.5) that there are essentially *two* bulk moduli for transversely isotropic materials

$$\kappa_1 \equiv \frac{EE_0(1+\nu_0)}{E_0(1-\nu)-2E\nu_0^2}, \quad \kappa_2 \equiv \frac{EE_0(\nu_0-1)+E_0^2(1-\nu)}{E_0(1-\nu)-2E\nu_0^2}. \tag{4.8}$$

Some indication of the size and sign of these constants is useful. Liao *et al.* [22] obtained the following values of the elastic constants for argillite rock:

$$E = 68.34\,\text{GPa},\ E_0 = 51.35\,\text{GPa},\ \mu_0 = 20.98\,\text{GPa},\ \nu = 0.196,\ \nu_0 = 0.163,$$

and therefore

$$\kappa_1 = 108.39\,\text{GPa}, \quad \kappa_2 = -21.70\,\text{GPa},$$

which shows that at least the second of the bulk moduli can be negative. Typical values of these parameters for soft tissue can be inferred from, for example, the experimental data of Morrow *et al.* [23] who found that

$$E_0 = 447\,\text{kPa}, \quad E = 22\,\text{kPa}, \tag{4.9}$$

using data from simple tension experiments on the extensor digitorum longus muscles of rabbits. Motivated by the limits (4.3) for ν, ν_0 derived earlier, it will be assumed that

$$\nu = 0.95, \quad \nu_0 = 0.49. \tag{4.10}$$

The corresponding values for the bulk moduli are therefore

$$\kappa_1 = 1243\,\text{kPa}, \quad \kappa_2 = 442\,\text{kPa}, \tag{4.11}$$

which values suggest that for strongly anisotropic soft tissue both bulk moduli are positive.

5 The standard model of slight compressibility

The interesting problems in biomechanics invariably involve complicated biological materials, boundaries and boundary conditions and the solutions to such problems therefore require numerical methods. Because of their high water-content, soft tissue is typically assumed to be incompressible. However there are major difficulties in enforcing this constraint globally and to avoid these difficulties soft tissue is usually assumed to be slightly compressible. There is no standard formulation of slight compressibility. The usual approach (see, for example, Ogden [24], Holzapfel [25]) is first to reformulate the kinematics in terms of the *modified* or *distortional* stretches, λ_i^*, defined as

$$\lambda_i^* \equiv J^{-1/3}\lambda_i. \tag{5.1}$$

The motivation for doing this is to develop a theory that has close parallels with the now classical theory of perfectly incompressible materials since $\lambda_1^*\lambda_2^*\lambda_3^* = 1$. The tensorial measures of deformation can therefore be multiplicatively decomposed into dilatational and volume-preserving parts as follows:

$$\begin{aligned}
\boldsymbol{F} &= \left(J^{1/3}\boldsymbol{I}\right)\boldsymbol{F}^* = J^{1/3}\boldsymbol{F}^*,\\
\boldsymbol{B} &= \left(J^{2/3}\boldsymbol{I}\right)\boldsymbol{B}^* = J^{2/3}\boldsymbol{B}^*,\\
\boldsymbol{C} &= \left(J^{2/3}\boldsymbol{I}\right)\boldsymbol{C}^* = J^{2/3}\boldsymbol{C}^*,
\end{aligned} \tag{5.2}$$

with the relationship between the two sets of invariants $\{I_1, I_2, I_3, I_4, I_5\}$ and $\{I_1^*, I_2^*, I_4^*, I_5^*; J\}$ given by

$$I_a^* = J^{-2/3}I_a, \quad a \in \{1,4\}, \qquad I_b^* = J^{-4/3}I_b, \quad b \in \{2,5\}, \qquad I_3^* = 1. \tag{5.3}$$

In terms of the modified stretches, the stress-strain relation (2.2) becomes

$$\begin{aligned}
J\boldsymbol{\sigma} =& J\frac{\partial W^*}{\partial J}\boldsymbol{I} + 2W_1^*\left(\boldsymbol{B}^* - \tfrac{1}{3}I_1^*\boldsymbol{I}\right) + 2W_2^*\left[I_1^*\boldsymbol{B}^* - (\boldsymbol{B}^*)^2 - \tfrac{2}{3}I_2^*\boldsymbol{I}\right]\\
&+ 2W_4^*\left(\boldsymbol{F}^*\boldsymbol{M}\otimes\boldsymbol{F}^*\boldsymbol{M} - \tfrac{1}{3}I_4^*\boldsymbol{I}\right) +\\
&+ 2W_5^*\left(\boldsymbol{F}^*\boldsymbol{M}\otimes\boldsymbol{B}^*\boldsymbol{F}^*\boldsymbol{M} + \boldsymbol{B}^*\boldsymbol{F}^*\boldsymbol{M}\otimes\boldsymbol{F}^*\boldsymbol{M} - \tfrac{2}{3}I_5^*\boldsymbol{I}\right),
\end{aligned} \tag{5.4}$$

where, now, the subscripts attached to $W^* = W^*(I_1^*, I_2^*, I_4^*, I_5^*; J)$ denote partial differentiation with respect to the modified invariants I_c^*. Although (5.4) is the general constitutive form assumed for slightly compressible materials reinforced with two families of fibres, this form is in fact valid for *all* materials reinforced with two families of fibres. The constitutive assumption that is widely assumed to be specific to slightly compressible materials is that the strain-energy function in (5.4) can be *additively decomposed* into dilatational and volume preserving parts as:

$$W^*(I_1^*, I_2^*, I_4^*, I_5^*; J) = f(J) + \mathcal{W}(I_1^*, I_2^*, I_4^*, I_5^*). \tag{5.5}$$

This is the form of slight compressibility adopted by most commercial Finite Element developers. The stress-strain relation for these slightly compressible materials will therefore have the form

$$\begin{aligned} J\boldsymbol{\sigma} =& Jf'(J)\boldsymbol{I} + 2\mathcal{W}_1\left(\boldsymbol{B}^* - \tfrac{1}{3}I_1^*\boldsymbol{I}\right) + 2\mathcal{W}_2\left[I_1^*\boldsymbol{B}^* - (\boldsymbol{B}^*)^2 - \tfrac{2}{3}I_2^*\boldsymbol{I}\right] \\ &+ 2\mathcal{W}_4\left(\boldsymbol{F}^*\boldsymbol{M}\otimes\boldsymbol{F}^*\boldsymbol{M} - \tfrac{1}{3}I_4^*\boldsymbol{I}\right) + \\ &+ 2\mathcal{W}_5\left(\boldsymbol{F}^*\boldsymbol{M}\otimes\boldsymbol{B}^*\boldsymbol{F}^*\boldsymbol{M} + \boldsymbol{B}^*\boldsymbol{F}^*\boldsymbol{M}\otimes\boldsymbol{F}^*\boldsymbol{M} - \tfrac{2}{3}I_5^*\boldsymbol{I}\right), \end{aligned} \tag{5.6}$$

where to ensure zero strain energy and stress in the reference configuration it will be required that

$$f(1) + \mathcal{W}(3,3,1,1) = 0, \quad f'(1) = 0, \quad \mathcal{W}_4(3,3,1,1) + 2\mathcal{W}_5(3,3,1,1) = 0.$$

This model will be called the standard model of slight compressibility.

The restrictions imposed by (3.9) on the standard model of slight compressibility (5.5) will now be obtained. Substituting this strain-energy into (3.9) yields

$$\begin{aligned} &9f''(1) = \frac{E_0^2(1-\nu) - 2EE_0(2+\nu_0)}{2E\nu_0^2 + E_0(\nu-1)}, \\ &\mathcal{W}_1^0 + \mathcal{W}_2^0 = \frac{E}{4(1+\nu)}, \quad \mathcal{W}_5^0 = \frac{2\mu_0(1+\nu) - E}{4(1+\nu)}, \\ &\mathcal{W}_{44}^0 + 4\mathcal{W}_{45}^0 + 4\mathcal{W}_{55}^0 + 2\mathcal{W}_5^0 = \frac{3EE_0(-\nu+\nu_0+\nu\nu_0) - 3E^2\nu_0^2}{4(1+\nu)(2E\nu_0^2 + E_0(\nu-1))}, \\ &\mathcal{W}_{44}^0 + 4\mathcal{W}_{45}^0 + 4\mathcal{W}_{55}^0 + 2\mathcal{W}_5^0 = \frac{EE_0(1-2\nu+2\nu_0+2\nu\nu_0) - 3E^2\nu_0^2 + E_0^2(\nu^2-1)}{4(1+\nu)(2E\nu_0^2 + E_0(\nu-1))}. \end{aligned} \tag{5.7}$$

The last two of these are compatible if, and only if, (4.7) holds. The constitutive assumption (5.5) therefore violates the modelling axiom of Section 3 and therefore will be rejected as a physically realistic model of slight compressibility for anisotropic materials. As demonstrated by Vergori *et al.* [9], this violation has an easily interpreted physical consequence: a hydrostatic tension causes a hydrostatic expansion for infinitesimal deformations which would seem at variance with our intuitive expectation of how a fibre-reinforced material should behave. Indeed, it was shown by Vergori *et al.* [9] that hydrostatic tension always results in hydrostatic strain for the standard representation of slight compressibility, even for non-linear deformations, if it is coupled with the condition that when $I_4^* < 1$ the material behaves isotropically, intuitively reflecting the notion that there is no mechanical contribution from the fibres when compressed. As was first observed by Ni Annaidh [7], it seems that even for possibly the simplest conceivable experiment, that of hydrostatic tension, commercial Finite Element codes cannot predict physically realistic response for transversely isotropic elastic materials. Some alternative models of slight compressibility that are consistent with the linear theory are proposed in Section 7.

6 The standard model and the hydrostatic tension of a cube

Vergori *et al.* [9] have shown that the hydrostatic tension or compression of a slightly compressible cube of a transversely isotropic material modelled by the standard model (5.5) results in the cuboid shape being maintained, on restriction to infinitesimal deformations. It will now be shown that this is *always* true, regardless of the size of the deformation.

Consider then the problem of a hydrostatic Cauchy stress of amount T exerted on the faces of a transversely isotropic cube, whose fibres are aligned parallel to one set of parallel faces. This is essentially a thought-experiment, being very difficult to implement practically, yet seems the most natural to consider when studying the modelling of slight compressibility. Assuming that the resulting deformation is homogeneous, let λ_f denote the stretch in the direction of the fibres, with the two stretches in the perpendicular direction being identical of amount λ. It therefore follows from (2.2) that for a general compressible material

$$\begin{aligned} JT &= 2W_1\lambda_f^2 + 4W_2\lambda^2\lambda_f^2 + 2W_3\lambda^4\lambda_f^2 + 2W_4\lambda_f^2 + 4W_5\lambda_f^4, \\ JT &= 2W_1\lambda^2 + 2W_2\lambda^2\left(\lambda^2 + \lambda_f^2\right) + 2W_3\lambda^4\lambda_f^2, \end{aligned} \tag{6.1}$$

where $J = \lambda^2\lambda_f$. Subtraction yields

$$W_1\left(\lambda_f^2 - \lambda^2\right) + W_2\lambda^2\left(\lambda_f^2 - \lambda^2\right) + W_4\lambda_f^2 + 2W_5\lambda_f^4 = 0, \tag{6.2}$$

the determining equation for the relationship between the two stretches. For the standard model of slight compressibility given in (5.5), this equation becomes

$$\mathcal{W}_1 J^{-2/3}\left(\lambda_f^2 - \lambda^2\right) + \mathcal{W}_2 J^{-4/3}\lambda^2\left(\lambda_f^2 - \lambda^2\right) + \mathcal{W}_4 J^{-2/3}\lambda_f^2 + 2\mathcal{W}_5 J^{-4/3}\lambda_f^4 = 0, \tag{6.3}$$

which is independent of the α term of the volumetric component of the strain-energy function. This equation can be re-written as

$$2\mathcal{W}_1 x^{-4/3}\left(x^2 - 1\right) + 2\mathcal{W}_2 x^{-2/3}\left(x^2 - 1\right) - 2\mathcal{W}_4 x^{-4/3} - 4\mathcal{W}_5 x^{-8/3} = 0, \quad x \equiv \frac{\lambda}{\lambda_f}, \tag{6.4}$$

where now

$$I_1^* = x^{-4/3}\left(1 + 2x^2\right), \quad I_2^* = x^{-2/3}\left(2 + x^2\right), \quad I_4^* = x^{-4/3}, \quad I_5^* = x^{-8/3}. \tag{6.5}$$

Noting that

$$\tilde{\mathcal{W}}'(x) = \frac{2}{3x}\left(2\mathcal{W}_1 x^{-4/3}\left(x^2-1\right) + 2\mathcal{W}_2 x^{-2/3}\left(x^2-1\right) - 2\mathcal{W}_4 x^{-4/3} - 4\mathcal{W}_5 x^{-8/3}\right), \quad (6.6)$$

where

$$\tilde{\mathcal{W}}(x) \equiv \mathcal{W}\left(x^{-4/3}\left(1+2x^2\right), x^{-2/3}\left(2+x^2\right), x^{-4/3}, x^{-8/3}\right),$$

(6.4) can be written more succinctly as

$$\tilde{\mathcal{W}}'(x) = 0. \quad (6.7)$$

It follows from (6.6) and the identities

$$2\mathcal{W}_4 x^{-4/3} + 4\mathcal{W}_5 x^{-8/3}|_{x=1} = 2\left(\mathcal{W}_4^0 + 2\mathcal{W}_5^0\right) = 0,$$

that $x = 1$ is a solution to (6.7), which is the isotropic solution with $\lambda = \lambda_f$.

Convexity of models of soft tissue was identified as being essential in Holzapfel *et al.* [26], for example. Assuming therefore here that $\tilde{\mathcal{W}}(x)$ is a convex function, it follows that $x = 1$ is the *unique* solution to (6.7). Therefore the standard model of slight compressibility predicts that when a transversely isotropic cube is subjected to hydrostatic tension or compression, the deformed state is also a cube. This prediction certainly seems unphysical as one would expect a contribution from the reinforcing fibres, particularly in tension and seems a serious defect of the standard model.

The deformation of a single cubic element of dimensions $1\times1\times1$ m^3 under hydrostatic tension has been simulated in two finite element packages: Abaqus® v6.12 and FEBio v1.8 [27]. The material was assumed transversely isotropic with fibres in the x-direction (see Fig. 1). Boundary conditions were applied to faces at $x = 0$, $y = 0$ and $z = 0$ to allow the cube to expand but prevent it from translating.

In Abaqus®, the following Holzapfel-Gasser-Ogden strain energy function was used :

$$W = C_{10}\left(I_1^* - 3\right) + \frac{1}{D}\left(\frac{J^2-1}{2} - \ln(J)\right) + \frac{k_1}{2k_2} e^{\left(I_4^*-1\right)^2 - 1} \quad (6.8)$$

with $C_{10} = 1$ Pa, $D = 1 \times 10^{-3}$ Pa^{-1} and $k_1 = k_2 = 1$. A hydrostatic tension of 1000 Pa was applied to the faces of the element (C3D8, 8-node linear brick), and a static analysis was performed using the Standard (implicit) solver. It was observed that the cube deformed into another cube, as shown in Figure 1. Equal stretches of 1.17 were obtained in each direction.

The same analysis was performed in FEBio. The following uncoupled transversely isotropic Mooney-Rivlin strain-energy function introduced by Weiss *et al.* [4] was used:

$$W = C_1\left(I_1^* - 3\right) + C_2\left(I_2^* - 3\right) + \frac{K}{2}\left(\ln(J)\right)^2 + F(I_4^*), \quad (6.9)$$

with $F(I_4^*)$ defined as

$$\lambda^* \frac{\partial F}{\partial \lambda^*} = C_3\left(e^{C_4(\lambda^*-1)} - 1\right), \quad (6.10)$$

with $I_4^* = (\lambda^*)^2$. The following parameter values were used: C_1 =1 Pa, C_2 =2 Pa, K =1000 Pa, C_3 =1 Pa, C_4 =1. A hydrostatic tension of 300 Pa was applied to the cube. Once again, it was verified that the cube deformed into another cube with uniform stretches of 1.21.

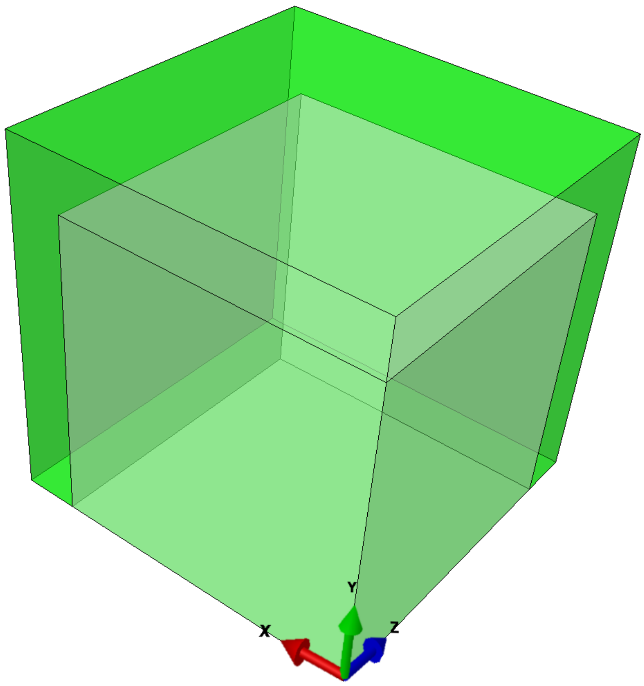

Figure 1: Superposition of the initial and final states of the simulation showing that in Abaqus®, a transversely isotropic cube under a hydrostatic tension deforms into another cube.

7 Alternative models

Before viable representations of slight compressibility are considered, a seemingly natural model will be removed from consideration. Merodio and Ogden [17, 28] have proposed the following additive decomposition of the strain-energy function for transversely isotropic materials into isotropic and anisotropic parts:

$$W = W_{iso}\left(I_1, I_2, I_3\right) + W_{an}\left(I_4, I_5\right). \tag{7.1}$$

These models suggest the attractive possibility of modelling slight compressibility using the isotropic invariants only and thereby extending the standard models of slightly compressible, *isotropic* elasticity to anisotropy by simply adding functions of anisotropic invariants using models, for example, of the form

$$W = f\left(J\right) + W_{iso}\left(I_1^*, I_2^*\right) + W_{an}\left(I_4, I_5\right). \tag{7.2}$$

However, it is easy to check that for materials of the form (7.1) the following identity applies:

$$W_{14} + 2W_{24} + 2W_{15} + W_{34} + 4W_{25} + 2W_{35} = 0,$$

and therefore the right-hand side of $(3.9)_4$ is identically zero. This means that a necessary condition for (7.1) to hold is that

$$E_0\left(\nu - \nu_0 - \nu\nu_0\right) + E\nu_0^2 = 0,$$

which therefore means that the linear theory isn't fully recoverable and that (7.1) isn't an allowable model of slightly compressible, anisotropic materials.

The goal here is to provide a rational approach to the modelling of slight compressibility for nonlinear, transversely isotropic, elastic materials, while maintaining as much of the structure of the standard compressible model (5.5) as possible to take advantage of the extensive computational infrastructure already in existence. To achieve this, use will be made of the following equivalence noted earlier by Sansour [5] and Ni Annaidh *et al.* [7]: if $W = W\left(I_1, I_2, I_3, I_4, I_5\right)$, then

$$\operatorname{tr}\boldsymbol{\sigma} = 3f'\left(J\right) \tag{7.3}$$

if, and only if,

$$W = f\left(J\right) + \mathcal{W}\left(I_1^*, I_2^*, I_4^*, I_5^*\right). \tag{7.4}$$

It is immediately evident then that hydrostatic tension of these slightly compressible materials causes a volume change that is independent of the anisotropy of the material, echoing the conclusions of Vergori *et al.* [9]. Another equivalence result will be used here to propose a model of slight compressibility for anisotropic materials. In order to preserve as much of the structure of the dominant model of slight compressibility given in (7.4) as possible, a constitutive assumption based on the form of $\operatorname{tr}\boldsymbol{\sigma}$ will again be used but now a dependence of the anisotropic invariants will also be assumed. Specifically it will be assumed that for all deformations

$$\operatorname{tr}\boldsymbol{\sigma} = f\left(I_3\right) + g\left(I_4\right), \tag{7.5}$$

where $f, g \in C^{\infty}(0, \infty)$. Noting (2.5), this results in the following linear partial differential equation in W:

$$2I_1W_1 + 4I_2W_2 + 6I_3W_3 + 2I_4W_4 + 4I_5W_5 = I_3^{1/2} f(I_3) + I_3^{1/2} g(I_4). \tag{7.6}$$

The solution to the homogenous equation is given by

$$W_h = \mathcal{W}(I_1^*, I_2^*, I_4^*, I_5^*), \quad \text{arbitrary } \mathcal{W},$$

while a particular integral is given by

$$W_p = \mathcal{A}(I_3) + I_3^{1/2}\mathcal{B}(I_4),$$

where

$$\mathcal{A} = \int \frac{f(I_3)}{6I_3^{1/2}} dI_3, \quad \mathcal{B} = \frac{1}{2I_4^{3/2}} \int g(I_4) I_4^{1/2} dI_4.$$

Therefore (7.5) holds if, and only if,

$$W = \mathcal{A}(I_3) + I_3^{1/2}\mathcal{B}(I_4) + \mathcal{W}(I_1^*, I_2^*, I_4^*, I_5^*). \tag{7.7}$$

This form is therefore proposed as a viable, simple generalisation of the standard form (5.5). To ensure zero strain energy and stress in the reference configuration, it will be required that

$$\begin{aligned} \mathcal{A}(1) + \mathcal{B}(1) + \mathcal{W}^0 &= 0, \\ 6\mathcal{A}'(1) + 3\mathcal{B}(1) + 2\mathcal{B}'(1) &= 0, \\ \mathcal{B}'(1) + \mathcal{W}_4^0 + 2\mathcal{W}_5^0 &= 0. \end{aligned} \tag{7.8}$$

The conditions that ensure compatibility with the linear theory are given by

$$\begin{aligned} &36\mathcal{A}''(1) - 9\mathcal{B}(1) - 16\mathcal{B}'(1) - 4\mathcal{B}''(1) = \frac{E_0^2(1-\nu) - 2EE_0(2+\nu_0)}{2E\nu_0^2 + E_0(\nu-1)}, \\ &\mathcal{W}_1^0 + \mathcal{W}_2^0 = \frac{E}{4(1+\nu)}, \\ &\mathcal{W}_5^0 = \frac{2\mu_0(1+\nu) - E}{4(1+\nu)}, \\ &5\mathcal{B}'(1) + 2\mathcal{B}''(1) = \frac{EE_0(1-\nu_0) + E_0^2(\nu-1)}{2(2E\nu_0^2 + E_0(\nu-1))}, \\ &\mathcal{B}''(1) + \mathcal{W}_{44}^0 + 4\mathcal{W}_{45}^0 + 4\mathcal{W}_{55}^0 + 2\mathcal{W}_5^0 = \frac{EE_0(1-2\nu+2\nu_0+2\nu\nu_0) - 3E^2\nu_0^2 + E_0^2(\nu^2-1)}{4(1+\nu)(2E\nu_0^2 + E_0(\nu-1))}. \end{aligned} \tag{7.9}$$

8 Specific forms for the compressibility functions

Specific forms for the compressibility functions $\mathcal{A}, \mathcal{B}$ are now proposed. First note that for the strain energies of the form (7.7)

$$\operatorname{tr}\boldsymbol{\sigma} = 6I_3^{1/2}\mathcal{A}'(I_3) + 3\mathcal{B}(I_4) + 2I_4\mathcal{B}'(I_4). \tag{8.1}$$

The very limited experimental data available on the compressibility of non-linear materials will be used together with this relation to motivate specific forms for the compressibility functions.

Despite the crucial need for accurate and reliable data on the compressibility of soft tissue, there is dearth of results in the literature. There are two exceptional studies, both of which suffer from some deficiencies if one wishes to model the compressibility of anisotropic, soft tissue. Carew *et al.* [3] measured the volume change that accompanies the internal pressurisation of dog arteries and found that the volume change is much less than 1%. Although this measurement of the size of the volume change is extremely valuable, the corresponding experiments involve inhomogeneous deformations and are therefore of limited interest from a constitutive modelling perspective. Chuong and Fung [29] conducted uniaxial compression experiments on rabbit thoracic arteries to observe their compressibility but their one-dimensional nature prevent these experiments yielding information about the interaction between compressibility and anisotropy which is of interest here. By far the most sophisticated measurements of the compressibility of non-linear elastic materials were performed by Penn [10], who measured the volume change accompanying simple tension experiments on vulcanised rubber. Volume changes much smaller than 1% were observed in specimens stretched by 100%. Horgan and Murphy [6] showed that Penn's data are consistent with a linear relationship between stress and volume change.

Motivated by these data, it will be assumed here that the trace of the Cauchy stress is a linear function of volume change. Similarly, in the absence of experimental data to suggest otherwise, it will also be assumed that the trace of the Cauchy stress is a linear function of the fibre stretch. Consequently it will be assumed that

$$\begin{aligned} 6I_3^{1/2}\mathcal{A}'\left(I_3\right) &= 6\alpha\left(I_3^{1/2}-1\right), \\ 3\mathcal{B}+2I_4\mathcal{B}'\left(I_4\right) &= 12\beta\left(I_4^{1/2}-1\right), \end{aligned} \tag{8.2}$$

where material constants are denoted by Greek symbols, and therefore

$$\begin{aligned} \mathcal{A}\left(I_3\right) &= \alpha\left(I_3-2I_3^{1/2}\right)+\gamma, \\ \mathcal{B}\left(I_4\right) &= \beta\left(3I_4^{1/2}-4\right)+\delta I_4^{-3/2}, \end{aligned} \tag{8.3}$$

noting that $(7.8)_2$ is identically satisfied. Requiring that $\mathcal{A}(1)=\mathcal{B}(1)=0$ then yields

$$\begin{aligned} \mathcal{A}\left(I_3\right) &= \alpha\left(I_3-2I_3^{1/2}+1\right)=\alpha\left(I_3^{1/2}-1\right)^2, \\ \mathcal{B}\left(I_4\right) &= \beta\left(3I_4^{1/2}-4+I_4^{-3/2}\right). \end{aligned} \tag{8.4}$$

It now follows from $(7.9)_4$ that

$$\beta=\frac{EE_0\left(1-\nu_0\right)+E_0^2\left(\nu-1\right)}{12\left(2E\nu_0^2+E_0\left(\nu-1\right)\right)}, \tag{8.5}$$

and then, from $(7.9)_1$, that

$$\alpha=-\frac{EE_0\left(1+\nu_0\right)}{6\left(2E\nu_0^2+E_0\left(\nu-1\right)\right)}. \tag{8.6}$$

Alternatively, these expressions for α, β in terms of the material constants could have obtained directly from the strain coefficients for the linear form of $\operatorname{tr} \boldsymbol{\sigma}$ given in (4.5). Assuming that the material constants are known, then the chosen compressibility functions are fully determined and the following strain-energy function is therefore proposed as a model of slightly compressible, transversely isotropic, non-linear elasticity:

$$W = \alpha \left(I_3^{1/2} - 1 \right)^2 + \beta I_3^{1/2} \left(3 I_4^{1/2} - 4 + I_4^{-3/2} \right) + \mathcal{W} \left(I_1^*, I_2^*, I_4^*, I_5^* \right). \tag{8.7}$$

The effect of including the necessary additional term $\beta I_3^{1/2} \left(3 I_4^{1/2} - 4 + I_4^{-3/2} \right)$ in the volumetric contribution to the strain energy will be the main focus in what follows. Setting $\beta \equiv 0$ recovers a typical form of the standard compressibility model that is currently widely used. It follows from (7.8), (7.9) that the deviatoric part of the strain-energy function must satisfy the following conditions in order to satisfy zero initial conditions in the reference configuration and to ensure compatibility with the linear theory:

$$\begin{gathered} \mathcal{W}^0 = 0, \quad \mathcal{W}_4^0 + 2\mathcal{W}_5^0 = 0, \quad \mathcal{W}_1^0 + \mathcal{W}_2^0 = \frac{\mu}{2}, \quad \mathcal{W}_5^0 = \frac{\mu_0 - \mu}{2}, \\ \mathcal{W}_{44}^0 + 4\mathcal{W}_{45}^0 + 4\mathcal{W}_{55}^0 = \frac{3 E E_0 \left(\nu_0 + \nu \nu_0 - \nu \right) - 3 E^2 \nu_0^2}{4 \left(1 + \nu \right) \left(2 E \nu_0^2 + E_0 \left(\nu - 1 \right) \right)} + \mu - \mu_0. \end{gathered} \tag{8.8}$$

9 The new compressibility factor and hydrostatic tension

To investigate the effect of the new compressibility factor β, the problem of a hydrostatic Cauchy stress of amount T exerted on the faces of a transversely isotropic cube, whose fibres are aligned parallel to one set of parallel faces, will again be considered. The determining equation for the relationship between the lateral and fibre stretches, λ and λ_f respectively, is given for all materials by (6.2). For the new model of slight compressibility given in (8.7), this equation becomes

$$\frac{\tilde{\mathcal{W}}'(x)}{2x} = \beta \left(\lambda_f^4 - 1 \right), \tag{9.1}$$

using the notation of Section 6. It follows immediately from the assumed convexity of $\tilde{\mathcal{W}}(x)$ and the fact that $\tilde{\mathcal{W}}'(1) = 0$, that, if a solution to this equation exists for a given $\lambda_f \neq 1$ and $\beta > 0$, then $(x-1)(\lambda_f - 1) > 0$.

To make further progress both the form of the deviatoric part of the strain energy and a physically realistic value for β must be specified. First note that $\beta = \kappa_2/12$, where κ_2 is a bulk modulus defined in Section 4. Using the κ_2 value given in (4.11), a physically realistic value for β is therefore 37. This value for κ_2 was obtained from simple asymptotics and using values for material constants for muscles obtained by Morrow *et al.* [23] from simple tension tests. These values for the parameters E, E_0, ν, ν_0 given in (4.9), (4.10) will also be used here for illustrative purposes, assuming additionally that $\mu_0 = 4\,\text{kPa}$, a value proposed by Morrow *et al.* [23] to match their data obtained from shearing testing of the the extensor digitorum longus muscles of rabbits. To simplify the analysis, the following particularly simple form of the deviatoric part of the strain-energy function is chosen:

$$\mathcal{W}\left(I_1^*, I_2^*, I_4^*, I_5^*\right) = c_1\left(I_1^* - 3\right) + c_2\left(2I_4^* - I_5^* - 1\right) + c_3\left(I_4^* - 1\right)^2, \quad c_1, c_2, c_3 \text{ constants,} \tag{9.2}$$

where, in order to satisfy (8.8),

$$c_1 = \frac{\mu}{2}, \quad c_2 = \frac{\mu - \mu_0}{2}, \quad c_3 - c_2 = \frac{3EE_0\left(\nu_0 + \nu\nu_0 - \nu\right) - 3E^2\nu_0^2}{8\left(1+\nu\right)\left(2E\nu_0^2 + E_0\left(\nu - 1\right)\right)}.$$

The values of these material parameters, in kPa, for the chosen values of $E, E_0, \nu, \nu_0, \mu_0$ are therefore

$$c_1 = 2.8, \quad c_2 = 0.8, \quad c_3 - c_2 = 1.0.$$

The defining equation (9.1) therefore becomes for this material

$$f(x) \equiv 2.8x^{-4/3} - 0.8x^{-10/3} - 2x^{-14/3} = 27.8\left(\lambda_f^4 - 1\right). \tag{9.3}$$

The nature of the solutions to this equation is most easily explored by first considering the plot of $f(x)$ in Figure 2, an important feature of which is the maximum value 1.36 occurring at $x = 1.48$.

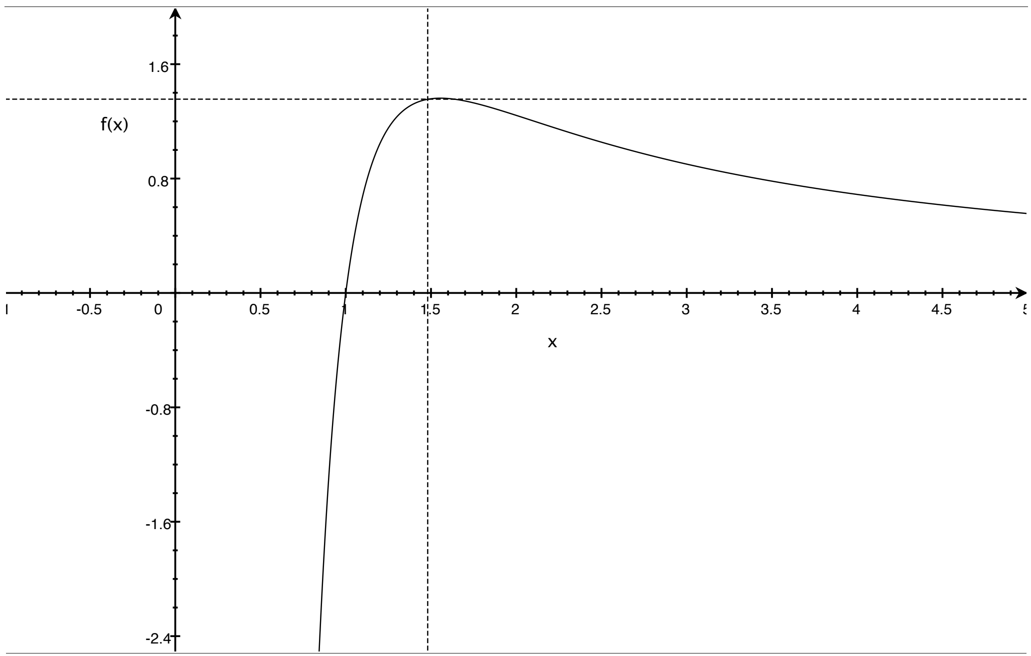

Figure 2: A plot of the function $f(x) = 2.8x^{-4/3} - 0.8x^{-10/3} - 2x^{-14/3}$, with the maximum point indicated.

Therefore there are three possibilities for solutions to (9.3):

1. if $\lambda_f > \left(1 + \frac{1.12}{27.8}\right)^{1/4} \approx 1.01$, there are no solutions;
2. for $\lambda_f \in (1.1.01]$, there are two solutions;
3. if $\lambda_f \leq 1$, there is a unique solution.

The upper limit on allowable fibre stretches seems a reasonable result physically as one would expect the fibres to suffer only an infinitesimal deformation in a typical application of the model. Since one would also expect only an infinitesimal contraction of the fibres before buckling, (9.3) is therefore only solved for $\lambda_f \in [0.99, 1.01]$. The solution curve for this range of fibre stretches, with the smaller of the two solutions for fibre stretches in the range $(1, 1.01]$ chosen, is given in Figure 3.

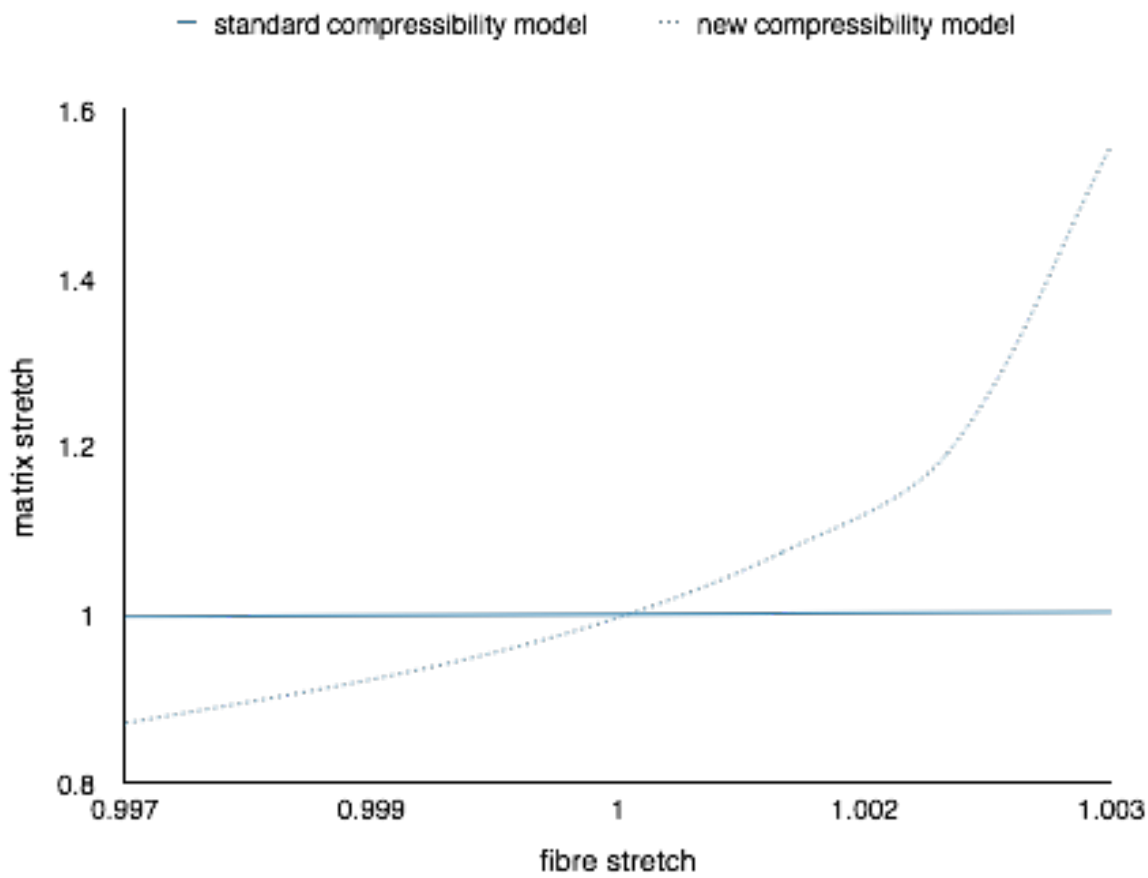

Figure 3: Plots of the relationships between the two stretches in hydrostatic tension and/or compression for both the standard compressibility model, which yields an isotropic response, and the new proposed model.

The two plots clearly demonstrate the more physically realistic response in hydrostatic tension and compression for the model proposed here: in tension, the transverse stretch is larger than the fibre stretch and smaller in compression. This is in contrast to the current standard model of slight compressibility which predicts that the two stretches should be the same.

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References

[1] Rivlin RS (1997) *Collected papers, Volume 1 (G. I. Barenblatt, Daniel D. Joseph Eds.)* New York: Springer.

[2] Vito RP, Dixon SA (2003). Blood vessel constitutive models-1995-2002. Annual review of Biomedical Engineering 5:413-439.

[3] Carew TE, Vaishnav RN, Patel DJ (1968) Compressibility of the arterial wall Circulation Research 23: 61-68.

[4] Weiss JA, Makerc BN, Govindjeed, S (1996) Finite element implementation of incompressible, transversely isotropic hyperelasticity. Comput. Methods Appl. Mech. Engrg. 135: 107-128

[5] Sansour C (2008) On the physical assumptions underlying the volumetric-isochoric split and the case of anisotropy. Eur J Mech A Solids 27: 28-39.

[6] Horgan CO, Murphy JG (2009) Constitutive modeling for moderate deformations of slightly compressible rubber. J. Rheol. 53: 153-168

[7] Ni Annaidh A, Destrade M, Gilchrist MD, Murphy JG (2013) Deficiencies in numerical models of anisotropic nonlinearly elastic materials, Biomechanics and Modeling in Mechanobiology 12:781-791.

[8] Helfenstein J, Jabareen M, Mazza E, Govindjee S (2010) On non-physical response in models for fiber-reinforced hyperelastic materials. International Journal of Solids and Structures 47: 2056-2061.

[9] Vergori L, Destrade M, McGarry P, Ogden RW (2013) On anisotropic elasticity and questions concerning its Finite Element implementation, Comput. Mech. 52:1185-1197.

[10] Penn RW (1070) Volume changes accompanying the extension of rubber. Trans. Soc. Rheol. 14: 509-517.

[11] Johnson VE, Stewart W, Smith DH (2013) Axonal pathology in traumatic brain injury. Experimental Neurology. 246:35-43.

[12] Ogden RW (2003) Nonlinear elasticity, anisotropy, material stability and residual stresses in soft tissue, In Biomechanics of Soft Tissue in Cardiovascular Systems, CISM Courses and Lectures Series no. 441, 65-108, Springer, Wien.

[13] Lubarda VA, Chen MC (2008) On the elastic moduli and compliances of transversely isotropic and orthotropic materials. J. Mech. Mat. Struct. 3:153-171.

[14] Rajagopal KB, Srinivasa AR (2009) On a class of non-dissipative solids that are not hyperelastic. Proc. Roy. Soc. Lond. A 465:493-500.

[15] Rajagopal KB (2010) On a new class of models in elasticity. Math. Comput. Applications 15:506-528.

[16] Bustamante R, Rajagopal KB (2010) A note on plane strain and stress for a new class of elastic bodies. Math. Mech. Solids 15:220-238.

[17] Merodio J, Ogden RW (2005) Remarks on instabilities and ellipticity for a fiber-reinforced compressible nonlinear elastic solid under plane deformation. Quart. Appl. Math. 63:325-333.

[18] Murphy JG (2013) Transversely isotropic biological, soft tissue must be modelled using both anisotropic invariants. Eur. J. Mech. A/Solids, 42:90-96.

[19] Destrade M, MacDonald B, Murphy JG, Saccomandi G (2013) At least three invariants are necessary to model the mechanical response of incompressible, transversely isotropic materials. Comput. Mech. 52:959-969.

[20] Feng Y, Okamoto RJ, Namani R, Genin GM, Bayly PV (2013) Measurements of mechanical anisotropy in brain tissue and implications for transversely isotropic material models of white matter. J. Mech. Behav. Biomed. Mater. 23:117-132.

[21] Musgrave MJP, Crystal Acoustics. Holden-Day (1970).

[22] Liao JJ, Hu TB, Chang CW (1997). Determination of dynamic elastic constants of transversely isotropic rocks using a single cylindrical specimen. Int. J. Rock Mech. Mining Sci. 34:1045-1054.

[23] Morrow DA, Haut Donahue TL, Odegard GM, Kaufman KR (2010) Transversely isotropic tensile material properties of skeletal muscle tissue, J. Mech. Behav. Biomed. Mater. 3:124-129.

[24] Ogden RW (1997) Nonlinear Elastic Deformations. Ellis Horwood, Chichester (1984). Reprinted by Dover, New York.

[25] Holzapfel GA (2000) Nonlinear Solid Mechanics, Wiley, Chichester.

[26] Holzapfel GA, Gasser TC, Ogden RW (2000) A new constitutive framework for arterial wall mechanics and a comparative study of material models. J. Elasticity 61:1-48.

[27] Maas SA, Ellis BJ, Ateshian GA, Weiss JA (2012) FEBio: finite elements for biomechanics. J. Biomed. Eng. 134: 011005.

[28] Merodio J, Ogden RW (2003) Instabilities and loss of ellipticity in fiber-reinforced compressible non-linearly elastic solids under plane deformation. Int. J. Solids Struct. 40:4707-4727.

[29] Chuong CJ, Fung YC (1984) Compressibility and constitutive equation of arterial wall in radial compression experiments. J. Biomechanics 17: 35-40.