ANGULAR DERIVATIVES ON BOUNDED SYMMETRIC DOMAINS

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ABSTRACT. In this paper we generalise the classical Julia–Wolff–Carathéodory theorem to holomorphic functions defined on bounded symmetric domains.

1. INTRODUCTION

Throughout, let $\mathsf{Hol}(A,B)$ denote the holomorphic functions from A to B, where A and B are domains in a complex Banach space and let Δ denote the open unit disc in $\mathbb{C}$. A classical theorem of complex analysis, known as the Julia–Wolff–Carathéodory theorem, is the following.

Theorem 1.1. *Let $f \in \mathsf{Hol}(\Delta,\Delta)$ satisfy*

$$\alpha := \liminf_{\zeta\to 1} \frac{1-|f(\zeta)|^2}{1-|\zeta|^2} < \infty$$

and $f(\zeta_n) \to 1$ for some sequence ζ_n converging to 1. Then the angular limit of f at 1 exists and equals 1, and the angular limit of the derivative f' at 1 exists and equals α.

If one transfers this theorem to the right half plane $\Pi \subset \mathbb{C}$ by means of the Cayley transform $\zeta \mapsto \frac{1+\zeta}{1-\zeta}$ then the statement becomes the following.

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Theorem 1.2. *Let $F \in \mathsf{Hol}(\Pi, \Pi)$ and let $a := \inf_{\Pi} \dfrac{\mathrm{Re}\, F(z)}{\mathrm{Re}\, z}$. Then*

$$a = \angle\text{-}\lim_{z\to\infty} \frac{\mathrm{Re}\, F(z)}{\mathrm{Re}\, z} = \angle\text{-}\lim_{z\to\infty} \frac{F(z)}{z} = \angle\text{-}\lim_{z\to\infty} F'(z).$$

Generalisations of these results have evolved in two directions. In 1980, Rudin [16] achieved a complete extension for a holomorphic map from B_n to B_m, where B_n denotes the open unit ball in the Euclidean norm of $\mathbb{C}^n$ (which is a strictly convex bounded symmetric domain). Let ∂B_n denote the boundary of B_n. Rudin's result contains the following.

Theorem 1.3. *Let $e \in \partial B_n$ and let $F = (f_1, \ldots, f_m) \in \mathsf{Hol}(B_n, B_m)$ satisfy $F(0) = 0$ and*

$$\alpha := \liminf_{z\to e} \frac{1 - \|F(z)\|^2}{1 - \|z\|^2} < \infty.$$

If $(z_k)_k \subset B$ satisfies $\lim_k z_k = e$,

$$\lim_k \frac{1 - \|F(z_k)\|^2}{1 - \|z_k\|^2} = \alpha$$

and $\lim_k F(z_k) = e' \in \partial B_m$, then the angular limit of $F(z)$ as $z \to e$ is e' and the restricted angular limit $F_1'(z)e$ is $\alpha e'$, where $F_1(z) = \langle F(z), e' \rangle$.

The Hilbert ball is a prototype for two important classes of domain, namely the strictly pseudoconvex domains and the bounded symmetric domains. The theory of angular limits and angular derivatives for functions on a strictly pseudoconvex domain is well developed (see [1] for a comprehensive account). We concentrate on the second category, that of bounded symmetric domains.

Fan [6], in 1986, proved the following Julia–Wolff–Carathéodory theorem for operator valued holomorphic functions on Δ. H denotes a complex Hilbert space and $\mathcal{L}(H)$ is the C^*-algebra of bounded linear operators from H to itself. $\Pi_{\mathcal{L}(H)} := \{T \in \mathcal{L}(H) : \mathrm{Re}\, T > 0\}$ is a generalised half-plane in $\mathcal{L}(H)$.

Theorem 1.4. *Let $F \in \mathsf{Hol}(\Pi, \Pi_{\mathcal{L}(H)})$. Suppose that there exists $A = A^* \in \mathcal{L}(H)$ with*

$$\frac{\operatorname{Re} F(z)}{\operatorname{Re} z} > A$$

for all $z \in \Pi$ and such that

$$\inf_{\Pi} \left\| \frac{\operatorname{Re} F(z)}{\operatorname{Re} z} - A \right\| = 0.$$

Then

$$A = \angle\text{-}\lim_{z\to\infty} \frac{\operatorname{Re} F(z)}{\operatorname{Re} z} = \angle\text{-}\lim_{z\to\infty} \frac{F(z)}{z} = \angle\text{-}\lim_{z\to\infty} F'(z).$$

Włodarczyk [18] generalises Theorem 1.4 by allowing in place of $\mathcal{L}(H)$ any J^*-algebra having a non-zero partial isometry. In [14] the authors further extend Theorem 1.4 to the case of JB^*-triples using the concept of Siegel domain in a JB^*-triple in place of $\Pi_{\mathcal{L}(H)}$. In this paper, we pursue a more general result for holomorphic functions between arbitrary bounded symmetric domains, more in the spirit of Theorem 1.3. Results of this type in the literature are sparse even in finite dimensions, with the case of the polydisc Δ^n, that is $\mathsf{Hol}(\Delta^n, \Delta^m)$, only being recently resolved [2]. A principal reason for this sparsity has been the lack of suitable analogues of either the classical Julia's lemma or Wolff's theorem for bounded symmetric domains. These were recently provided in [15] and play a crucial role in achieving angular limit and angular derivative results, in particular motivating the definition of angular approach region for bounded symmetric domains. The existence of angular limits over these regions is then reduced by an extension of the classical Lindelöf–Čirka principle (cf. [16]) to the existence of certain radial limits. A crucial tool throughout is the concept of a Bergman operator on a JB^*-triple.

2. Bounded Symmetric Domains and JB^*-triples

Let D be a bounded domain in a complex Banach space E. A symmetry at a point $a \in D$ is a biholomorphic map s on E for which a is an isolated fixed point of s and $s = s^{-1}$. D is said to be a **bounded symmetric domain** if there is a symmetry at every $a \in D$. Every bounded symmetric domain (with fixed base-point) is biholomorphically equivalent to the unit ball of a (unique) JB^*-triple.

Definition 2.1. A JB^*-triple is a complex Banach space Z with a continuous map $\{\cdot,\cdot,\cdot\} : Z^3 \to Z$, $(x, y, z) \to \{x, y, z\}$, which is complex linear and symmetric in x and z, anti-linear in y and satisfies

(i) the operator $x \,\square\, x$ has spectrum in $[0, \infty)$,

(ii) $\exp(i\, x \,\square\, x\,)$ is both an algebraic automorphism and an isometry,

(iii) $\|\{x, x, x\}\| = \|x\|^3$,

for all $x \in Z$, where $x \,\square\, y$ denotes the linear map $z \mapsto \{x, y, z\}$.

The equality

(1)
$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$$

which ensues from (ii) for all a, b, x, y and $z \in Z$ is known as the Jordan triple identity, and provides a weak form of associativity for the triple product. The inequality

$$\|\{x, y, z\}\| \leq \|x\|\|y\|\|z\| \tag{2}$$

is proved in [7].

Any C^*-algebra, and more generally any J^*-algebra, is a JB^*-triple with triple product given by $\{x, y, z\} = \frac{1}{2}(xy^*z+zy^*x)$ where x^* denotes the usual operator adjoint of x. In particular, every complex Hilbert space is a JB^*-triple whose triple product is given by $\{x, y, z\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x)$.

The Bergman operator $B(x, y) \in \mathcal{L}(Z)$ defined by

$$B(x, y) = \mathsf{Id} - 2\, x \,\square\, y\, + Q_x Q_y$$

where $Q_x(z) = \{x, z, x\}$ is an important tool, encoding much of the geometry of Z in the same way that an inner product does for a Hilbert space. On a C^*-algebra, the Bergman operator reduces to $B(x, y)z = (1 - xy^*)z(1 - y^*x)$.

An element $e \in Z$ for which $\{e, e, e\} = e$ is called a tripotent and, from (iii) above, a non-zero tripotent has norm one. For example, a tripotent of a C^*-algebra is an element v satisfying $v = vv^*v$, that is, a partial isometry. Each tripotent induces a splitting of Z, called the Peirce decomposition, into $Z = Z_1 \oplus Z_{\frac{1}{2}} \oplus Z_0$ where Z_k is the k-eigenspace of $e \,\square\, e$, with mutually orthogonal projections P_k onto the subspaces Z_k,

$$\begin{aligned} P_1 &= Q_e Q_e, \\ P_{\frac{1}{2}} &= 2\, e \,\square\, e \, - 2 Q_e Q_e, \\ P_0 &= B(e,e), \end{aligned}$$

satisfying $P_1 + P_{\frac{1}{2}} + P_0 = \mathsf{Id}$. Where the need arises, we write P_j^e rather that P_j to highlight the tripotent in question. The tripotent e is called MAXIMAL if $Z_0 = \{0\}$ and this is the case precisely when e is an extreme point of the unit ball of Z [10]. The tripotent is called UNITARY if $P_0 = P_{\frac{1}{2}} = 0$. We say $z \in Z$ is proportional to a tripotent if $z = \lambda e$ for some tripotent $e \in Z$ and some $\lambda \in \mathbb{C}$. We point out that since the triple product is continuous, the set of tripotents forms a closed subset of Z.

3. GENERALISED HOROCYCLES AND THE ANGULAR LIMIT

In Δ the sequence $(\xi_n)_n$ is said to approach $a \in \partial\Delta$ non-tangentially if $\xi_n \to a$ and if, for some $k > 0$, $(\xi_n)_n$ is eventually in the angular region

$$\Delta_k(a) = \{\xi \in \Delta : \frac{|a-\xi|}{1-|\xi|^2} < k\}.$$

Let $f \in \mathsf{Hol}(\Delta, \Delta)$. Then f is said to have angular limit b at a if $f(\xi_n) \to b$ whenever $\xi_n \to a$ non-tangentially. This is written as $\angle\text{-}\lim_{\xi \to a} f(\xi) = b$.

Let B be a bounded symmetric domain, which we may take to be the open unit ball of a JB^*-triple Z [8]. For $e \in \partial B$ and $k > 0$, we define a GENERALISED ANGULAR REGION by

$$D_k(e) := \{w \in B : \|B(w,e)Q_e Q_e\|^{\frac{1}{2}} < k(1 - \|w\|^2)\}.$$

Of course, if $B = \Delta$ then $D_k(1) = \Delta_k(1)$. We note that e is in the boundary of $D_k(e)$ only if $B(e,e)Q_e Q_e = 0$, a condition which is equivalent to e being a tripotent. As we wish to consider approach paths to a boundary point e which are contained in the angular region $D_k(e)$ (such approach paths will be called non-tangential), we take it that from this point on,

e denotes a tripotent of the JB^*-triple Z. The angular region $D_k(e)$ can then be rewritten as

$$D_k(e) := \{w \in B : \|B(w,e)|_{Z_1}\|^{\frac{1}{2}} < k(1 - \|w\|^2)\}$$

where Z_1 is the Peirce 1-space of the tripotent e. Note that if Z is a Hilbert space this reduces to $D_k(e) := \{w \in B : |1 - \langle w, e\rangle| < k(1 - \|w\|^2)\}$.

Another type of domain which appears naturally in any discussion of angular limits in Δ is the horocycle, which is an open disc in Δ internally tangent to the boundary. The horocycle

$$\mathcal{E}_\lambda(a) = \{\xi \in \Delta : \frac{|a - \xi|^2}{1 - |\xi|^2} < \lambda\}$$

is a Euclidean disc of radius $\frac{\lambda}{\lambda+1}$ internally tangent to $\partial\Delta$ at a. In Figure 1, we sketch the angular region $\Delta_{1.2}(1)$ and the horocycle $\mathcal{E}_{\frac{1}{2}}(1)$ in Δ. Bergman operators are used to generalise these horocycles to a bounded

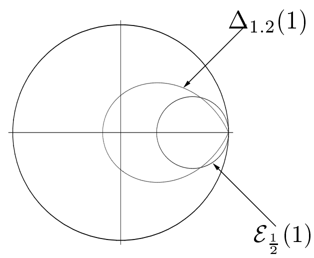

FIGURE 1. Angular region and horocycle in the unit disc

symmetric domain B. For a tripotent e in B and $\lambda > 0$, define

$$E_\lambda(e) := \{w \in B : \|B_w^{-1}B(w,e)Q_eQ_e\| < \lambda\}$$

where $B_w := B(w,w)^{\frac{1}{2}}$ is invertible for $\|w\| < 1$ by [8]. We will frequently use the fundamental identity [9] which holds for all $x \in B$,

$$\|B_x^{-1}\| = \frac{1}{1 - \|x\|^2}. \tag{3}$$

The next result is a concrete realisation of the horocycles $E_\lambda(e)$. The proof is a distillation of various results and techniques in [15].

Proposition 3.1. *Let $e \in \partial B$ be a tripotent. For all $\lambda > 0$, the horocycle $E_\lambda(e)$ has the form*

$$E_\lambda(e) = \left(\frac{1}{1+\lambda}\right) e + B(se, se)(B)$$

for $s > 0$ satisfying $(1-s^2)^2 = \frac{\lambda}{1+\lambda}$. Moreover, for any $y \in B$, $y \in \partial E_{\lambda_y}(e)$ where $\lambda_y := \|B_y^{-1} B(y,e) Q_e Q_e\| > 0$.

We note that $B(se, se) \in \mathsf{GL}(Z)$, the group of invertible bounded linear operators on Z. In particular, $E_\lambda(e)$ is a convex domain in B containing e in its boundary.

Proof. It is easy to calculate

$$B(\beta e, \beta e) = P_0 + (1-|\beta|^2) P_{\frac{1}{2}} + (1-|\beta|^2)^2 P_1 \tag{4}$$

and

$$B(\beta e, e) = P_0 + (1-\beta) P_{\frac{1}{2}} + (1-\beta)^2 P_1 \tag{5}$$

for all $\beta \in \mathbb{C}$. Choose a sequence $(\alpha_k)_k$, $0 < \alpha_k < 1$ with $\lim_k \alpha_k = 1$. Then

$$\begin{aligned} B^{-1}_{\alpha_k e} &= B(\alpha_k e, \alpha_k e)^{-\frac{1}{2}} \\ &= (P_0 + (1-\alpha_k^2) P_{\frac{1}{2}} + (1-\alpha_k^2)^2 P_1)^{-\frac{1}{2}} \\ &= P_0 + \frac{1}{\sqrt{1-\alpha_k^2}} P_{\frac{1}{2}} + \frac{1}{1-\alpha_k^2} P_1 \end{aligned}$$

and therefore

$$\lim_k (1-\alpha_k^2) B^{-1}_{\alpha_k e} = P_1 = Q_e Q_e. \tag{6}$$

Take $w \in E_\lambda(e)$, that is, $\|B_w^{-1} B(w,e) Q_e Q_e\| < \lambda$. Then

$$\|B_w^{-1} B(w, \alpha_k e) B^{-1}_{\alpha_k e}\| < \frac{\lambda}{1-\alpha_k^2}$$

for all k large. In the notation of [15], and using [15, Corollary 3.2], we have that w belongs to the Kobayashi ball $D_{\alpha_k e, r_k}$ for all k large where $r_k > 0$ satisfies $(1-r_k^2)^{-1} = \lambda(1-\alpha_k^2)^{-1}$. Proposition 2.3 of [15] then implies that $w \in D_{\alpha_k e, r_k} = c_k + T_k(B)$ for all k large where

$$c_k = (1-r_k^2) B^{-1}_{r_k \alpha_k e}(\alpha_k e) \in B \quad \text{and} \quad T_k = r_k B_{\alpha_k e} B^{-1}_{r_k \alpha_k e} \in \mathsf{GL}(Z).$$

From (4) one can then calculate that

$$c_k = \frac{\alpha_k}{\lambda + \alpha_k^2} e \quad \text{and} \quad T_k = B(s_k e, s_k e)$$

for $s_k > 0$ satisfying $(1 - s_k^2)^2 = \lambda(\lambda + \alpha_k^2)^{-1}$. Since $c := \lim_k c_k = (1+\lambda)^{-1} e$ and $T := \lim_k T_k = B(se, se) \in \mathsf{GL}(Z)$ for $s > 0$ satisfying $(1 - s^2)^2 = \lambda(1+\lambda)^{-1}$, the above shows that $w \in \overline{c + T(B)}$. Therefore, $E_\lambda(e) \subset \overline{c + T(B)}$. One may reverse this argument to show that $c + T(B) \subset \overline{E_\lambda(e)}$ and hence $E_\lambda(e) = c + T(B)$. As $T \in \mathsf{GL}(Z)$, it follows that $c + T(B)$ is a convex domain in B.

$$T(e) = B(se, se)e = (1 - s^2)^2 e = \lambda(1+\lambda)^{-1} e$$

implies that $e \in \partial E_\lambda(e)$.

Fix $y \in B$. Since $\|y\|\|e\| < 1$, $B(y, e)$ is invertible [8] and hence

$$\lambda_y := \|B_y^{-1} B(y, e) Q_e Q_e\| \geq \frac{\|Q_e Q_e\|}{\|B(y, e)^{-1} B_y\|} = \frac{1}{\|B(y, e)^{-1} B_y\|} > 0.$$

It is then clear from the definition that $y \in \partial E_{\lambda_y}(e)$. □

Note 3.2. For a tripotent e, one can see from the above formulation of $E_\lambda(e)$ that $\|x - \frac{e}{1+\lambda}\| < \frac{\lambda}{1+\lambda}$ for $x \in P_1(E_\lambda(e))$. In particular, $\|x - e\| < \frac{2\lambda}{1+\lambda} \leq 2\lambda$ for $x \in P_1(E_\lambda(e))$.

The following Julia type lemma for a holomorphic function between bounded symmetric domains B and B', contained in JB^*-triples Z and Z', is of a type first proved in [15].

Lemma 3.3. *Let $f\colon B \to B'$ be holomorphic. Let $e \in \partial B$ be a tripotent. If there exists $(z_k)_k \subset B$ such that $z_k \to e$ and $f(z_k) \to e' \in \partial B'$ where each z_k and each $f(z_k)$ is proportional to a tripotent, and*

$$\alpha := \liminf_{k \to \infty} \frac{1 - \|f(z_k)\|^2}{1 - \|z_k\|^2} < \infty$$

then $f(E_\lambda(e)) \subset E_{\alpha\lambda}(e')$.

Proof. Since the tripotents form a closed set in any JB^*-triple it follows immediately that e' is also a tripotent. Since $z_k = \alpha_k e_k$ for $\alpha_k \in \mathbb{C}$ and

e_k a tripotent, we have from (4) that

$$(1-\|z_k\|^2)B_{z_k}^{-1} = (1-|\alpha_k|^2)P_0^{e_k} + (1-|\alpha_k|^2)^{\frac{1}{2}}P_{\frac{1}{2}}^{e_k} + P_1^{e_k}$$

and thus $(1-\|z_k\|^2)B_{z_k}^{-1} \to P_1^e = Q_eQ_e$. Similarly,

$$(1-\|f(z_k)\|^2)B_{f(z_k)}^{-1} \to P_1^{e'} = Q_{e'}Q_{e'}.$$

The following Schwarz–Pick type result holds [15, Corollary 3.3] as a consequence of the Schwarz lemma,

$$\|B_{f(w)}^{-1}B(f(w),f(z))B_{f(z)}^{-1}\| \leq \|B_w^{-1}B(w,z)B_z^{-1}\| \tag{7}$$

for all z and w in B. In particular this holds for $z = z_k$ and so for all k

$$\begin{aligned}&\left\|B_{f(w)}^{-1}B(f(w),f(z_k))\,(1-\|f(z_k)\|^2)B_{f(z_k)}^{-1}\right\| \\ &\qquad\qquad \leq \frac{1-\|f(z_k)\|^2}{1-\|z_k\|^2}\left\|B_w^{-1}B(w,z_k)\,(1-\|z_k\|^2)B_{z_k}^{-1}\right\|.\end{aligned}$$

We take a limit over k to obtain

$$\|B_{f(w)}^{-1}B(f(w),e')Q_{e'}Q_{e'}\| \leq \alpha\|B_w^{-1}B(w,e)Q_eQ_e\|$$

for all $w \in B$. That is, $f(E_\lambda(e)) \subset E_{\alpha\lambda}(e')$. □

Note 3.4. The α appearing in the previous result depends of course on the sequence $(z_k)_k$ chosen. We will have occasion later to choose the least possible α and so the following reformulation may be more appropriate.

Corollary 3.5. *Let $f\colon B \to B'$ be holomorphic. Let $e \in \partial B$ be a tripotent and let*

$$\alpha := \liminf_{z\to e}\frac{1-\|f(z)\|^2}{1-\|z\|^2} < \infty.$$

If there exists a sequence $(z_k)_k$ in B converging to e such that $f(z_k) \to e' \in \partial B'$, each z_k and each $f(z_k)$ is proportional to a tripotent and $(z_k)_k$ satisfies $\liminf_{k\to\infty}\frac{1-\|f(z_k)\|^2}{1-\|z_k\|^2} = \alpha$ then $f(E_\lambda(e)) \subset E_{\alpha\lambda}(e')$ for all $\lambda >$ 0.

Not surprisingly when $B' = \Delta$ and the hypothesis of the above result is satisfied, the sequence $(z_k)_k$ may be taken to be of the form $(r_ke)_k$ where each $r_k \in [0,1)$. The proof is similar to that of [1, Lemma 3.2].

Lemma 3.6. *Let $f \in \mathsf{Hol}(B, \Delta)$. Let $e \in \partial B$ be a tripotent and suppose there exists a sequence $(z_k)_k \in B$ converging to e each element of which is proportional to a tripotent and for which*

$$\liminf_{k\to\infty} \frac{1-|f(z_k)|^2}{1-\|z_k\|^2} = \liminf_{z\to e} \frac{1-|f(z)|^2}{1-\|z\|^2}.$$

Then

$$\liminf_{z\to e} \frac{1-|f(z)|^2}{1-\|z\|^2} = \liminf_{\zeta\to 1} \frac{1-|f(\zeta e)|^2}{1-|\zeta|^2} = \liminf_{t\to 1^-} \frac{1-|f(te)|^2}{1-t^2}.$$

Proof. Clearly,

$$\begin{aligned}\alpha := \liminf_{z\to e} \frac{1-|f(z)|^2}{1-\|z\|^2} &\le \liminf_{\zeta\to 1} \frac{1-|f(\zeta e)|^2}{1-|\zeta|^2} \\ &\le \liminf_{t\to 1^-} \frac{1-|f(te)|^2}{1-t^2} =: \beta\end{aligned}$$

and so we may suppose that $\alpha < \infty$. We need only show therefore that $\alpha \ge \beta$. Corollary 3.5 implies that $f(E_\lambda(e)) \subset E_{\alpha\lambda}(e')$ for some $e' \in \partial\Delta$, for all $\lambda > 0$. Without loss of generality, we may assume that $e' = 1$. Therefore for all $w \in B$,

$$\|B_{f(w)}^{-1} B(f(w), e') Q_{e'} Q_{e'}\| \le \alpha \|B_w^{-1} B(w, e) Q_e Q_e\|$$

and, in particular for $w = \omega e$, $\omega \in \Delta$. Then (4) and (5) imply

$$\frac{|1-f(\omega e)|^2}{1-|f(\omega e)|^2} \le \alpha \frac{|1-\omega|^2}{1-|\omega|^2}.$$

Let $t_k = \frac{k-1}{k+1}$. Then $\frac{|1-t_k|^2}{1-|t_k|^2} = \frac{1}{k}$ and so

$$\frac{|1-f(t_k e)|^2}{1-|f(t_k e)|^2} \le \frac{\alpha}{k}.$$

That is, $f(t_k e) \in \overline{E_{\alpha/k}(1)}$. We use Note 3.2 to write

$$1 - |f(t_k e)| \le |1 - f(t_k e)| \le \frac{2(\alpha/k)}{1 + (\alpha/k)} = \frac{2\alpha}{\alpha + k}.$$

As $1-|t_k|^2 = 4k/(k+1)^2$ we get

$$\beta = \liminf_{t\to 1} \frac{1-|f(te)|^2}{1-t^2} \leq \liminf_{k\to\infty} \frac{1-|f(t_k e)|^2}{1-t_k^2} \leq \lim_{k\to\infty} \frac{(4\alpha)/(\alpha+k)}{4k/(k+1)^2} = \alpha$$

as required. $\square$

We now obtain the existence of an angular limit where we take as our hypothesis the conclusion of Lemma 3.3.

Theorem 3.7. *Let $f \in \mathsf{Hol}(B, B')$ and let $e \in \partial B$ be a tripotent. Let e' be an extreme point in B'. If there exists $\alpha > 0$ such that $f(E_\lambda(e)) \subset E_{\alpha\lambda}(e')$ for all $\lambda > 0$ then*

$$\angle\text{-}\lim_{x\to e} f(x) = e'.$$

Proof. Fix an angular region $D_k(e)$ and let $\varepsilon > 0$. If $w \in D_k(e)$ then

$$\begin{aligned}\|B_w^{-1} B(w,e) Q_e Q_e\| &\leq \|B_w^{-1}\| \|B(w,e) Q_e Q_e\| \\ &= \frac{\|B(w,e)Q_eQ_e\|}{1-\|w\|^2} \\ &< k^2(1-\|w\|^2).\end{aligned}$$

It follows that if $(w_n)_n \subset D_k(e)$ converges to e then there exists n_ε such that $w_n \in E_\varepsilon(e)$ for all $n \geq n_\varepsilon$ and thus, by hypothesis, we have $f(w_n) \in E_{\alpha\varepsilon}(e')$ for all $n \geq n_\varepsilon$. In other words,

$$\|B_{f(w_n)}^{-1} B(f(w_n), e') Q_{e'} Q_{e'}\| < \alpha\varepsilon \text{ for all } n \geq n_\varepsilon.$$

Since ε is arbitrary, $\|B_{f(w_n)}^{-1} B(f(w_n), e') Q_{e'} Q_{e'}\| \to 0$. This implies, from Lemma 3.8 below, that $f(w_n) \to e'$. Since k is arbitrary this implies that $\angle\text{-}\lim_{x\to e} f(x) = e'$. $\square$

Lemma 3.8. *Let e be an extreme point of B. Let $(x_n)_n \subset B$ satisfy*

$$\lim_n \|B_{x_n}^{-1} B(x_n, e) Q_e Q_e\| = 0.$$

Then $\lim_n x_n = e$.

Proof. We consider the domains $E_{\gamma_n}(e)$ for $\gamma_n = \|B_{x_n}^{-1}B(x_n,e)Q_eQ_e\|$. We know from Proposition 3.1 that x_n and e are elements of $\partial E_{\gamma_n}(e)$. It follows from [15, Proposition 3.15] that $E_{\gamma_n}(e)$ has the alternative description

$$E_{\gamma_n}(e) = \{z \in B : \left\| \frac{1}{t_n} P_{\frac{1}{2}}(z) + \frac{P_1(z) - e}{t_n^2} + e \right\| < 1\}$$

where $t_n = \sqrt{\frac{\gamma_n}{1+\gamma_n}}$. This gives

$$\left\| \frac{P_{\frac{1}{2}}(x_n)}{t_n} + \frac{P_1(x_n) - e}{t_n^2} + e \right\| = 1 \tag{8}$$

for all n. Since $t_n \to 0$, one sees easily that $P_1(x_n) \to e$. Now consider the Peirce reflection $g = \exp(2\pi i\, e \square e\,)$ which, by definition of a JB^*-triple, is a linear isometry. It is not difficult to see that g acts as the identity Id on Z_1 and as $-\mathsf{Id}$ on $Z_{\frac{1}{2}}$. Then from (8)

$$\left\| \frac{-P_{\frac{1}{2}}(x_n)}{t_n} + \frac{P_1(x_n) - e}{t_n^2} + e \right\| = 1$$

and this implies that $\left\| \frac{P_{\frac{1}{2}}(x_n)}{t_n} \right\| < 1$. As $t_n \to 0$, we must have $P_{\frac{1}{2}}(x_n) \to 0$ and so $\lim_n x_n = \lim_n P_1(x_n) + \lim_n P_{\frac{1}{2}}(x_n) = e$. $\square$

The following rather technical lemma generalises [6, Proposition 2.1] and [16, Lemma 8.5.5(i)]. It provides useful information about the geometry of the angular regions $D_k(e)$ and is used several times in the sequel.

Lemma 3.9. *Suppose* $1 < \beta < \alpha$ *and* $\|b\| = 1$, *so that* $D_\beta(b) \subsetneqq D_\alpha(b)$. *There exists* $\delta > 0$ *such that for* $x \in D_\beta(b)$ *and* $\|y\| \le 1$,

$$|\lambda| \le \delta \|B(x,b)Q_bQ_b\|^{\frac{1}{2}} \implies x + \lambda y \in D_\alpha(b).$$

Proof. Let $\delta = (1/\beta - 1/\alpha)/10$ and $x \in D_\beta(b)$. Of course, $\delta < \frac{1}{10}$ so $|\lambda| \le \delta\|B(x,b)Q_bQ_b\|^{\frac{1}{2}}$ implies that $|\lambda| < 1$ because from (2)

$\|B(u,v)\| \le (1+\|u\|\|v\|)^2$ for any u and v. We have

$$\begin{aligned}\|x+\lambda y\|^2 &\le \|x\|^2 + 2\|x\||\lambda| + |\lambda|^2 \\ &\le \|x\|^2 + 3|\lambda|.\end{aligned}$$

A standard identity for Bergman operators [12, JP34] lets us write

$$B(x+\lambda y, b)Q_bQ_b = B(\lambda y, b^x)B(x,b)Q_bQ_b$$

where $b^x = B(x,b)^{-1}(x - Q_xb)$ denotes the quasi-inverse of b with respect to x (see [5, 12]). Thus

$$\|B(x+\lambda y, b)Q_bQ_b\| \le \|B(x,b)Q_bQ_b\|(1+|\lambda|\|b^x\|)^2. \tag{9}$$

Since $\|b\|\|x\| < 1$, it follows easily from the series expansion (see [12])

$$b^x = \sum_{j=0}^{\infty} (\, b \,\square\, x\,)^j b$$

that

$$\|b^x\| \le \frac{\|b\|}{1-\|b\|\|x\|} = \frac{1}{1-\|x\|} = \frac{1+\|x\|}{1-\|x\|^2} < \frac{2}{1-\|x\|^2}. \tag{10}$$

Now we use the fact that $x \in D_\beta(b)$ to write

$$\|b^x\| < 2(1-\|x\|^2)^{-1} < 2\beta\|B(x,b)Q_bQ_b\|^{-\frac{1}{2}}.$$

Therefore

$$\begin{aligned}\|B(x+\lambda y,b)Q_bQ_b\|^{\frac{1}{2}} &\le \|B(x,b)Q_bQ_b\|^{\frac{1}{2}}\Big(1+|\lambda|\frac{2\beta}{\|B(x,b)Q_bQ_b\|^{\frac{1}{2}}}\Big) \\ &\le \|B(x,b)Q_bQ_b\|^{\frac{1}{2}} + 2|\lambda|\beta. \qquad (11)\end{aligned}$$

Since $\frac{\beta}{\alpha} < 1$, we have

$$\|x+\lambda y\|^2 + \frac{1}{\alpha}\|B(x+\lambda y,b)Q_bQ_b\|^{\frac{1}{2}} \le$$
$$\|x\|^2 + 3|\lambda| + \frac{1}{\alpha}\|B(x,b)Q_bQ_b\|^{\frac{1}{2}} + 2|\lambda|$$

which gives

$$\|x+\lambda y\|^2 + \frac{1}{\alpha}\|B(x+\lambda y,b)Q_bQ_b\|^{\frac{1}{2}} \le$$
$$\|x\|^2 + (5\delta + \frac{1}{\alpha})\|B(x,b)Q_bQ_b\|^{\frac{1}{2}}.$$

Finally, as $5\delta + \frac{1}{\alpha} < \frac{1}{\beta}$ and $x \in D_\beta(b)$, we can write

$$\begin{aligned}\|x + \lambda y\|^2 + \frac{1}{\alpha}\|B(x + \lambda y, b)Q_b Q_b\|^{\frac{1}{2}} &\leq \|x\|^2 + \frac{1}{\beta}\|B(x, b)Q_b Q_b\|^{\frac{1}{2}} \\ &< 1.\end{aligned}$$

That is, $x + \lambda y \in D_\alpha(b)$. □

The following estimates are required in the sequel.

Lemma 3.10. *Let Z be a JB^*-triple and let $e \in Z$ be a tripotent. Then,*

$$\|x - e\|^2 \leq \|B(x, e)Q_e Q_e\| \leq 3\|x - e\|^2$$

for any $x \in Z_1$.

Proof. The space $Z_1 = P_1(Z)$ is a JB^*-algebra with respect to the product $x \circ y := \{x, e, y\}$ and involution $x^* := \{e, x, e\}$ (cf. [17]). The Bergman operator $B(x, e)|_{Z_1}$ may be represented as $2L^2_{e-x} - L_{(e-x)^2}$ where $L_x(y) := x \circ y = \{x, e, y\}$. Thus the right hand inequality is immediate using (2).

For any JB^*-algebra A, we have that $\|2y(y^*y) - y^2y^*\| = \|y\|^3$ for all $y \in A$. Therefore, for $x \in Z_1$,

$$\begin{aligned}\|B(x, e)|_{Z_1}\| &\geq \|[2L^2_{e-x} - L_{(e-x)^2}]\frac{(e - x)^*}{\|e - x\|}\| \\ &= \|2(e - x)((e - x)^*(e - x)) - (e - x)^2(e - x)^*\|/\|e - x\| \\ &= \|e - x\|^3/\|e - x\| = \|e - x\|^2.\end{aligned}$$

□

4. THE LINDELÖF PRINCIPLE

The Lindelöf principle [11] allows one to deduce the existence of an angular limit at $b \in \partial\Delta$ for a bounded function on Δ from the existence of a limit along any one approach curve to b. The following result is a version of this principle for functions mapping Δ to a JB^*-triple Z. The proof, which we include for completeness, is a very slight modification of the classical one provided in [16]. For a bounded symmetric domain B and $b \in \partial B$, we define a b-CURVE to be a continuous curve $\sigma : [0,1) \to B$ such that $\sigma(t) \to b$ as $t \to 1$.

Theorem 4.1. *Let $f\colon \Delta \to Z$ be holomorphic and bounded and $\gamma : [0,1) \to \Delta$ be a 1-curve. If $\lim_{t\to 1^-} f(\gamma(t)) = l \in Z$ then f has angular limit l at 1.*

Proof. Without loss of generality, $\|f\| = 1$ and $l = 0$. Let Σ denote the strip $\{z \in \mathbb{C} : |\mathrm{Re}\, z| < 1\}$ and let $\varphi : \Delta \to \Sigma$ be a conformal mapping for which $\varphi(0) = 0$. Let $\Gamma = \varphi \circ \gamma$ and $F = f \circ \varphi^{-1}$. Then $\mathrm{Im}\,(\Gamma(t)) \to \infty$ and $F(\Gamma(t)) \to 0$ as $t \to 1$. In this setting, an angular limit of $f(t)$ as $t \to 1$ in Δ is equivalent to a uniform limit of $F(x+iy)$ as $y \to \infty$ in the strip $\{x+iy \in \mathbb{C} : |x| \le 1-\delta\}$ for any $\delta > 0$.

Given $\delta \in (0,1)$ we have to show therefore that $F(x+iy) \to 0$ uniformly as $y \to \infty$ for $|x| \le 1-\delta$. If we fix $\varepsilon \in (0,1)$, and choose $y_0 > \mathrm{Im}\,\Gamma(0)$ such that $\|F(\Gamma(t))\| < \varepsilon$ for $\mathrm{Im}\,\Gamma(t) \ge y_0$, the proof will follow from the statement that

$$\|F(x+iy_0)\| \le \varepsilon^{\delta/6} \text{ if } |x| \le 1-\delta. \tag{12}$$

To show (12) we can assume, by a vertical translation of Σ, that $y_0 = 0$. Choosing t_0 with $\mathrm{Im}\,\Gamma(t_0) = 0$ and $\mathrm{Im}\,\Gamma(t) > 0$ for all $1 > t > t_0$, we let $E = \{\Gamma(t) : t_0 \le t < 1\}$ and $\overline{E}$ be the reflection of E in the x-axis. Let $x_0 = \Gamma(t_0) \in \mathbb{R}$. Suppose $x_0 \le x \le 1-\delta$. Then we can define, for $\eta \in (0, \frac{1}{3})$ and $z \in \Sigma$,

$$G_\eta(z) = \{F(z), F(\overline{z}), F(z)\} \frac{\varepsilon^{(1+z)/2}}{1+\eta(1+z)}. \tag{13}$$

G_η is a holomorphic function and is bounded on Σ.

Since $\|F(z)\| < \varepsilon$ on E and $\|F(\overline{z})\| < \varepsilon$ on $\overline{E}$, we have that

$$\|\{F(z), F(\overline{z}), F(z)\}\| < \varepsilon$$

on $E \cup \overline{E}$. It follows that $\|G_\eta(z)\| < \varepsilon$ on $E \cup \overline{E}$. On the right hand boundary of Σ, $\|G_\eta(z)\| < \varepsilon$ and for $|\mathrm{Im}\, z|$ sufficiently large we also have $\|G_\eta(z)\| < \varepsilon$. Now we apply the maximum principle in the component of the strip bounded by $\operatorname{Re} z = 1$, $E \cup \overline{E}$ and $\pm\infty$ to get that $\|G_\eta(z)\| < \varepsilon$ in this component. In particular,

$$\begin{aligned}\|F(x)\|^3 \frac{\varepsilon^{(1+x)/2}}{1+\eta(1+x)} &= \|\{F(x), F(x), F(x)\} \frac{\varepsilon^{(1+x)/2}}{1+\eta(1+x)}\| \\ &= \|G_\eta(x)\| < \varepsilon\end{aligned}$$

and so $\|F(x)\|^3 < \varepsilon.\varepsilon^{-(1+x)/2}(1+\eta(1+x))$. We let $\eta \to 0$ to get $\|F(x)\|^3 \leq \varepsilon^{(1-x)/2} < \varepsilon^{\delta/2}$ and so $\|F(x)\| < \varepsilon^{\delta/6}$ for $x_0 \leq x < 1-\delta$. On the other hand, if $x_0 > x > -1+\delta$ then we simply replace $1+z$ in equation (13) by $1-z$ and repeat the argument to conclude that (12) holds. $\square$

Note. We can slightly change the statement of this theorem to the following: *Let $f\colon \Delta \to Z$ be holomorphic and bounded in the angular region $\Delta_k(1)$. Let $\alpha > \beta$ and $\gamma : [0,1) \to \Delta$ be a 1-curve contained in $\Delta_\alpha(1)$. If $\lim_{t\to 1^-} f(\gamma(t)) = l \in Z$ then f has the limit l at 1 along any approach curve in $\Delta_\beta(1)$.*

Čirka [4] extended Lindelöf's result to functions of several variables and we show that this can be improved to include functions defined on a bounded symmetric domain B. For $b \in \partial B$, a key tool required is a continuous linear projection $\pi = \pi_b : B \to b\Delta = \{\zeta b : \zeta \in \Delta\}$ of the bounded symmetric domain onto the one dimensional subspace containing b which satisfies $\pi(D_k(b)) \subset D_k(b)$ for all $k > 0$ and $\pi(b) = b$. Following the terminology of [2], we will call π_b a PROJECTIVE DEVICE at b. Throughout the remainder of this paper, we restrict attention to a boundary point b of a bounded symmetric domain B for which a projective device π at b exists. In many situations, there is a canonical choice for the projective device. For example, if B is the n-dimensional Hilbert ball then one can take π to be the orthogonal projection of B onto $b\Delta$. A canonical choice for π when $B = \Delta^n$ is given [2] by $\pi(x) = \frac{1}{d}\langle x, \widehat{b}\rangle b$ where d is the cardinality of the set $\{j : |b_j| = 1\}$, $b = (b_1, \ldots, b_n)$ and $\widehat{b} = (b_1', \ldots, b_n')$ where

$$b_j' = \begin{cases} b_j, & |b_j| = 1 \\ 0, & |b_j| < 1 \end{cases}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{C}^n$.

In the following $\Gamma : [0,1) \to B$ will be an e-curve for some tripotent e. An e-curve is called non-tangential if it lies eventually in $D_k(e)$ for some $k > 0$. Clearly, if Γ is a non-tangential e-curve then the projection of Γ under $\pi = \pi_e$, denoted γ, is also a non-tangential e-curve since $\pi(D_k(e)) \subset D_k(e)$ for all $k > 0$. The proof of the following theorem is a modification of that of Čirka's result given in [16].

Theorem 4.2. *Let e be a tripotent in Z having projective device π_e and let Γ be a non-tangential e-curve. Let $\gamma = \pi_e \circ \Gamma$. Let $f\colon B \to Z'$ be holomorphic and bounded in every angular region $D_k(e)$. If*

$$\lim_{t\to 1^-} \frac{1 - \|\gamma(t)\|}{\|\Gamma(t) - \gamma(t)\|} = \infty$$

then

$$\lim_{t\to 1^-} f(\Gamma(t)) - f(\gamma(t)) = 0.$$

Proof. Define $R(t) := \frac{1-\|\gamma(t)\|}{\|\Gamma(t)-\gamma(t)\|}$. Fix $\lambda \in \mathbb{C}$ and choose t_0 such that $R(t) > \sqrt{R(t)} > |\lambda|$ for $t > t_0$. Then for $t > t_0$,

$$C_\lambda(t) := \lambda\Gamma(t) + (1-\lambda)\gamma(t)$$

is an element of B. This is due to the fact that $\|\lambda(\Gamma(t) - \gamma(t)) + \gamma(t)\| < R\|\Gamma(t) - \gamma(t)\| + \|\gamma(t)\| = 1$. As Γ is non-tangential, there exists $k > 0$ such that for t sufficiently close to 1, say $t > t_1 \geq t_0$, $\Gamma(t) \in D_k(e)$ and so $\gamma(t) \in D_k(e)$.

Fix $k' > k$. Notice that $\|B(\gamma(t), e)Q_eQ_e\| \geq \|\gamma(t) - e\|^2$ by Lemma 3.10 and so from Lemma 3.9, there exists $\delta > 0$ such that

$$|\mu| < \delta\|e - \gamma(t)\| \implies \gamma(t) + \mu y \in D_{k'}(e) \text{ for } \|y\| = 1. \tag{14}$$

Also, for $t > t_1$ we have $|\lambda| < \sqrt{R(t)}$, and so

$$\begin{aligned} \|\lambda(\Gamma(t) - \gamma(t))\| &< \sqrt{R(t)}\frac{1 - \|\gamma(t)\|}{R(t)} \\ &= \frac{1 - \|\gamma(t)\|}{\sqrt{R(t)}}. \end{aligned} \tag{15}$$

Since $\sqrt{R(t)} \to \infty$ there exists $t_2 \geq t_1$ such that for $t > t_2$ we have $\delta\sqrt{R(t)} > 1 \geq \frac{1-\|\gamma(t)\|}{\|e-\gamma(t)\|}$ and so

$$\frac{1-\|\gamma\|}{\sqrt{R(t)}} < \delta\|e-\gamma(t)\|.$$

From (15), this gives,

$$\|\lambda(\Gamma(t)-\gamma(t))\| < \delta\|e-\gamma(t)\|$$

for all $t > t_2$. This suggests a choice of μ in (14). Taking $y = \frac{\lambda(\Gamma-\gamma)}{\|\lambda(\Gamma-\gamma)\|}(t)$ and $\mu = \|\lambda(\Gamma(t)-\gamma(t))\|$ we conclude that $\gamma + \lambda(\Gamma-\gamma) \in D_{k'}(e)$ whenever $t > t_2$. In other words, $C_\lambda(t) \in D_{k'}(e)$ for all $t > t_2$.

The rest of the proof proceeds as in the classical case [16] which we now recall. Define the holomorphic function g_t from the open disc of radius $\sqrt{R(t)}$ in $\mathbb{C}$ to the ball of radius $\|f\|_{D_{k'}}$ in Z' by

$$g_t(\lambda) := f\big(\lambda\Gamma(t) + (1-\lambda)\gamma(t)\big).$$

Of course $g_t(1) = f(\Gamma(t))$ and $g_t(0) = f(\gamma(t))$. The map $h_t(\lambda) := g_t(\lambda) - g_t(0)$ is a holomorphic function from the complex disc of radius $\sqrt{R(t)}$ to the ball of radius $2\|f\|_{D_{k'}(e)}$ which maps 0 to 0. The Schwarz Lemma then gives

$$\|h_t(\lambda)\| \leq \frac{2\|f\|_{D_{k'}}}{\sqrt{R(t)}}|\lambda|$$

and, in particular, $\|h_t(1)\| \leq 2\|f\|_{D_{k'}(e)}/\sqrt{R(t)}$ for all $t \geq t_2$. Since $R(t) \to \infty$ we have that $\lim_{t\to 1^-} h_t(1) = 0$ and from this it follows that $\lim_{t\to 1^-} f(\Gamma(t)) - f(\gamma(t)) = 0$. $\square$

The terminology given below is modelled on that of Rudin [16] although our definitions do not agree exactly with those made in the Hilbert space case. (When working in a Hilbert space one may square the quotient factor occurring in (16) below. This allows a larger collection of restricted curves.) Our definition seems a more appropriate one for bounded symmetric domains.

Definition 4.3. We say a b-curve Γ is SPECIAL if

$$\lim_{t\to 1^-} \frac{1-\|\gamma(t)\|}{\|\Gamma(t)-\gamma(t)\|} = \infty, \tag{16}$$

where $\gamma = \pi_b \circ \Gamma$. Further, we say a b-curve Γ is RESTRICTED if it is special and $\gamma([0,1))$ is eventually contained in $b\Delta_k(1)$ for some $k > 0$. In other words, Γ is a restricted curve if it is special and the projection of Γ onto the disc $b\Delta$ has non-tangential approach to b.

We say that $f\colon B \to Z'$ has RESTRICTED ANGULAR LIMIT l AT b if

$$\lim_{t\to 1^-} f(\Gamma(t)) = l$$

for every restricted b-curve Γ. For this we write R-$\lim_{z\to b} f(z) = l$.

It is not obvious from the definition, but any restricted b-curve is non-tangential if b is a tripotent.

Lemma 4.4. *Let e be a tripotent and let Γ be a restricted e-curve. Then Γ is a non-tangential e-curve.*

Proof. Let Γ be a restricted e-curve. Then $\lim_{t\to 1^-} \frac{\|\Gamma(t)-\gamma(t)\|}{1-\|\gamma(t)\|} = 0$ and there exists $k > 0$ and $t_0 \in (0,1)$ such that $\frac{\|\gamma(t)-e\|}{1-\|\gamma(t)\|^2} < k$ for $t > t_0$.

$$\begin{aligned}
\|B(\Gamma(t),e)Q_eQ_e\| &= \|B(\gamma + (\Gamma-\gamma), e)Q_eQ_e\| \\
&= \|B(\Gamma-\gamma, e^\gamma)\, B(\gamma,e)Q_eQ_e\| \qquad \text{using [12, JP34]} \\
&\le \|B(\gamma,e)Q_eQ_e\|\, \|B(\Gamma-\gamma, e^\gamma)\| \\
&\le \|B(\gamma,e)Q_eQ_e\| (1 + \|\Gamma-\gamma\|\|e^\gamma\|)^2
\end{aligned}$$

which by Lemma 3.10 and (10) is

$$\begin{aligned}
&\le 3\|\gamma-e\|^2 \Big(1 + \|\Gamma-\gamma\| \frac{\|e\|}{1-\|e\|\|\gamma\|}\Big)^2 \\
&\le 3\|\gamma-e\|^2 \Big(1 + \frac{\|\Gamma-\gamma\|}{1-\|\gamma\|}\Big)^2 \\
&< 4\|\gamma-e\|^2
\end{aligned}$$

for $t > t_1 \in [t_0, 1)$ since $\frac{\|\Gamma(t)-\gamma(t)\|}{1-\|\gamma(t)\|} \to 0$ as $t \to 1^-$. Thus, for $t > t_1$,

$$\|B(\Gamma(t),e)Q_eQ_e\|^{\frac{1}{2}} < 2\|\gamma(t)-e\| < 2k(1-\|\gamma(t)\|^2).$$

Since $\|\Gamma\| \leq \|\Gamma - \gamma\| + \|\gamma\|$ we have

$$\begin{aligned} 1 - \|\Gamma(t)\| \geq 1 - \|\Gamma - \gamma\| - \|\gamma\| &= 1 - \frac{\|\Gamma - \gamma\|}{1 - \|\gamma\|}(1 - \|\gamma\|) - \|\gamma\| \\ &= (1 - \|\gamma\|)(1 - \frac{\|\Gamma - \gamma\|}{1 - \|\gamma\|}) \\ &\geq \frac{1}{2}(1 - \|\gamma(t)\|) \end{aligned}$$

for $t \geq t_2 \in [t_1, 1)$. Since $\lim_{t \to 1^-} \|\Gamma(t)\| = \lim_{t \to 1^-} \|\gamma(t)\| = 1$ we can choose $t_3 \in [t_2, 1)$ such that $1 + \|\Gamma(t)\| \geq \frac{1}{2}(1 + \|\gamma(t)\|)$ for $t \geq t_3$. Thus, for $t \geq t_3$, $1 - \|\Gamma(t)\|^2 \geq \frac{1}{4}(1 - \|\gamma(t)\|^2)$. The resulting inequality

$$\frac{\|B(\Gamma(t), b)Q_e Q_e\|^{\frac{1}{2}}}{1 - \|\Gamma(t)\|^2} < 8k$$

for $t \geq t_3$ implies that $\Gamma(t) \in D_{8k}(e)$ for $t \geq t_3$. That is, Γ is eventually in some angular region at e, and thus Γ is a non-tangential approach curve to e. $\square$

The RADIAL limit of a function f at e is the limit of f along the radial path $\Gamma(t) = te$, $t \in [0, 1)$. Notice that this radial path is restricted and contained in $D_k(e)$ for $k > 1$. We will use the notation $\lim_{t \to 1^-} f(te)$ for the radial limit of f at e.

Corollary 4.5. *Let e be a tripotent and let $f \in \mathsf{Hol}(B, Z')$ be bounded in every angular region $D_k(e)$. If the radial limit of f exists at e then the restricted angular limit of f exists at e.*

Proof. If Γ is any restricted e-curve then, by Theorem 4.2 and Lemma 4.4, f has the same limit along Γ as it does along its projection $\gamma = \pi_e \circ \Gamma$. Define $\varphi \colon \Delta \to Z'$ by $\varphi(\zeta) = f(\zeta e)$. Then φ is bounded in every $\Delta_k(1)$ and $\lim_{t \to 1^-} \varphi(t)$ exists. Let l be the value of this limit. Theorem 4.1 implies that φ has non-tangential limit l at 1. As γ is a non-tangential e-curve in $e\Delta$, $\lim_{t \to 1^-} f(\gamma(t))$ may be identified with a non-tangential limit of φ in Δ. As the value of this limit is l, we conclude that the limit of f along the restricted curve Γ is also l. $\square$

Corollary 4.5 implies that f has an angular derivative at a tripotent $e \in \partial B$ (by which we mean a restricted angular limit of the function $z \mapsto f'(z)e$) if the map $z \mapsto f'(z)e$ is, firstly, bounded in every angular region $D_k(e)$ and, secondly, has a radial limit at e. A natural simplification is to consider instead the 'projected' map $z \mapsto P_1' f'(z)e$ where P_1' is the Peirce 1-projection of e', where $e' \in Z'$ is the tripotent that arises as the angular limit of f at e.

In the classical setting, $B = B' = \Delta$, one shows that the function $z \mapsto f'(z)$ is bounded in every angular region $\Delta_k(e)$ by using the Cauchy integral formaula to write $f'(z)$ as an integral of the incremental ratios $\frac{f(z)-e'}{z-e}$ where $e, e' \in \partial\Delta$ and $e' = \angle\text{-}\lim_{z\to e} f(z)$ and then showing that these incremental ratios are themselves bounded in every angular region. Our approach is essentially the same, using the Banach space version of Cauchy's integral formula (see, for example, [3]) and a natural analogue for the ratios $\frac{f(z)-e'}{z-e}$. We take as this analogue

$$\frac{\|B(f(z), e')P_1'\|}{\|B(z, e)P_1\|}$$

where $P_1 = Q_e Q_e$ and $P_1' = Q_{e'} Q_{e'}$ are the Peirce 1-projections corresponding to the tripotents e and e' respectively. (This reduces to $\left|\frac{f(z)-e'}{z-e}\right|^2$ when $B = \Delta$.) Part of the argument therefore in proving that $z \mapsto P_1' f'(z)e$ is bounded in every angular region $D_k(e)$ is to show that the map

$$z \mapsto \frac{\|B(P_1' f(z), e')P_1'\|}{\|B(z, e)P_1\|}$$

is bounded in every $D_k(e)$. As we require e' to be $\angle\text{-}\lim_{z\to e} f(z)$ we adopt here the hypothesis of Theorem 3.7.

Theorem 4.6. *Let $f \in \mathsf{Hol}(B, B')$. Suppose that $e \in \partial B$ and $e' \in \partial B'$ are tripotents and that for some $\alpha > 0$,*

$$f(E_r(e)) \subset E_{\alpha r}(e')$$

for all $r > 0$. Then the following functions are bounded in every angular region $D_k(e)$.

(i) $\quad z \mapsto \dfrac{\|B(P_1' f(z), e')P_1'\|}{\|B(z, e)P_1\|}$, $\qquad$ (ii) $\quad z \mapsto P_1' f'(z)e$, *where $P_1 = Q_eQ_e$ and $P_1' = Q_{e'}Q_{e'}$ are the Peirce 1-projections of the tripotents e and e' respectively.*

Proof. (i) Fix an angular region $D_k(e) = \{z \in B : \|B(z,e)Q_eQ_e\|^{\frac{1}{2}} < k(1-\|z\|^2)\}$. Let $z \in D_k(e)$ and let $r = k\|B(z,e)Q_eQ_e\|^{\frac{1}{2}}$. Then

$$\|B(z,e)Q_eQ_e\| = \frac{r}{k}\|B(z,e)Q_eQ_e\|^{\frac{1}{2}} < r(1-\|z\|^2)$$

and hence from (9)

$$\|B_z^{-1}B(z,e)Q_eQ_e\| \leq \frac{\|B(z,e)Q_eQ_e\|}{1-\|z\|^2} < r,$$

that is, $z \in E_r(e)$. By hypothesis therefore, $f(z) \in E_{\alpha r}(e')$ and $P_1'f(z) \in P_1'E_{\alpha r}(e')$. From Note 3.2, $\|P_1'f(z) - e'\| < 2\alpha r$ and by Lemma 3.10,

$$\|B(P_1'f(z), e')Q_{e'}Q_{e'}\| \leq 3\|P_1'f(z) - e'\|^2 < 12\alpha^2r^2.$$

Thus $\|B(P_1'f(z), e')Q_{e'}Q_{e'}\| < 12\alpha^2r^2 = 12\alpha^2k^2\|B(z,e)Q_eQ_e\|$ and we have

$$\frac{\|B(P_1'f(z), e')P_1'\|}{\|B(z,e)P_1\|} < 12\alpha^2k^2$$

giving (i).

(ii) Fix an angular region $D_k(e)$ and fix $k' > k$. By Lemma 3.9 choose $\delta > 0$ so that for $z \in D_k(e)$, $|\lambda| \leq \delta\|B(z,e)Q_eQ_e\|^{\frac{1}{2}}$ implies $z + \lambda e \in D_{k'}(e)$. Let $r = \delta\|B(z,e)Q_eQ_e\|^{\frac{1}{2}}$. By the Cauchy integral formula

$$\begin{aligned}
P_1'f'(z)e &= \frac{1}{2\pi i}\int_{|\lambda|=r} \frac{P_1'f(z+\lambda e)}{\lambda^2}d\lambda \\
&= \frac{1}{2\pi i}\int_{|\lambda|=r} \frac{P_1'f(z+\lambda e) - e'}{\lambda^2}d\lambda \\
&= \frac{1}{2\pi i}\int_{|\lambda|=r} A(\lambda)\, B(\lambda)\, C(\lambda)\, d\lambda
\end{aligned}$$

where

$$A(\lambda) = \frac{P_1' f(z+\lambda e) - e'}{\|B(P_1' f(z+\lambda e), e') P_1'\|^{\frac{1}{2}}}\,,$$

$$B(\lambda) = \frac{\|B(P_1' f(z+\lambda e), e') P_1'\|^{\frac{1}{2}}}{\|B(z+\lambda e, e) P_1\|^{\frac{1}{2}}}\,,$$

$$C(\lambda) = \frac{\|B(z+\lambda e, e) P_1\|^{\frac{1}{2}}}{\lambda^2}\,.$$

Therefore

$$\|P_1' f'(z) e\| \leq \Big(\sup_{|\lambda|=r} \|A(\lambda)\|\Big)\Big(\sup_{|\lambda|=r} B(\lambda)\Big)\Big(\sup_{|\lambda|=r} r|C(\lambda)|\Big).$$

Lemma 3.10 implies that $\sup_{|\lambda|=r} \|A(\lambda)\|$ is bounded (that is, uniformly bounded over $z \in D_k(e)$). By part (i), $\sup_{|\lambda|=r} B(\lambda)$ is bounded, since $z+\lambda e$ is contained in the angular region $D_{k'}(e)$. A glance back at (11) in the proof of Lemma 3.9 shows that $r|C(\lambda)| = \frac{1}{|\lambda|}\|B(z+\lambda e, e)P_1\|^{\frac{1}{2}}$ is bounded by $2k + \frac{1}{\delta}$ and hence (ii) is proved. □

The above result, combined with Corollary 4.5 applied to $z \mapsto P_1' f'(z) e$, allows us to conclude the existence of a restricted angular limit of $P_1' f'(z) e$ from the existence of a radial limit. To examine this radial limit,

$$\lim_{t \to 1^-} P_1' f'(te) e$$

we again use Cauchy's integral formula to write

$$\begin{aligned}
P_1' f'(te) e &= \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{P_1' f((t+\lambda)e) - e'}{\lambda^2} d\lambda \\
&= \frac{1}{2\pi i} \int_{|\lambda|=r} \left(\frac{P_1 f((t+\lambda)e) - e}{1-(t+\lambda)}\right)\left(\frac{1-(t+\lambda)}{\lambda}\right)\frac{d\lambda}{\lambda} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{P_1 f((t+r\mathrm{e}^{i\theta})e) - e}{t + r\mathrm{e}^{i\theta} - 1}\,(1 - \frac{1-t}{r\mathrm{e}^{i\theta}})\, d\theta
\end{aligned}$$

Fix $1 < k < k'$. As $te \in D_k(e)$ we again use Lemma 3.9 to choose $\delta > 0$ such that

$$|\lambda| \leq \delta \|B(te, e) Q_e Q_e\|^{\frac{1}{2}}$$

implies $(t+\lambda)e \in D_{k'}(e)$. In particular, $|\lambda| \leq \delta(1-t)$ implies that $(t+\lambda)e \in D_{k'}(e)$. If we take $r = \delta(1-t)$ then

$$P_1'f'(te)e = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{P_1'f((t+r\mathrm{e}^{i\theta})e) - e'}{t + r\mathrm{e}^{i\theta} - 1}(1 - \frac{\mathrm{e}^{-i\theta}}{\delta})d\theta. \tag{17}$$

We know from part (i) of Theorem 4.6 that the first factor in this integral is bounded since $(t + r\mathrm{e}^{i\theta})e$ lies in the angular region $D_{k'}(e)$. Moreover, if the radial limit of these incremental ratios exists and equals l, that is if

$$\lim_{t\to 1^-} \frac{P_1'f(te) - e'}{t-1} = l \tag{18}$$

then we apply Theorem 4.1 to the function

$$g(\zeta) = \frac{P_1'f(\zeta e) - e'}{\zeta - 1}$$

for $\zeta \in \Delta$ to get that the angular limit

$$\angle\text{-}\lim_{z\to 1} \frac{P_1'f(\zeta e) - e'}{\zeta - 1}$$

exists and equals l. Hence, the integral (17) above has the limit l as $t \in [0,1)$ tends to 1. We have proved the following.

Theorem 4.7. *Let $f \in \mathsf{Hol}(B, B')$ and let $e \in \partial B$, $e' \in \partial B'$ be tripotents. Suppose that for some $\alpha > 0$, $f(E_r(e)) \subset E_{\alpha r}(e')$ for all $r > 0$. If the radial limit*

$$\lim_{t\to 1^-} \frac{e' - P_1'f(te)}{1-t} = l$$

then the restricted angular limit, R-$\lim_{z\to e} P_1'f'(z)e = l$.

We now turn our attention to proving that the limit in (18) above does indeed exist in a reasonably general setting.

Theorem 4.8. *Let $f \in \mathsf{Hol}(B, \Delta)$ and let $e \in \partial B$ be a tripotent such that*

$$\alpha := \liminf_{z\to e} \frac{1 - |f(z)|^2}{1 - \|z\|^2} < \infty.$$

Suppose there exists a sequence $(z_k)_k$ *in* B *converging to* e*, each element of which is proportional to a tripotent and which satisfies*

$$\liminf_k \frac{1-|f(z_k)|^2}{1-\|z_k\|^2} = \alpha$$

and $f(z_k) \to e' \in \Delta$*. Then the radial limit*

$$\lim_{t\to 1^-} \frac{e' - f(te)}{1-t} = \alpha e'$$

and hence R- $\lim_{z\to e} f'(z)e = \alpha e'$*.*

Proof. Define $\varphi \in \mathsf{Hol}(\Delta, \Delta)$ by $\varphi(\zeta) = f(\zeta e)$. Lemma 3.6 implies that

$$\liminf_{\zeta\to 1} \frac{1-|\varphi(\zeta)|^2}{1-|\zeta|^2} = \alpha$$

and so we can apply the one dimensional result (Theorem 1.1) to φ to get $\angle$- $\lim_{\zeta\to 1} \varphi'(\zeta) = \alpha e'$. In particular,

$$\angle\text{-}\lim_{\zeta\to 1} \frac{e' - \varphi(\zeta)}{1-\zeta} = \alpha e'$$

which gives the result. □

If we add the condition that $f(0) = 0$ then the above result can be extended to include the case where the range of f is a Hilbert space.

Theorem 4.9. *Let* $f \in \mathsf{Hol}(B, B')$*, where* B' *is the open unit ball of a Hilbert space, satisfy* $f(0) = 0$*. Let*

$$\alpha := \liminf_{z\to e} \frac{1-\|f(z)\|^2}{1-\|z\|^2} < \infty.$$

Suppose there exists a sequence $(z_k)_k$ *in* B*, each element of which is proportional to a tripotent and which satisfies* $\liminf_k \frac{1-\|f(z_k)\|^2}{1-\|z_k\|^2} = \alpha$ *and* $f(z_k) \to e'$ *where* $e' \in \partial B'$ *is a tripotent. Then the radial limit*

$$\lim_{t\to 1^-} \frac{e' - P_1' f(te)}{1-t} = \alpha e'$$

and hence R- $\lim_{z\to e} P_1' f(z)e = \alpha e'$*.*

Proof. Lemma 3.6 implies that

$$\alpha = \liminf_{z\to e} \frac{1-\|f(z)\|^2}{1-\|z\|^2} = \liminf_{t\to 1^-} \frac{1-\|f(te)\|^2}{1-t^2}.$$

Let $c = \frac{1}{2}(1-t)$. From Proposition 3.1, $te \in \partial E_{\frac{c}{1-c}}(e)$ and so by Lemma 3.3 $f(te) \in \overline{E_{\alpha\frac{c}{1-c}}(e')}$ and thus (recall Note 3.2) $\|P_1'f(te)-e'\| \le \frac{2\alpha c}{1-(1-\alpha)c}$. Since $f(0)=0$, the Schwarz lemma implies that $\alpha \ge 1$ and thus $\|P_1'f(te)-e'\| \le 2\alpha c$.

We now have that

$$\begin{aligned} 1-\|f(te)\|^2 &\le 1-\|P_1'f(te)\|^2 \\ &\le (1+\|P_1'f(te)\|)\|P_1'f(te)-e'\| \\ &\le 2\alpha c(1+t) = \alpha(1-t^2) \end{aligned}$$

giving

$$\frac{1-\|f(te)\|^2}{1-t^2} \le \frac{1-\|P_1'f(te)\|^2}{1-t^2} \le \alpha.$$

Hence, $\liminf_{t\to1^-} \frac{1-\|P_1'f(te)\|^2}{1-t^2} \le \alpha$. Thus,

$$\begin{aligned} \alpha = \liminf_{z\to e} \frac{1-\|f(z)\|^2}{1-\|z\|^2} &\le \liminf_{z\to e} \frac{1-\|P_1'f(z)\|^2}{1-\|z\|^2} \\ &\le \liminf_{t\to 1^-} \frac{1-\|P_1'f(te)\|^2}{1-t^2} \le \alpha. \end{aligned}$$

We conclude that

$$\liminf_{z\to e} \frac{1-\|P_1'f(z)\|^2}{1-\|z\|^2} = \alpha.$$

As P_1' is nothing but the orthogonal projection of B' onto e', $P_1'f$ may be identified with a holomorphic function from B to Δ and the rest of the proof proceeds as in Theorem 4.8. $\square$

For example, we have the following.

Corollary 4.10. *Let $f \in \mathsf{Hol}(B, \Delta)$ satisfy $f(0) = 0$, where B is the open unit ball of a C^*-algebra with identity $\mathbf{1}$. If*

$$\alpha := \liminf_{z \to \mathbf{1}} \frac{1 - |f(z)|^2}{1 - \|z\|^2} < \infty$$

and there exists a sequence $z_k \to \mathbf{1}$ such that each z_k is a scalar multiple of a partial isometry, $\lim_{k\to\infty} \frac{1-\|f(z_k)\|^2}{1-\|z_k\|^2} = \alpha$ and $f(z_k) \to 1$ then the angular limit $\angle\text{-}\lim_{z\to\mathbf{1}} f(z) = 1$ and the restricted angular limit R-$\lim_{z\to\mathbf{1}} f'(z)\mathbf{1} = \alpha$.

In particular, if B is the n-dimensional polydisc we obtain the following, which is contained in [2].

Corollary 4.11. *Let $f \in \mathsf{Hol}(\Delta^n, \Delta)$ satisfy $f(0) = 0$. If*

$$\alpha := \liminf_{z \to \mathbf{1}=(1,\dots,1)} \frac{1 - |f(z)|^2}{1 - \|z\|^2} < \infty$$

and there exists a sequence $z_k \to \mathbf{1}$ such that each z_k is a scalar multiple of an extreme point, $\lim_{k\to\infty} \frac{1-\|f(z_k)\|^2}{1-\|z_k\|^2} = \alpha$ and $f(z_k) \to 1$ then the angular limit $\angle\text{-}\lim_{z\to\mathbf{1}} f(z) = 1$ and the restricted angular limit R-$\lim_{z\to\mathbf{1}} f'(z)\mathbf{1} = \alpha$.

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