A NOTE ON MILNOR-WITT $K$-THEORY AND A THEOREM OF SUSLIN

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Abstract. We give a simple presentation of the additive Milnor-Witt $K$-theory groups $K_{n}^{MW}(F)$ of the field $F$, for $n \geq 2$, in terms of the natural small set of generators. When $n = 2$, this specialises to a theorem of Suslin which essentially says that $K_2^{MW}(F) \cong H_2(\text{Sp}(F), \mathbb{Z})$.

1. Introduction

In [7], Suslin proved that for an infinite field $F$, $H_2(\text{Sl}(2, F), \mathbb{Z})$ is isomorphic to the fibre product $K_2^M(F) \times_{I_2/F} I^2(F)$, where $K_n^M(F)$ is the $n$-th Milnor $K$-group of $F$ and $I = I(F)$ is the ideal of even-dimensional forms in the Witt ring $W(F)$. The proof uses the Matsumoto-Moore presentation of the group $H_2(\text{Sp}(F), \mathbb{Z}) = H_2(\text{Sl}(2, F), \mathbb{Z})$ as well as the characterisation of the 2-torsion of $K_2^M(F)$ as the set of all elements of the form $\{-1, a\}$. (More recently, Mazzoleni, [3], has given an alternative proof of this theorem which by-passes the theorem of Matsumoto-Moore.)

More recently, F. Morel has introduced the Milnor-Witt $K$-theory, $K_*^{MW}(F)$ ([4], [5]). This is a graded algebra given by a simple presentation, due to Morel and M. Hopkins, from which the following properties are easily deduced: $K_n^{MW}(F) \cong W(F)$ for all $n < 0$; $K_0^{MW}(F) \cong GW(F)$, the Grothendieck-Witt ring of isometry classes of quadratic forms over $F$; there is an element $\eta$, of degree $-1$, such that $K_*^{MW}(F)/\langle \eta \rangle \cong K_*^M(F)$. The main result about Milnor-Witt $K$-theory is that it gives an exact description of certain operations in stable motivic homotopy theory; namely there is a natural isomorphism of graded rings

$$K_*^{MW}(F) \cong [S^0, (\mathbb{G}_m)^*]$$

where $S^0$ is the 'motivic' sphere spectrum, and $[ , ]$ denotes the group of morphisms in the stable $\mathbb{A}^1$-homotopy category ([4], section 6).

Morel has shown (see [5], for example) that, for all $n \geq 0$,

$$K_n^{MW}(F) \cong K_n^M(F) \times_{I^n/F} I^{n+1}(F).$$

In fact this result is essentially a reformulation of some of the main results of Arason and Elman, [1], on the powers of $I(F)$. Their work, in turn, relies heavily on the work of Voevodsky, Orlov and Vishik on the Milnor conjecture. In view of Morel’s result, Suslin’s theorem can be re-formulated as the statement that $H_2(\text{Sl}(2, F), \mathbb{Z}) \cong K_2^M(F)$, at least when $F$ is infinite. Elsewhere ([6]), Morel has sketched a direct proof of this fact, using the machinery of $\mathbb{A}^1$-homotopy theory.

In this note, which is more elementary in nature than any of the references above, we prove that the Matsumoto-Moore relations give a simple presentation of the additive
group $K_n^{MW}(F)$, for all $n \geq 2$, in terms of the natural set of generators. When $n = 2$, this statement specializes to Suslin’s theorem, as re-formulated above.

As another application of our main theorem, we give an abstract additive presentation of the group $I^n(F)$ with $n$-fold Pfister forms as generators. (Corollary 2.16).

2. Milnor-Witt $K$-theory

**Definition 2.1** (Hopkins-Morel, [4]). The Milnor-Witt $K$-theory of the field $F$ is the graded associative ring $K_n^{MW}(F)$ generated by the symbols $\langle u \rangle$, $u \in F^\times$, of degree $+1$ and one symbol $\eta$ of degree $-1$ subject to the following relations:

1. For each $a \in F^\times \setminus \{1\}$, $[a] \cdot [1-a] = 0$.
2. For each $a, b \in F^\times$, $[ab] = [a] + [b] + \eta[a][b]$.
3. For each $u \in F^\times$, $[u] \eta = \eta[u]$.
4. $\eta^2[-1] + 2\eta = 0$.

The following result is easily deduced ([6], Lemma 2.4):

**Lemma 2.2.** For all $n \in \mathbb{Z}$, $K_n^{MW}(F)$ has the following presentation as an additive group:

It is generated by the elements $\eta^m[a_1] \cdots [a_r]$, $m \geq 0$, $r = n + m \geq 0$ subject to the following relations:

1. $\eta^m[a_1] \cdots [a_r] = 0$ if $r \geq 2$ and $a_{i-1} + a_i = 1$ for some $i \geq 2$.
2. $\eta^m[a_1] \cdots [a_{i-1}][a_i][a_{i+1}] \cdots [a_r] = \eta^m[a_1] \cdots [a_{i-1}][a_i][a_{i+1}] \cdots [a_r] + \eta^{m+1}[a_1] \cdots [a_{i-1}][a_i][a_{i+1}] \cdots [a_r]$.
3. $\eta^{m+2}[a_1] \cdots [a_{i-1}][-1][a_{i+1}] \cdots [a_{r+2}] + 2\eta^{m+1}[a_1] \cdots [a_{i-1}][a_{i+1}] \cdots [a_{r+2}] = 0$.

However, in view of the relation $\eta[a_1][a_2] = [a_1a_2] - [a_1] - [a_2]$, it is clear that $K_n^{MW}(F)$ can be generated by the elements $\langle a_1 \rangle \cdots \langle a_n \rangle$ whenever $n \geq 1$. Our main theorem is a presentation of $K_n^{MW}(F)$ in terms of these generators when $n \geq 2$.

The theorem of Matsumoto and Moore ([2]), for the case of the symplectic group $Sp(F)$, gives a presentation of the group $H_2(\text{Sp}(F), \mathbb{Z})$. It has the following form: The generators are symbols $\langle a_1, a_2 \rangle$, $a_i \in F^\times$, subject to the relations:

1. $\langle a_1, a_2 \rangle = 0$ if $a_i = 1$ for some $i$
2. $\langle a_1, a_2 \rangle = \langle a_2^{-1}, a_1 \rangle$
3. $\langle a_1, a_2 a_3 \rangle + \langle a_2, a_3 \rangle = \langle a_1 a_2, a_3 \rangle + \langle a_1, a_2 \rangle$
4. $\langle a_1, a_2 \rangle = \langle a_1, -a_2 \rangle$
5. $\langle a_1, a_2 \rangle = \langle a_1, (1-a_i)a_2 \rangle$

This motivates the following (provisional) definition:

**Definition 2.3.** Let $n \geq 2$. For a field $F$, $K_n^{MM}(F)$ (MM is for ‘Matsumoto-Moore’) will denote the additive group which has the following presentation: the generators are $\langle a_1, \ldots, a_n \rangle$, $a_i \in F^\times$ subject to the following relations:

1. $\langle a_1, \ldots, a_n \rangle = 0$ if $a_i = 1$ for some $i$
2. $\langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_i^{-1}, a_{i-1}, \ldots, a_n \rangle$
3. $\langle a_1, \ldots, a_{n-1}, a_n a_n' \rangle + \langle a_1, \ldots, a_n, a_n' \rangle = \langle a_1, \ldots, a_{n-1}a_n, a_n' \rangle + \langle a_1, \ldots, a_{n-1}, a_n \rangle$
4. $\langle a_1, \ldots, a_{n-1}, a_n \rangle = \langle a_1, \ldots, a_{n-1}, -a_{n-1}a_n \rangle$
5. $\langle a_1, \ldots, a_{n-1}, a_n \rangle = \langle a_1, \ldots, a_{n-1}, (1-a_{n-1})a_n \rangle$

**Remark 2.4.** In particular, $K_2^{MM}(F) \cong H_2(\text{Sp}(F), \mathbb{Z})$ (= $H_2(\text{SL}(2, F), \mathbb{Z}$) if $F$ is infinite or at least sufficiently large) by the theorem of Matsumoto-Moore.
Observe that, using relation (ii) together with (iii), (iv) and (v), we easily deduce the following relations in $K_n^{\text{MM}}(F)$:

\begin{align*}
(iii') \ & \langle a_1, \ldots, a_{i-1}, a_i a'_i, \ldots, a_n \rangle + \langle a_1, \ldots, a_i, a'_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_{i-1} a_i, a'_i, \ldots, a_n \rangle + \\
(iv') \ & \langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_{i-1}, -a_{i-1} a_i, \ldots, a_n \rangle \\
(v') \ & \langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_{i-1}, (1 - a_{i-1}) a_i, \ldots, a_n \rangle
\end{align*}

**Theorem 2.5.** $K_n^{\text{MW}}(F) \cong K_n^{\text{MM}}(F)$ for all $n \geq 2$ via an isomorphism sending $[a_1] \cdots [a_n]$ to $\langle a_1, \ldots, a_n \rangle$.

**Proof.** The theorem follows from Lemmas 2.8 and 2.15 below. \hfill \Box

**Corollary 2.6.** For all infinite fields $F$, $K_2^{\text{MW}}(F) \cong H_2(\text{Sl}(2, F), \mathbb{Z})$.

**Lemma 2.7.** The relations $[a][-a] = 0$ and $[a][b] = [b^{-1}][a]$ hold in $K^*_\text{MW}(F)$ for all $a, b \in F^\times$.

**Proof.** Using the identity

$$-a(1 - a^{-1}) = 1 - a$$

together with 2. gives

$$[-a] = [1 - a] - [1 - a^{-1}] - \eta[-a][1 - a^{-1}]$$

Hence

$$[a][-a] = -[a](1 + \eta[-a])[1 - a^{-1}] \quad \text{(using 1.)}$$
$$= -(1 + \eta[-a])[a][1 - a^{-1}] \quad \text{(since $\eta[u][v] = \eta[v][u]$ by 2.)}$$
$$= (1 + \eta[-a])(1 + \eta[a])[a^{-1}][1 - a^{-1}] \quad \text{(since $[a] = (1 + \eta[a])[a^{-1}]$ by 2.)}$$
$$= 0 \quad \text{(by 1.)}$$

We will need the identity

$$\eta([a] + [-a])[-1] = -2[-1]$$

obtained by letting $x = a$ and $x = -a$ in the identity $[x] = [-1] + [-x] + \eta[-x][-1]$ and adding.

Observe also that $[a]^2 = [a][-a]$ for all $a \in F^\times$ (let $x = a$ above and use $[a][-a] = 0$).

Now, for any $a, b \in F^\times$ we have

$$0 = [ab][-ab]$$
$$= ([a] + [b] + \eta[a][b])([-a] + [b] + \eta[-a][b])$$
$$= [a][b] + [b][-a] + \eta([a] + [-a])[b^2] + [b]^2$$
$$= [a][b] + [b][-a] + \eta([a] + [-a])[-1][b] + [b][-1]$$
$$= [a][b] + [b][-a] - [b][-1]$$
$$= [a][b] + [b]([-a] - [-1])$$
$$= [a][b] + [b][a](1 + \eta[-1])$$

and hence

$$[a][b] = -[b][a](1 + \eta[-1]) = -(1 + \eta[-1])[b][a].$$
Thus
\[
[a][b] - [b^{-1}][a] = -((1 + \eta[-1])[b] + [b^{-1}])[a] \\
= -([b] + [b^{-1}] + \eta[-1][b])[a] \\
= -([b] + [b^{-1}] + \eta[b^{-1}][b])[a] = -[1][a] = 0
\]
(where we have used \([-1][b] = [b^{-1}][b]\) which follows from \([-1] = [-b] + [b^{-1}] + \eta[-b][b^{-1}]\) and \([b][-b] = 0\)).

\[\square\]

**Lemma 2.8.** Let \(n \geq 2\). The map \(\phi\) which sends the element \(\langle a_1, \ldots, a_n \rangle\) of \(K_n^{\text{MM}}(F)\) to \([a_1] \cdots [a_n]\) in \(K_n^{\text{MW}}(F)\) extends uniquely to a well-defined epimorphism of groups.

**Proof.** Well-definedness is the issue; we must prove that relations (i)–(v) are preserved by \(\phi\).

Relation (i): This follows from the identity \([1] = 0\) in \(K_2^{\text{MW}}(F)\) (since \([1] = 2[-1] + \eta[-1]^2\) by 2. and hence \(\eta[1] = 0\) by 4. and thus \([1] = 2[1]\) by 2. again).

Relation (ii): This follows immediately from the relation \([a_{i-1}]a_i = [a_i^{-1}]a_{i-1}\) (Lemma 2.7).

Relation (iii): We have
\[
[a_{n-1}]a_na'_n + [a_n]a'_n = [a_{n-1}][a_n] + [a'_n] + \eta[a_n][a'_n] + [a_n][a'_n] \\
= ([a_{n-1}] + [a_n] + \eta[a_{n-1}][a_n])[a'_n] + [a_{n-1}][a_n] \\
= [a_{n-1}a_n][a'_n] + [a_{n-1}][a_n]
\]

Relation (iv): We have \([a_{n-1}][-a_{n-1}a_n] = [a_{n-1}][-a_{n-1} + [a_n] + \eta[-a_{n-1}][a_n]] = [a_{n-1}][a_n]\) by Lemma 2.7.

Relation (v): Similarly, \([a_{n-1}][(1 - a_{n-1})a_n] = [a_{n-1}][a_n]\) using 1. and 2.

\[\square\]

**Lemma 2.9.** Let \(n \geq 2\). For \(a_1, \ldots, a_n, x \in F^x\) let
\[
\rho_x(\langle a_1, \ldots, a_n \rangle) := \langle a_1, \ldots, a_n x \rangle - \langle a_1, \ldots, a_n \rangle - \langle a_1, \ldots, x \rangle.
\]

Then \(\rho_x\) extends uniquely to an endomorphism of \(K_n^{\text{MM}}(F)\).

**Proof.** We must prove that \(\rho_x\) preserves defining relations (i)-(v).

Relation (i) is clear.

Relation (ii): When \(i < n\) in (ii), the result is clear. For the case \(i = n\), we find:
\[
\rho_x(\langle a_1, \ldots, a_{n-1}, a_n \rangle) = \langle a_1, \ldots, a_{n-1}, x a_n \rangle - \langle a_1, \ldots, a_{n-1}, a_n \rangle - \langle a_1, \ldots, a_{n-1}, x \rangle \\
= \langle a_1, \ldots, a_{n-1}, x, a_n \rangle + \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, x, a_n \rangle - \langle a_1, \ldots, a_{n-1}, x \rangle \\
= \langle a_1, \ldots, a_{n-1}, x, a_n \rangle - \langle a_1, \ldots, x, a_n \rangle - \langle a_1, \ldots, a_{n-1}, a_n \rangle \\
= \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, a_{n-1}, a_n \rangle \\
= \rho_x(\langle a_1, \ldots, a_{n-1}, a_n \rangle).
\]
Relation (iii):

\[ \rho_x([a_1, \ldots, a_{n-1}, a_n, a'_n]) + \rho_x([a_1, \ldots, a_n]) = [a_1, \ldots, a_n] \]

For \( n \geq 2 \), we will denote the element \( \rho_x([a_1, \ldots, a_n]) \) in \( K_n^{MM}(F) \) by \( [a_1, \ldots, a_n, x] \).

**Lemma 2.10.** Let \( n \geq 2 \). For any permutation, \( \sigma \), of \( \{1, \ldots, n+1\} \), \( [a_1, \ldots, a_{n+1}] = [a_{\sigma(1)}, \ldots, a_{\sigma(n+1)}] \).

**Proof.** It is immediate from the definition that \( [a_1, \ldots, a_n, a_{n+1}] = [a_1, \ldots, a_{n+1}, a_n] \); i.e. the result is true when \( \sigma \) is the transposition \( (n \ n+1) \). From this it follows that

\[ [a_1, \ldots, a_{n-1}, a_n, x] = \rho_x([a_1, \ldots, a_n]) \]

This fact, together with relation (ii), now implies that

\[ [a_1, \ldots, a_{i-1}, a_i, \ldots, a_{n}, x] = \rho_x([a_1, \ldots, a_{i-1}, a_i, \ldots, a_n]) \]

proving the lemma.

**Lemma 2.11.** For all \( x, y \in F^\times \), \( \rho_x \rho_y = \rho_y \rho_x \).

**Proof.** Let \( a_1, \ldots, a_n \in F^\times \). Then

\[ \rho_y([a_1, \ldots, a_n]) = [a_1, \ldots, a_n, y] = [y, a_1, \ldots, a_n] \]
and hence

\[ \rho_x(\rho_y((a_1, \ldots, a_n))) = \rho_x([y, a_1, \ldots, a_n]) \]
\[ = \rho_x([y, a_1, \ldots, a_{n-1}, a_n]) - \rho_x([y, a_1, \ldots, a_{n-1}]) - \rho_x([y, a_1, \ldots, a_{n-2}, a_n]) \]
\[ = [y, \ldots, a_{n-1}a_n, x] - [y, a_1, \ldots, a_{n-1}, x] - [y, a_1, \ldots, a_{n}, x] \]
\[ = [x, \ldots, a_{n-1}a_n, y] - [x, a_1, \ldots, a_{n-1}, y] - [x, a_1, \ldots, a_n, y] \]
\[ = \rho_y(\rho_x((a_1, \ldots, a_n))). \]

\[ \square \]

More generally, we define elements \([a_1, \ldots, a_r] \in K_n^{MM}(F) \) \((r > n)\) recursively by the formula

\[ [a_1, \ldots, a_r, a_{r+1}] := \rho_{a_{r+1}}([a_1, \ldots, a_r]). \]

We will also use the notation \([a_1, \ldots, a_n] := \langle a_1, \ldots, a_n \rangle \) \((i.e., \text{ when } r = n)\).

**Corollary 2.12.** Fix \(n \geq 2\). For all \(r > n\) and for all permutations, \(\sigma\), of \(\{1, \ldots, r\}\)

\[ [a_1, \ldots, a_r] = [a_{\sigma(1)}, \ldots, a_{\sigma(r)}]. \]

**Proof.** We use induction on \(r\). The case \(r = n + 1\) has already been proved.

For permutations of \(\{1, \ldots, r+1\}\) which fix \(r+1\), the result holds by induction since

\[ [a_1, \ldots, a_r, a_{r+1}] = \rho_{a_{r+1}}([a_1, \ldots, a_r]). \]

On the other hand, when \(r > n\), the result holds for the permutation \(\sigma = (r+1)\) since

\[ [a_1, \ldots, a_r, a_{r+1}] = \rho_{a_{r+1}}(\rho_{a_r}([a_1, \ldots, a_{r-1}])) = \rho_{a_r}(\rho_{a_{r+1}}([a_1, \ldots, a_{r-1}])) = [a_1, \ldots, a_r, a_{r+1}]. \]

\[ \square \]

**Remark 2.13.** Observe that it follows that the relations \((i)-(v)\) extend to the symbols \([a_1, \ldots, a_r] \) \((r \geq n)\) since we can always permute the key entries to before the \(n\)-th position and then use the fact that \([a_1, \ldots, a_r] = \phi([a_1, \ldots, a_n])\) for an appropriate endomorphism \(\phi\). Furthermore, property \((ii)\) and symmetry (Corollary 2.12) imply that

\[ [a_1, \ldots, a_i, \ldots, a_r] = [a_1, \ldots, a_i^{-1}, \ldots, a_r] \] for any \(i\).

**Corollary 2.14.** Let \(n \geq 2\). If \(r > n\) and if \(a_1, \ldots, a_r \in F^\times\) then \([a_1, \ldots, a_r] = 0\) if \(a_i\) is a square in \(F^\times\) for some \(i \leq r\).

**Proof.** By symmetry we can suppose that \(i > 1\). Suppose that \(a_i = b_i^2\). We thus have

\[ [a_1, \ldots, a_{i-1}, b_i^2, \ldots] = [a_1, \ldots, a_{i-1}b_i, b_i, \ldots] + [a_1, \ldots, a_{i-1}, b_i, \ldots] - [a_1, \ldots, b_i, b_i, \ldots] \]
\[ = [a_1, \ldots, a_{i-1}b_i, b_i^{-1}, \ldots] + [a_1, \ldots, a_{i-1}, b_i, \ldots] - [a_1, \ldots, b_i, b_i^{-1}, \ldots] \]
\[ = [a_1, \ldots, a_{i-1}, b_i^{-1}, \ldots] = [a_1, \ldots, a_{i-1}, 1, \ldots] = 0. \]

\[ \square \]

**Lemma 2.15.** Let \(n \geq 2\). There is a unique epimorphism \(\lambda : K_n^{MW}(F) \to K_n^{MM}(F)\)

satisfying

\[ \lambda(\eta^m[a_1] \cdots [a_r]) = [a_1, \ldots, a_r] \quad (r = n + m) \]
Proof. We must show that $\lambda$ preserves relations (1)-(3) of Lemma 2.2. Relation (1) follows from (i) and (v) (see Remark 2.13).

Relation (2): We must prove that, for $r \geq n$ and $i \leq r$,

$$[a_1, \ldots, a_i a_i', \ldots, a_r] = [a_1, \ldots, a_i, \ldots, a_r] + [a_1, \ldots, a_i', \ldots, a_r] + [a_1, \ldots, a_i, a_i', \ldots, a_r].$$

By symmetry we can assume $i \leq n$ and thus we reduce to the key case $r = n$; i.e. we must prove

$$\langle a_1, \ldots, a_i a_i', \ldots, a_n \rangle = \langle a_1, \ldots, a_i, \ldots, a_n \rangle + \langle a_1, \ldots, a_i', \ldots, a_n \rangle + \langle a_1, \ldots, a_i, a_i', \ldots, a_n \rangle.$$

By property (ii), and by symmetry, we can assume $i = n$. The identity is now just the definition of $[a_1, \ldots, a_n, a_n']$.

Relation (3): We must prove that for $r \geq n$

$$[a_1, \ldots, a_i-1, -1, a_{i+1}, \ldots, a_{r+2}] = -2[a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{r+2}].$$

By symmetry, we can suppose that $r = n$ and $i = n + 1$. So we must show that

$$[a_1, \ldots, a_n, -1, a_{n+2}] = -2[a_1, \ldots, a_n, a_{n+2}].$$

Now

$$[a_1, \ldots, a_n, -1] = [a_1, \ldots, a_n, a_n] \quad (by \ (iv))$$

and thus

$$[a_1, \ldots, a_n, -1, a_{n+2}] = [a_1, \ldots, a_n^2, a_{n+2}] - 2[a_1, \ldots, a_n, a_{n+2}] = -2[a_1, \ldots, a_n, a_{n+2}]$$

by Corollary 2.14. \qed

As an application, we derive a simple additive presentation of the ideals $I^n(F)$, $n \geq 2$, in the Witt Ring of a field $F$:

**Corollary 2.16.** For any field $F$, let $I(F)$ be the ideal of even-dimensional forms in the Witt Ring, $W(F)$, of the field $F$. As an additive group, $I^n(F) = I(F)^n$ has the following abstract presentation:

*It is generated by the classes of Pfister forms $<< a_1, \ldots, a_n >>$, $a_i \in F^\times$ subject to the following relations:*

- (i) $<< a_1, \ldots, a_n >> = 0$ if $a_i$ is a square for some $i$
- (ii) $<< a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n >> = << a_1, \ldots, a_i, a_{i+1}, \ldots, a_n >>$
- (iii) $<< a_1, \ldots, a_{n-1}, a_n a_n' >> + << a_1, \ldots, a_n, a_n' >> = << a_1, \ldots, a_{n-1}a_n, a_n' >>$
- (iv) $<< a_1, \ldots, a_{n-1}, a_n >> = << a_1, \ldots, a_{n-1}, (1-a_{n-1})a_n >>$

**Proof.** Morel’s theorem ([5], Théorème 5.3), shows that there is an exact sequence

$$0 \rightarrow K_n^M(F)^2 \rightarrow K_n^{MW}(F) \rightarrow I^n(F) \rightarrow 0$$

where the first (nontrivial) homomorphism maps $\{a_1, \ldots, a_n\}^2 = \{a_1, \ldots, a_i^2, \ldots, a_n\}$ to $[a_1] \cdots [a_i^2] \cdots [a_n]$ (for any $i$) and the next homomorphism sends $[a_1] \cdots [a_n]$ to the class of the Pfister form $<< a_1, \ldots, a_n >>$. Combining this with Theorem 2.5 give the result, since the identity $-a = (1 - a)/(1 - a^{-1})$ shows that (i) and (iv) imply the identity $<< a_1, \ldots, a_{n-1}, a_n >> = << a_1, \ldots, a_{n-1}, -a_{n-1}a_n >>$. \qed

**Remark 2.17.** Compare this with the presentation of $I^n(F)$ given by Arason and Elman ([1], Theorem 3.1). Of course, Corollary 2.16 – like the result of Arason and Elman – requires the proof of the Milnor conjecture (since it is needed for Morel’s theorem), and conversely easily implies the Milnor conjecture.
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