

A convergence theorem for harmonic measures with applications to Taylor series

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Abstract

Let f be a holomorphic function on the unit disc, and (S_{n_k}) be a subsequence of its Taylor polynomials about 0. It is shown that the nontangential limit of f and $\lim_{k \rightarrow \infty} S_{n_k}$ agree at almost all points of the unit circle where they simultaneously exist. This result yields new information about the boundary behaviour of universal Taylor series. The key to its proof lies in a convergence theorem for harmonic measures that is of independent interest.

1 Introduction

Let f be a holomorphic function on the unit disc $\mathbb{D}$. We assume that its Taylor series about 0 has radius of convergence 1 and denote by S_n the partial sum of this series up to degree n . It is natural to ask how the boundary behaviour of f at a subset A of the unit circle $\mathbb{T}$ constrains the functions on A that can arise as $\lim_{k \rightarrow \infty} S_{n_k}$ for some subsequence (S_{n_k}) of (S_n) .

It turns out that even in the simplest situation, where f is holomorphic on $\mathbb{C} \setminus \{1\}$, the sequence (S_n) typically has chaotic behaviour on Dirichlet subsets of $\mathbb{T}$, that is, compact sets on which (z^{n_k}) converges uniformly to 1 for some subsequence (n_k) of the natural numbers. More precisely, Beise, Meyrath and Müller [2] have shown recently that, given any Dirichlet set $A \subset \mathbb{T} \setminus \{1\}$, there is a residual subset of the space of holomorphic functions on $\mathbb{C} \setminus \{1\}$ (endowed with the topology of local uniform convergence), each member f of which has the properties that:

- (i) for each continuous function h on A there is a subsequence (S_{n_k}) that converges uniformly to h on A ;
- (ii) there is a subsequence (S_{m_k}) that converges locally uniformly to f on $\mathbb{T} \setminus \{1\}$.

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Dirichlet sets A can have Hausdorff dimension 1 but cannot have positive arc length measure $\sigma(A)$ (see, for example, p.171 of [9]). This leaves open the question of whether property (i) above can occur on subsets $A \subset \mathbb{T}$ of positive measure. We show below that this cannot happen, even where the boundary values of f exist merely as nontangential limits. Let $\text{nt} \lim_{z \rightarrow \zeta} f(z)$ denote the nontangential limit of f at a point $\zeta \in \mathbb{T}$, wherever it exists (finitely).

Theorem 1 *Given a holomorphic function f on $\mathbb{D}$ and a subsequence (S_{n_k}) of the partial sums of its Taylor series about 0, let*

$$E = \{\zeta \in \mathbb{T} : S(\zeta) := \lim_{k \rightarrow \infty} S_{n_k}(\zeta) \text{ exists}\}$$

and

$$F = \{\zeta \in \mathbb{T} : f(\zeta) := \text{nt} \lim_{z \rightarrow \zeta} f(z) \text{ exists}\}.$$

Then $S = f$ almost everywhere (σ) on $E \cap F$.

A classical result in this area is Abel's Limit Theorem, which says that, if $(S_n(\zeta))$ converges for some $\zeta \in \mathbb{T}$, then $\text{nt} \lim_{z \rightarrow \zeta} f(z)$ exists, and the two limits agree. If we merely know that a subsequence $(S_{n_k}(\zeta))$ converges, no conclusion about the boundary behaviour of f at ζ may be drawn. Indeed, for a typical holomorphic function f on $\mathbb{D}$, any continuous function on $\mathbb{T}$ is the pointwise limit of a suitable subsequence (S_{n_k}) . (See the properties of the collection $\mathcal{U}_0(\mathbb{D}, 0)$ noted below.) Nevertheless, Theorem 1 still shows that $\lim_{k \rightarrow \infty} S_{n_k}(\zeta)$ and $\text{nt} \lim_{z \rightarrow \zeta} f(z)$ must agree almost everywhere on the set where they simultaneously exist.

Theorem 1 fails if we replace nontangential limits by radial limits. To see this, let F be a closed nowhere dense subset of $\mathbb{T}$ such that $\sigma(F) > 0$. Then, by Theorem 1.2 of Costakis [3], there is a holomorphic function f on $\mathbb{D}$ which has radial limit 0 at each point of F and such that some subsequence (S_{n_k}) converges pointwise to 1 on $\mathbb{T}$.

Now let f be a holomorphic function on a proper subdomain ω of $\mathbb{C}$, let $\xi \in \omega$, $r_0 = \text{dist}(\xi, \mathbb{C} \setminus \omega)$ and D_0 denote the open disc $D(\xi, r_0)$ of centre ξ and radius r_0 . Further, let $S_n(f, \xi)$ denote the partial sum up to degree n of the Taylor series of f about ξ . Following Nestoridis [13] we call this series *universal*, and write $f \in \mathcal{U}(\omega, \xi)$, if for every compact set $K \subset \mathbb{C} \setminus \omega$ that has connected complement, and every continuous function h on K that is holomorphic on K° , there is a subsequence $(S_{n_k}(f, \xi))$ that converges uniformly to h on K . Similarly, we write $f \in \mathcal{U}_0(\omega, \xi)$ if f satisfies the corresponding condition in which we only consider compact sets $K \subset \partial D_0 \setminus \omega$. Clearly $\mathcal{U}(\omega, \xi) \subset \mathcal{U}_0(\omega, \xi)$, with equality if $\mathbb{C} \setminus \omega \subset \partial D_0$. When ω is simply connected, $\mathcal{U}(\omega, \xi)$ is a dense G_δ subset of the space of all holomorphic functions on ω , endowed with the topology of local uniform convergence [14].

For multiply connected domains ω this collection is often empty. (Important known exceptions, in addition to the case where $\mathbb{C} \setminus \omega \subset \partial D_0$ [15], are where $\mathbb{C} \setminus \omega$ is either discrete or a continuum [11].) However, Nestoridis and Papachristodoulos [15] have shown that $\mathcal{U}_0(\omega, \xi)$ is always a dense G_δ subset of the space of all holomorphic functions on ω . Further, even for simply connected domains, the inclusion $\mathcal{U}(\omega, \xi) \subset \mathcal{U}_0(\omega, \xi)$ is generally strict; for, if $\omega \neq D_0$, we can choose a function in $\mathcal{U}_0(\mathbb{C} \setminus (\partial\omega \cap \partial D_0), \xi)$ and restrict it to ω to get a function in $\mathcal{U}_0(\omega, \xi)$, yet functions in $\mathcal{U}(\omega, \xi)$ cannot be holomorphically extended to any larger domain [12]. The authors of [15] observed that, if $f \in \mathcal{U}_0(\omega, \xi)$ and $\partial D_0 \setminus \omega$ contains a nondegenerate arc, then f does not extend continuously to $\omega \cup \partial D_0$. We can now give:

Corollary 2 *Let $f \in \mathcal{U}_0(\omega, \xi)$ and suppose that $\sigma(\partial D_0 \setminus \omega) > 0$. Then, for σ -almost every $\zeta \in \partial D_0 \setminus \omega$, the set $f(\Gamma)$ is dense in $\mathbb{C}$ for every open triangle $\Gamma \subset D_0$ which has a vertex at ζ and is symmetric about $[0, \zeta]$.*

This follows immediately from Theorem 1, because Plessner's theorem (Theorem 2.5 of [8]) tells us that at σ -almost every point of $\partial D_0 \setminus \omega$ either f has a finite nontangential limit or $f(\Gamma)$ is dense in $\mathbb{C}$ for every such triangle Γ . The special case of this corollary where $\omega = D_0$ was recently established in [5]. (It was stated there for $f \in \mathcal{U}(\omega, \xi)$, but the proof is valid also for $f \in \mathcal{U}_0(\omega, \xi)$.) For functions in $\mathcal{U}(\mathbb{D}, 0)$ much more can be said about boundary behaviour: see [6].

Theorem 1 of [7] tells us that, if $\zeta \in \partial D_0 \setminus \omega$ and a function f in $\mathcal{U}(\omega, \xi)$ is bounded in $D(\zeta, \rho) \cap \omega$ for some $\rho > 0$, then $\mathbb{C} \setminus (\omega \cup \overline{D_0})$ must be polar. Corollary 2 yields the additional information that $(\partial D_0 \setminus \omega) \cap D(\zeta, \rho)$ must have zero arc length measure.

Our proof of Theorem 1 relies on the following subtle convergence result for harmonic measures, which is of interest in its own right. In what follows Ω denotes a domain in $\mathbb{R}^N$ ($N \geq 2$) possessing a Green function $G_\Omega(\cdot, \cdot)$. For any (non-empty) open set ω , any Borel set A and any point x in $\mathbb{R}^N$, we denote by $\mu_x^\omega(A)$ the harmonic measure of $A \cap \partial\omega$ for ω evaluated at x . (If $x \notin \omega$, this measure is assigned the value 0.)

Theorem 3 *Let $\xi_0 \in \Omega$ and ω be an open subset of Ω . Suppose that (v_k) is a decreasing sequence of subharmonic functions on ω such that $v_1/G_\Omega(\xi_0, \cdot)$ is bounded above and $\lim_{k \rightarrow \infty} v_k < 0$ on ω . If $\mu_{x_1}^\omega(\partial\Omega) > 0$ for some x_1 , then $\mu_{x_1}^{\{v_k < 0\}}(\partial\Omega) > 0$ for all sufficiently large k .*

The above result fails without the upper boundedness hypothesis on $v_1/G_\Omega(\xi_0, \cdot)$, as can be seen from the following examples (there are obvious analogues in higher dimensions):

- (a) $\Omega = \omega = \mathbb{D}$ and $v_k(z) = 1 + k \log |z|$, so $\{v_k < 0\} = \{|z| < e^{-1/k}\}$.

- (b) $\Omega = \omega = \mathbb{D}$ and $v_k(z) = 1 - k \frac{1 - |z|^2}{|1 - z|^2}$, so $\{v_k < 0\}$ is a disc internally tangent to $\mathbb{T}$ at 1.

A weaker version of this result, where Ω is a simply connected plane domain and each function v_k is harmonic on all of Ω , was established in [4]. We will use a substantially different argument to prove this more general theorem. When $N = 2$ the result is valid for domains in the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In the application of Theorem 3 to the proof of Theorem 1 it is crucial that, in contrast to the above two examples, the sequence (v_k) need only have a negative limit on a suitable open subset ω of Ω , namely one for which $\mu_x^\omega(\partial\Omega) > 0$.

Theorem 3 and its proof are based on Chapter 6 of the second author's doctoral thesis [10].

2 Proof of Theorem 1

Let f , (S_{n_k}) , E and F be as in the statement of Theorem 1, and let

$$D = \{\zeta \in E \cap F : S(\zeta) \neq f(\zeta)\}.$$

Also, let $\Gamma(1)$ denote the open triangular region with vertices $1, (1 \pm i)/2$ (say), and let

$$\Gamma(\zeta) = \{\zeta z : z \in \Gamma(1)\} \quad (\zeta \in \mathbb{T}).$$

Now suppose, for the sake of contradiction, that the conclusion of the theorem fails. Then we may choose a positive number a sufficiently large to ensure that $\sigma(A_a) > 0$, where

$$A_a = \{\zeta \in D : |f| \leq a \text{ on } \Gamma(\zeta) \text{ and } |S_{n_k}(\zeta)| \leq a \text{ for all } k\}.$$

It follows, on multiplication by a suitable unimodular constant, that we can choose a compact set K of A_a such that $\inf_K \operatorname{Re}(S - f) > 0$ and $0 < \sigma(K) < 2\pi$. The domain $\Omega = \widehat{\mathbb{C}} \setminus K$ then possesses a Green function, by Myrberg's theorem (Theorem 5.3.8 of [1]), since K is non-polar.

We put $\omega = \cup_{\zeta \in K} \Gamma(\zeta)$, and reduce K , if necessary, to ensure that ω is a simply connected domain. Clearly $|f| \leq a$ on ω . Since the triangles $\Gamma(\zeta)$ are congruent, the boundary of ω is a rectifiable Jordan curve. Thus $\mu_z^\omega(K) > 0$ when $z \in \omega$ by the F. and M. Riesz theorem (Theorem VI.1.2 of [8]), in view of the fact that $\sigma(K) > 0$. Let $g : \mathbb{D} \rightarrow \omega$ be a conformal map. It extends to a continuous bijection $g : \mathbb{D} \rightarrow \overline{\omega}$, by Carathéodory's theorem. The function g' belongs to the Hardy space H^1 by Theorem VI.1.1 of [8], so the F. and M. Riesz theorem shows further that, for almost every $\zeta \in K$, the function g is conformal at $g^{-1}(\zeta)$ and $\lim_{w \rightarrow g^{-1}(\zeta)} (f \circ g)(w) = f(\zeta)$. Since $f \circ g$

is a bounded holomorphic function on $\mathbb{D}$, we know that $f \circ g = H_{f \circ g}^{\mathbb{D}}$, using the usual notation for Dirichlet solutions, whence

$$f = H_{f \circ g}^{\mathbb{D}} \circ g^{-1} = H_f^{\omega} \quad \text{on } \omega. \quad (1)$$

We define

$$u_k = \frac{1}{n_k} \log \frac{|S_{n_k} - f|}{2a} \quad \text{on } \mathbb{D} \quad (k \in \mathbb{N}).$$

Noting from Bernstein's lemma (Theorem 5.5.7 of [16]) that

$$|S_{n_k}| \leq ae^{n_k G_{\Omega}(\infty, \cdot)} \quad \text{on } \Omega,$$

we see that $u_k \leq G_{\Omega}(\infty, \cdot)$ on ω . Now $\limsup_{k \rightarrow \infty} u_k(z) \leq \log |z|$ on $\mathbb{D}$, so we can choose a sequence (r_k) in $[0, 1)$ such that $r_k \uparrow 1$ and

$$u_j(z) \leq \frac{1}{2} \log |z| \quad (|z| \leq r_k, j \geq k).$$

Let $v_k = H_{\psi_k}^{\omega}$, where

$$\psi_k(z) = \begin{cases} \frac{1}{2} \log |z| & \text{on } \partial\omega \cap D(0, r_k) \\ G_{\Omega}(\infty, z) & \text{on } \partial\omega \cap (\mathbb{D} \setminus D(0, r_k)) \\ 0 & \text{on } \partial\omega \cap \mathbb{T} \end{cases}.$$

Then $u_k \leq v_k$ on ω and (v_k) is a decreasing sequence of harmonic functions on ω with limit $\frac{1}{2} \log |\cdot|$ on $\partial\omega$.

By Theorem 3, and the fact that $\mu_z^{\omega}(K) > 0$ when $z \in \omega$, there exists $k' \in \mathbb{N}$ such that the open set $\omega_1 := \omega \cap \{v_{k'} < 0\}$ is non-empty and

$$\mu_w^{\omega_1}(\partial\Omega) > 0 \quad \text{for some } w \in \omega_1. \quad (2)$$

Clearly $u_k < 0$ on ω_1 for all $k \geq k'$. Thus $|S_{n_k} - f| \leq 2a$, and so $|S_{n_k}| \leq 3a$, on ω_1 for all $k \geq k'$. Now $S_{n_k} = H_{S_{n_k}}^{\omega_1}$ on ω_1 , so by dominated convergence

$$f = H_{\phi}^{\omega_1} \quad \text{on } \omega_1, \quad \text{where } \phi = \begin{cases} f & \text{on } \partial\omega_1 \cap \mathbb{D} \\ S & \text{on } \partial\omega_1 \cap \mathbb{T} \subset K \end{cases}.$$

However, we also know from (1) that

$$f = H_f^{\omega} = H_{H_f^{\omega}}^{\omega_1} = H_f^{\omega_1} \quad \text{on } \omega_1$$

(see Theorem 6.3.6 of [1]). Thus, by (2) and our choice of K , we arrive at the contradiction that there is a point w in ω_1 satisfying

$$0 = \operatorname{Re} H_{\phi-f}^{\omega_1}(w) \geq \inf_K \operatorname{Re}(S - f) \mu_w^{\omega_1}(\partial\Omega) > 0,$$

Theorem 1 is now established, subject to verification of Theorem 3.

3 Proof of Theorem 3

We will employ some results concerning the Martin boundary and the minimal fine topology, which are expounded in Chapters 8 and 9 of the book [1]. Let $\widehat{\Omega} = \Omega \cup \Delta$ denote the Martin compactification of a Greenian domain Ω in $\mathbb{R}^N$, let $M(\cdot, y)$ denote the Martin kernel with pole at $y \in \Delta$, and let Δ_1 denote the set of minimal elements of Δ . Thus

$$M(x, y) = \lim_{z \rightarrow y} \frac{G_{\Omega}(x, z)}{G_{\Omega}(x_0, z)} \quad (x \in \Omega, y \in \Delta),$$

where x_0 denotes the reference point for the compactification. A set $E \subset \Omega$ is said to be minimally thin at a point $y \in \Delta_1$ if $R_{M(\cdot, y)}^E \neq M(\cdot, y)$, where R_u^L denotes the usual reduction of a positive superharmonic function u on Ω relative to a set $L \subset \Omega$. Further, a function f is said to have minimal fine limit l at y if there is a set E , minimally thin at y , such that $f(x) \rightarrow l$ as $x \rightarrow y$ in $\Omega \setminus E$. Limit notions with respect to the minimal fine topology will be prefixed by “mf”. The main work lies in establishing the following result, which develops ideas from [5].

Proposition 4 *Let $\xi_0 \in \Omega, y \in \Delta_1$ and ω be an open subset of Ω such that $\Omega \setminus \omega$ is minimally thin at y . Suppose that (v_k) is a decreasing sequence of subharmonic functions on ω such that $v_1/G_{\Omega}(\xi_0, \cdot)$ is bounded above and $\lim_{k \rightarrow \infty} v_k < 0$ on ω . Then there exists $k' \in \mathbb{N}$ such that,*

$$\text{mf} \lim_{z \rightarrow y} \frac{v_k(z)}{G_{\Omega}(\xi_0, z)} < 0 \quad (k \geq k').$$

Proof. Without loss of generality we may assume that ξ_0 coincides with the reference point x_0 for the Martin compactification of Ω , and that $x_0 \notin \bar{\omega}$. For each $k \in \mathbb{N}$ we define

$$\tilde{v}_k(z) = \frac{v_k(z)}{G_{\Omega}(x_0, z)} \quad (z \in \omega).$$

By hypothesis there is a positive constant c such that the function $cG_{\Omega}(x_0, \cdot) - v_1$ is positive and superharmonic on ω . Hence, by Theorem 9.6.2(ii) of [1], each function $\tilde{v}_k$ has a minimal fine limit in the range $[-\infty, c)$ at y . We denote this limit by $\tilde{v}_k(y)$. Thus, for each k , there is a set L_k , minimally thin at y , such that

$$\tilde{v}_k(z) \rightarrow \tilde{v}_k(y) \quad (z \rightarrow y \text{ in } \widehat{\Omega}, z \in \Omega \setminus L_k).$$

By Lemma 9.3.1 of [1] we can now choose a single set $F \subset \Omega$, minimally thin at y , such that

$$\tilde{v}_k(z) \rightarrow \tilde{v}_k(y) \quad (z \rightarrow y \text{ in } \widehat{\Omega}, z \in \Omega \setminus F) \quad \text{for all } k. \quad (3)$$

By Corollary 8.2.9 and Theorem 8.3.1 of [1] we can find an open neighbourhood U of $\Delta \setminus \{y\}$ in $\widehat{\Omega}$ such that U is minimally thin at y , and hence a closed subneighbourhood L of $\Delta \setminus \{y\}$ with the same property. (A more detailed explanation of this step may be found in Lemma 7.2.3 of [10].) By removing L from ω we can ensure that the closure $\overline{\omega}^{\widehat{\Omega}}$ of ω in $\widehat{\Omega}$ meets Δ precisely at y . Next, by Lemma 9.2.2(iii) of [1], we can find an open neighbourhood of $\partial\omega \cap \Omega$ that is minimally thin at y , and hence a subneighbourhood F_0 of $\partial\omega \cap \Omega$ that is closed relative to Ω and has the same property. We now define the open set $\omega_0 = \omega \setminus F_0$. Thus $\overline{\omega}_0 \cap \Omega \subset \omega$, the set $\Omega \setminus \omega_0$ is minimally thin at y , and $\overline{\omega}_0^{\widehat{\Omega}} \cap \Delta = \{y\}$. We are going to construct a probability measure ν on the boundary $\partial\widehat{\Omega}\omega_0$ of ω_0 in $\widehat{\Omega}$ satisfying $\nu(\overline{\omega}_0^{\widehat{\Omega}} \cap \Omega) = 1$, whence $\nu(\{y\}) = 0$, and also

$$\tilde{v}_k(y) \leq \int_{\overline{\omega}_0^{\widehat{\Omega}}} \tilde{v}_k(\zeta) d\nu(\zeta) \quad (k \in \mathbb{N}).$$

Let (Ω_m) be an exhaustion of Ω by bounded open sets satisfying $\overline{\Omega}_m \subset \Omega_{m+1}$ for all m , and define $m(z) = \min\{m : z \in \Omega_m\}$ for $z \in \Omega$. For each $z \in \omega_0$ we define a measure on $\partial(\Omega_{m(z)} \cap \omega_0)$ by writing

$$d\mu_z^*(\zeta) = \frac{G_{\Omega}(x_0, \zeta)}{G_{\Omega}(x_0, z)} d\mu_z^{\Omega_{m(z)} \cap \omega_0}(\zeta).$$

Since $G_{\Omega}(x_0, \cdot)$ is harmonic on a neighbourhood of $\overline{\Omega_{m(z)} \cap \omega_0}$,

$$\mu_z^*(\overline{\omega}_0^{\widehat{\Omega}}) = \frac{1}{G_{\Omega}(x_0, z)} \int_{\partial(\Omega_{m(z)} \cap \omega_0)} G_{\Omega}(x_0, \zeta) d\mu_z^{\Omega_{m(z)} \cap \omega_0}(\zeta) = 1.$$

Later we will arrive at the desired measure ν as a w^* -limit of a suitable sequence of measures $(\mu_{z_n}^*)$.

As a first step we show that there is a potential u on Ω and a set $E_0 \subset \Omega$, minimally thin at y , such that

$$\frac{u(z)}{G_{\Omega}(x_0, z)} \rightarrow \infty \quad (z \rightarrow y, z \in \Omega \setminus \omega_0) \quad (4)$$

and

$$\frac{u(z)}{G_{\Omega}(x_0, z)} \rightarrow 1 \quad (z \rightarrow y, z \in \Omega \setminus E_0). \quad (5)$$

To see this, we note from Theorem 9.2.7 of [1] that, since $\Omega \setminus \omega_0$ is minimally thin at y , there is a potential $G_{\Omega}\mu$ such that

$$a := \int_{\Omega} M(x, y) d\mu(x) < \infty$$

and

$$\frac{G_{\Omega}\mu(z)}{G_{\Omega}(x_0, z)} \rightarrow \infty \quad (z \rightarrow y, z \in \Omega \setminus \omega_0).$$

Also, Fatou's lemma implies that

$$\liminf_{z \rightarrow y} \frac{G_\Omega \mu(z)}{G_\Omega(x_0, z)} \geq \int_\Omega \liminf_{z \rightarrow y} \frac{G_\Omega(x, z)}{G_\Omega(x_0, z)} d\mu(x) = \int_\Omega M(x, y) d\mu(x) = a,$$

while the reverse inequality follows from the result cited above and the fact that Ω is not minimally thin at y . Hence, by Theorem 9.3.3 of [1], there is a set $E_0 \subset \Omega$, minimally thin at y , such that

$$\frac{G_\Omega \mu(z)}{G_\Omega(x_0, z)} \rightarrow a \quad (z \rightarrow y, z \in \Omega \setminus E_0).$$

We now obtain (4) and (5) by setting $u = a^{-1} G_\Omega \mu$.

Let $\varepsilon > 0$. Using the above fact, we can find $r_\varepsilon > 0$ such that

$$u(z) > \frac{G_\Omega(x_0, z)}{\varepsilon} \quad \text{if } z \in (\Omega \setminus \omega_0) \cap B_M(y, r_\varepsilon)$$

and

$$u(z) < 2G_\Omega(x_0, z) \quad \text{if } z \in (\Omega \setminus E_0) \cap B_M(y, r_\varepsilon),$$

where $B_M(y, r)$ denotes the open ball of centre y and radius $r > 0$ with respect to some metric compatible with the Martin topology. Since $\overline{\Omega_{m(z)} \cap \omega_0} \subset \Omega$ and u is positive and superharmonic on Ω , we deduce that, for each $z \in (\omega_0 \setminus E_0) \cap B_M(y, r_\varepsilon)$,

$$\begin{aligned} \mu_z^*(\partial^{\widehat{\Omega}} \omega_0 \cap B_M(y, r_\varepsilon)) &= \frac{1}{G_\Omega(x_0, z)} \int_{\partial^{\widehat{\Omega}} \omega_0 \cap B_M(y, r_\varepsilon)} G_\Omega(x_0, \zeta) d\mu_z^{\Omega_{m(z)} \cap \omega_0}(\zeta) \\ &\leq \frac{1}{G_\Omega(x_0, z)} \int_{\partial(\Omega_{m(z)} \cap \omega_0)} \varepsilon u(\zeta) d\mu_z^{\Omega_{m(z)} \cap \omega_0}(\zeta) \\ &\leq \frac{\varepsilon u(z)}{G_\Omega(x_0, z)} \leq 2\varepsilon. \end{aligned}$$

Since $E_0 \cup F$ and $\Omega \setminus \omega_0$ are both minimally thin at y , we can choose a sequence (z_n) in $\omega_0 \setminus (E_0 \cup F)$ such that $z_n \rightarrow y$. Thus, recalling (3), we see that

$$\tilde{v}_k(z_n) \rightarrow \tilde{v}_k(y) \quad (n \rightarrow \infty) \tag{6}$$

and

$$\mu_{z_n}^*(\partial^{\widehat{\Omega}} \omega_0 \cap B_M(y, r_\varepsilon)) \leq 2\varepsilon \quad \text{for all sufficiently large } n. \tag{7}$$

Further, since $(\mu_{z_n}^*)$ is a sequence of probability measures on the compact set $\overline{\omega_0^{\widehat{\Omega}}}$, there is a subsequence $(\mu_{z_{n_j}}^*)$ which is w^* -convergent to some measure ν . Since every upper bounded upper semicontinuous function ϕ on $\overline{\omega_0^{\widehat{\Omega}}}$ is the pointwise limit of a decreasing sequence of continuous functions, the monotone convergence theorem yields

$$\limsup_{j \rightarrow \infty} \int_{\overline{\omega_0^{\widehat{\Omega}}}} \phi d\mu_{z_{n_j}}^* \leq \int_{\overline{\omega_0^{\widehat{\Omega}}}} \phi d\nu. \tag{8}$$

Clearly ν is a probability measure with support in $\partial^{\widehat{\Omega}}\omega_0$. Also, for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that, by (7),

$$\nu(\{y\}) \leq \nu(\partial^{\widehat{\Omega}}\omega_0 \cap B_M(y, r_\varepsilon)) \leq 2\varepsilon,$$

so $\nu(\{y\}) = 0$. Since $\overline{\omega}_0^{\widehat{\Omega}} \cap \Delta = \{y\}$, we conclude that $\nu(\partial^{\widehat{\Omega}}\omega_0 \cap \Omega) = 1$.

The subharmonicity of v_k on ω implies that

$$\begin{aligned} \tilde{v}_k(z_{n_j}) &= \frac{v_k(z_{n_j})}{G_\Omega(x_0, z_{n_j})} \\ &\leq \frac{1}{G_\Omega(x_0, z_{n_j})} \int_{\partial(\Omega_{m(z_{n_j})} \cap \omega_0)} v_k(\zeta) d\mu_{n_j}^{\Omega_{m(z_{n_j})} \cap \omega_0}(\zeta) \\ &= \int_{\overline{\omega}_0^{\widehat{\Omega}}} \tilde{v}_k(\zeta) d\mu_{z_{n_j}}^*(\zeta). \end{aligned} \quad (9)$$

Also, the functions $\tilde{v}_k$ are upper semicontinuous on ω and bounded above (by c) on $\overline{\omega}_0^{\widehat{\Omega}}$. Hence, defining $\phi = \tilde{v}_k$ on $\overline{\omega}_0^{\widehat{\Omega}} \cap \Omega$ and $\phi = c$ at y , we see from (8) that

$$\limsup_{j \rightarrow \infty} \int_{\overline{\omega}_0^{\widehat{\Omega}}} \tilde{v}_k(\zeta) d\mu_{z_{n_j}}^*(\zeta) \leq \int_{\overline{\omega}_0^{\widehat{\Omega}}} \tilde{v}_k(\zeta) d\nu(\zeta).$$

From (6) and (9) we conclude that

$$\tilde{v}_k(y) \leq \int_{\overline{\omega}_0^{\widehat{\Omega}}} \tilde{v}_k(\zeta) d\nu(\zeta) \quad (k \in \mathbb{N}).$$

Finally, $(\tilde{v}_k)$ is a decreasing sequence of upper bounded functions on $\overline{\omega}_0^{\widehat{\Omega}}$, so we can apply the monotone convergence theorem to conclude that

$$\lim_{k \rightarrow \infty} \tilde{v}_k(y) \leq \int_{\overline{\omega}_0^{\widehat{\Omega}}} \lim_{k \rightarrow \infty} \tilde{v}_k(\zeta) d\nu(\zeta).$$

Since $\nu(\overline{\omega}_0^{\widehat{\Omega}} \cap \Delta) = 0$ and $\lim_{k \rightarrow \infty} \tilde{v}_k < 0$ on ω , we conclude that $\lim_{k \rightarrow \infty} \tilde{v}_k(y) < 0$. Thus $\tilde{v}_k(y) < 0$ for all sufficiently large k , as required. ■

Proof of Theorem 3. Without loss of generality we may assume that ω is connected. There is a (unique) probability measure μ_1 on Δ_1 such that

$$1 = \int_{\Delta_1} M(x, y) d\mu_1(y) \quad (x \in \Omega).$$

Hence

$$\mu_x^\omega(\Omega) = R_1^{\Omega \setminus \omega}(x) = \int_{\Delta_1} R_{M(\cdot, y)}^{\Omega \setminus \omega}(x) d\mu_1(y) \quad (x \in \omega), \quad (10)$$

by Theorem 6.9.1 and Corollary 9.1.4 of [1]. Thus

$$\mu_x^\omega(\partial\Omega) = 1 - \mu_x^\omega(\Omega) = \int_A \left\{ M(x, y) - R_{M(\cdot, y)}^{\Omega \setminus \omega}(x) \right\} d\mu_1(y) \quad (x \in \omega),$$

where

$$A = \{y \in \Delta_1 : R_{M(\cdot, y)}^{\Omega \setminus \omega} \neq M(\cdot, y)\};$$

that is, A is the set of points in Δ_1 at which $\Omega \setminus \omega$ is minimally thin. Our hypothesis that $\mu_{x_1}^\omega(\partial\Omega) > 0$ shows that $\mu_1(A) > 0$.

Let

$$A_k = \{y \in A : R_{M(\cdot, y)}^{\Omega \setminus \{v_k < 0\}} \neq M(\cdot, y)\} \quad (k \in \mathbb{N}).$$

Proposition 4 tells us that, if $y \in A$, then $\Omega \setminus \{v_k < 0\}$ is minimally thin at y for all sufficiently large k . Hence (A_k) increases to A , and so we can choose k' such that $\mu_1(A_{k'}) > 0$. On each connected component of the open set $\{v_{k'} < 0\}$ either $R_{M(\cdot, y)}^{\Omega \setminus \{v_{k'} < 0\}} = M(\cdot, y)$ or $R_{M(\cdot, y)}^{\Omega \setminus \{v_{k'} < 0\}} < M(\cdot, y)$. Thus we can choose a component ω' of $\{v_{k'} < 0\}$ on which $R_{M(\cdot, y)}^{\Omega \setminus \{v_{k'} < 0\}} < M(\cdot, y)$ for all y in a subset of $A_{k'}$ of positive μ_1 -measure. Further, we can arrange that $x_1 \in \omega'$ by choosing k' large enough. The preceding calculation, applied to $\{v_{k'} < 0\}$ and $A_{k'}$ in place of ω and A , now shows that

$$\mu_x^{\{v_{k'} < 0\}}(\partial\Omega) = \int_{A_{k'}} \left\{ M(x, y) - R_{M(\cdot, y)}^{\Omega \setminus \{v_{k'} < 0\}}(x) \right\} d\mu_1(y) > 0 \quad (x \in \omega'),$$

as required. ■

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