A reflection result for harmonic functions which vanish on a cylindrical surface

Stephen J. Gardiner* and Hermann Render
School of Mathematics and Statistics, University College Dublin,
Belfield, Dublin 4, Ireland.

Abstract

Suppose that a harmonic function h on a finite cylinder U vanishes on the curved part A of the boundary. It was recently shown that h then has a harmonic continuation to the infinite strip bounded by the hyperplanes containing the flat parts of the boundary. This paper examines what can be said if the above function h is merely harmonic near A (and inside U). It is shown that h then has a harmonic extension to a larger domain formed by radial reflection.

1 Introduction

Let $N \geq 3$ and $a > 0$, and let B' denote the open unit ball in $\mathbb{R}^{N-1}$. The following harmonic extension result for cylinders was recently established in [4].

Theorem 1 *Any harmonic function on the finite cylinder $B' \times (-a, a)$ which continuously vanishes on $\partial B' \times (-a, a)$ has a harmonic extension to $\mathbb{R}^{N-1} \times (-a, a)$.*

In the case where $N = 2$, Theorem 1 is easily verified by repeated application of the Schwarz reflection principle. In higher dimensions the result was proved by a detailed analysis of series expansions involving Bessel functions. It is natural to ask whether some sort of extension result still holds when the given harmonic function is merely defined near the curved boundary (and inside the cylinder). The corresponding assertion certainly holds when $N = 2$, as can again be seen by Schwarz reflection. However, there is an obstacle to this approach in higher dimensions, since Ebenfelt and

*Corresponding author

[0]2010 *Mathematics Subject Classification* 31B05.
Keywords: harmonic continuation, Green function, cylindrical harmonics
E-mail addresses: stephen.gardiner@ucd.ie, hermann.render@ucd.ie

Khavinson [3] (see also Chapter 10 of [5]) have shown that a point-to-point reflection law in $\mathbb{R}^3$ can only hold for planar or spherical surfaces. Nevertheless, as will be seen below, it is still possible to establish harmonic extension to a "radial reflection" of the original domain.

Let (x', x_N) denote a typical point of $\mathbb{R}^{N-1} \times \mathbb{R}$ and $\|x'\|$ denote the Euclidean norm of x'. Our main result is as follows.

Theorem 2 *Let $\phi : (-a, a) \to [0, 1)$ be upper semicontinuous. Then any harmonic function on the domain*

$$\left\{(x', x_N) : |x_N| < a \quad \text{and} \quad \phi(x_N) < \left\|x'\right\| < 1\right\} \tag{1}$$

which continuously vanishes on $\partial B' \times (-a, a)$ has a harmonic extension to the domain

$$\left\{(x', x_N) : |x_N| < a \quad \text{and} \quad \phi(x_N) < \left\|x'\right\| < 2 - \phi(x_N)\right\}. \tag{2}$$

The sharpness of the upper bound $2 - \phi(x_N)$ in (2) is demonstrated by the example below.

Example Let $N = 4$ and $\phi : [-a, a] \to [0, 1)$ be continuous. Then there is a harmonic function on the domain (1) which continuously vanishes on $\partial B' \times (-a, a)$ and does not have a harmonic extension beyond the domain (2).

To see this, let $\omega = \{(s, t) \in \mathbb{R}^2 : \phi(t) < s < 1, |t| < a\}$, let u be the logarithmic potential of a measure comprising a dense sequence of point masses in the set $\{(s, t) \in \partial\omega : s \neq 1\}$, and let $v(s, t) = u(s, t) - u(2 - s, t)$. Thus v is harmonic on the domain $\{(s, t) : \phi(t) < s < 2 - \phi(t), |t| < a\}$, is unbounded near each boundary point, and vanishes on $\{1\} \times (-a, a)$. The function

$$(x', x_4) \mapsto \left\|x'\right\|^{-1} v(\left\|x'\right\|, x_4) \qquad (x' \in \mathbb{R}^3 \backslash \{0\})$$

is now easily seen (by computation of the Laplacian) to be harmonic on the domain (2) and to vanish on $\partial B' \times (-a, a)$, yet it does not have a harmonic extension beyond (2).

Remark. The special case of Theorem 2 where the harmonic function is of the form $f(\|x'\|, x_N)$ follows easily from known reflection results in the plane (see Lewy [7]), since $\Delta f + (N - 2)s^{-1}\partial f/\partial s = 0$ on the domain $\{(s, t) : |t| < a, \phi(t) < s < 1\}$ and $f = 0$ on the boundary line segment $\{1\} \times (-a, a)$. There are even explicit formulae for the extension in this case: see Savina [8]. (We are grateful to Dima Khavinson for these references.)

The proof of Theorem 2 will combine results from [4] with several additional arguments.

2 Preparatory material

Let J_ν and Y_ν denote the usual Bessel functions of order $\nu \geq 0$, of the first and second kinds (see Watson [11]). Further, let $(j_{\nu,m})_{m\geq 1}$ denote the sequence of positive zeros of J_ν, arranged in increasing order. We collect below a few facts that we will need.

Lemma 3 *(i)* $\dfrac{d}{dz}\dfrac{J_\nu(z)}{z^\nu} = -\dfrac{J_{\nu+1}(z)}{z^\nu}$.
(ii) $|J_\nu(t)| \leq 1 \quad (t > 0)$.
(iii) $j_{\nu,m} \geq (m + 3/4)\pi + \nu \quad (m \geq 1)$.
(iv) If $y(t) = \sqrt{t}J_\nu(kt)$, *where* k *is a non-zero constant, then*

$$\frac{d^2y}{dt^2} + \left(k^2 + \frac{\frac{1}{4} - \nu^2}{t^2}\right) y = 0 \qquad (t > 0).$$

(v) $j_{\nu,m}^2 \left\{J_{\nu+1}(j_{\nu,m})\right\}^2 \geq \dfrac{2}{\pi}\sqrt{j_{\nu,m}^2 - \nu^2} \quad (m \geq 1)$.

Proof. (i) See p.45 of Watson [11].
(ii) See p.406, (10) of [11].
(iii) Laforgia and Muldoon (see (2.4) of [6]) showed that $j_{\nu,m} \geq j_{0,m} + \nu$, and we know from p.489 of [11] that $j_{0,m} \geq (m + 3/4)\pi$.
(iv) See p.17, (1.8.9) of Szegö [10].
(v) See Lemma 3(i) of [4]. ∎

Next, we recall a result of Wimp and Colton [12], which was established using the theory of Volterra integral equations.

Lemma 4 *Let* $q \in C[-\delta, \delta]$, *where* $\delta > 0$, *let* (c_n) *be a sequence of non-zero real numbers, and let* (y_n) *be a sequence of functions on* $[-\delta, \delta]$ *satisfying*

$$\frac{d^2y_n}{dt^2} + \left(c_n^2 - q(t)\right) y_n = 0, \quad y_n(0) = 0, \quad y_n'(0) = c_n.$$

If there is a sequence (a_n) *of real numbers satisfying* $\sum |a_n| < \infty$ *and*

$$\sum a_n y_n(t) = 0 \qquad (0 \leq t \leq \delta), \tag{3}$$

then

$$\sum a_n y_n(t) = 0 \qquad (-\delta \leq t \leq 0), \tag{4}$$

and the series in (3) and (4) converge uniformly in t.

The identities in the following result were previously known to hold when $s \leq t \leq 1$. We will now extend their range of validity using Lemma 4.

Lemma 5 *Let $0 < s < 1$.*
(a) If $\nu > 0$, then

$$4\nu \sum_{m=1}^{\infty} \frac{J_\nu(j_{\nu,m}s)J_\nu(j_{\nu,m}t)}{j_{\nu,m}^2 \left\{J_{\nu+1}(j_{\nu,m})\right\}^2} = s^\nu(t^{-\nu} - t^\nu) \qquad (s \le t \le 2-s), \tag{5}$$

and the series converges uniformly in t.
(b) In the case where $\nu = 0$ we have

$$2 \sum_{m=1}^{\infty} \frac{J_0(j_{0,m}s)J_0(j_{0,m}t)}{j_{0,m}^2 \left\{J_1(j_{0,m})\right\}^2} = -\log t \qquad (s \le t \le 2-s),$$

and the series converges uniformly in t.

Proof. (a) Let $\nu > 0$, and let

$$y_{\nu,m}(t) = \sqrt{t}J_\nu(j_{\nu,m}t), \quad c_m = \sqrt{j_{\nu,m}^2 + 1}, \quad q(t) = \frac{\nu^2 - \frac{1}{4}}{t^2} + 1 \quad (m \ge 1, t > 0).$$

Then, for $m \ge 1$,

$$\frac{d^2 y_{\nu,m}}{dt^2} + \left(c_m^2 - q(t)\right) y_{\nu,m} = \frac{d^2 y_{\nu,m}}{dt^2} + \left(j_{\nu,m}^2 + \frac{\frac{1}{4} - \nu^2}{t^2}\right) y_{\nu,m} = 0,$$

by Lemma 3(iv). Further,

$$y_{\nu,m}(1) = 0 \text{ and } y'_{\nu,m}(1) = j_{\nu,m}J'_\nu(j_{\nu,m}) = -j_{\nu,m}J_{\nu+1}(j_{\nu,m}),$$

by Lemma 3(i). Also, if we define

$$y_{\nu,0}(t) = \sqrt{t}(t^{-\nu} - t^\nu) \ \ (t > 0) \ \text{ and } \ c_0 = 1,$$

then

$$\frac{d^2 y_{\nu,0}}{dt^2} + \left(c_0^2 - q(t)\right) y_{\nu,0} = \left\{\frac{d^2}{dt^2} + \frac{\frac{1}{4} - \nu^2}{t^2}\right\} \left(t^{\frac{1}{2}-\nu} - t^{\nu+\frac{1}{2}}\right) = 0,$$

$$y_{\nu,0}(1) = 0 \text{ and } y'_{\nu,0}(1) = -2\nu.$$

We know from Lemma 5(a) of [4] that equation (5) holds when $t \in [s, 1]$; that is,

$$\sum_{m=0}^{\infty} a_m \left\{b_m y_{\nu,m}(t)\right\} = 0 \qquad (s \le t \le 1),$$

where

$$b_m = -\frac{\sqrt{j_{\nu,m}^2 + 1}}{j_{\nu,m}J_{\nu+1}(j_{\nu,m})} \quad (m \ge 1), \qquad b_0 = -\frac{1}{2\nu},$$

$$a_m = -4\nu \frac{J_\nu(j_{\nu,m}s)}{j_{\nu,m}J_{\nu+1}(j_{\nu,m})\sqrt{j_{\nu,m}^2+1}} \qquad (m \geq 1), \qquad a_0 = 2\nu s^\nu.$$

Now $b_m y'_{\nu,m}(1) = c_m$ when $m \geq 0$. Also, when $m \geq 1$, we note that

$$\frac{|a_m|}{4\nu} \leq \frac{|J_\nu(j_{\nu,m}s)|}{j_{\nu,m}^2 |J_{\nu+1}(j_{\nu,m})|} \leq \sqrt{\frac{\pi}{2}} \frac{1}{j_{\nu,m}\left(j_{\nu,m}^2 - \nu^2\right)^{1/4}} \leq \frac{1}{m^{3/2}},$$

by parts (ii), (iii) and (v) of Lemma 3, whence $\sum |a_m| < \infty$. We can thus apply Lemma 4, on replacing t by $1-t$, to deduce that (5) holds and that the convergence is uniform in t.

(b) The argument is directly analogous to part (a). ■

If $\lambda > 0$, let $P_n^{(\lambda)}$ be the usual ultraspherical (Gegenbauer) polynomial defined by the expansion

$$(1-2tu+u^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(t)u^n \qquad (|t| \leq 1, |u| < 1).$$

(See Section 4.7 of Szegö [10], or Chapter IV of Stein and Weiss [9].) Also, let $T_n(t)$ be the Chebychev polynomial given by $\cos(n\cos^{-1} t)$ when $|t| \leq 1$. We define

$$\nu_n = n + \frac{N-3}{2} \qquad (n \geq 0),$$

and note the following basic facts for future reference. (The case of positive exponents may be found on p.54 of [1], and the remaining case follows on applying the Kelvin transformation.)

Lemma 6 *Let $y' \in \mathbb{R}^{N-1}\backslash\{0'\}$. Then the functions*

$$\begin{aligned} x' &\mapsto \left\|x'\right\|^{\frac{3-N}{2}} P_n^{\left(\frac{N-3}{2}\right)}\left(\frac{\langle x', y'\rangle}{\|x'\|\,\|y'\|}\right)\left\|x'\right\|^{\pm\nu_n} \qquad (N \geq 4), \\ x' &\mapsto T_n\left(\frac{\langle x', y'\rangle}{\|x'\|\,\|y'\|}\right)\left\|x'\right\|^{\pm n} \qquad (N = 3) \end{aligned}$$

are harmonic on $\mathbb{R}^{N-1}\backslash\{0'\}$.

The following result is taken from Lemma 11 in [4]. (The formula for the distributional Laplacian is stated there only on $B' \times \mathbb{R}$, but the same reasoning applies on all of $\mathbb{R}^N$.)

Lemma 7 *For any $n \geq 0, m \geq 1$ and any $y \in (B'\backslash\{0'\}) \times \mathbb{R}$, let $u_{n,m,y}$ be the function defined by*

$$x \mapsto \left\|x'\right\|^{\frac{3-N}{2}} P_n^{\left(\frac{N-3}{2}\right)}\left(\frac{\langle x', y'\rangle}{\|x'\|\,\|y'\|}\right) \frac{J_{\nu_n}(j_{\nu_n,m}\|x'\|)J_{\nu_n}(j_{\nu_n,m}\|y'\|)}{j_{\nu_n,m}\left\{J_{\nu_n+1}(j_{\nu_n,m})\right\}^2} e^{-j_{\nu_n,m}|x_N - y_N|} \qquad (N \geq 4),$$

$$x \mapsto T_n\left(\frac{\langle x', y'\rangle}{\|x'\|\,\|y'\|}\right)\frac{J_{\nu_n}(j_{\nu_n,m}\|x'\|)J_{\nu_n}(j_{\nu_n,m}\|y'\|)}{j_{\nu_n,m}\left\{J_{\nu_n+1}(j_{\nu_n,m})\right\}^2}e^{-j_{\nu_n,m}|x_N-y_N|} \qquad (N=3).$$

Then $u_{n,m,y}$
(i) is harmonic on $\left(\mathbb{R}^{N-1}\backslash\{0'\}\right)\times(\mathbb{R}\backslash\{y_N\})$;
(ii) has distributional Laplacian on $\mathbb{R}^N$ given by

$$-2j_{\nu_n,m}u_{n,m,y}(z',y_N)dz' \quad \text{on} \quad \mathbb{R}^{N-1}\times\{y_N\};$$

(iii) satisfies

$$|u_{n,m,y}(x)| \le \frac{1-\|y'\|}{\|x'\|^{N/2-1}}\binom{n+N-4}{n}\frac{e^{-j_{\nu_n,m}|x_N-y_N|}}{\left\{J_{\nu_n+1}(j_{\nu_n,m})\right\}^2},$$

where the binomial coefficient is interpreted as 1 when $N=3$.

3 Proof of Theorem 2

Let Ω denote the infinite cylinder $B'\times\mathbb{R}$, let $G_\Omega(\cdot,\cdot)$ denote its Green function, and let $y\in\Omega$. It will also be convenient to write $a_N=\sigma_N(N-2)$ when $N\ge 3$, and $a_2=\sigma_2$, where σ_N denotes the surface area of the unit sphere in $\mathbb{R}^N$. A natural intermediate step towards proving Theorem 2 is to establish a harmonic extension result for the function $G_\Omega(\cdot,y)$. We already know from Theorem 13 of [4] that $G_\Omega(\cdot,y)$ has a harmonic extension $\widetilde{G}_\Omega(\cdot,y)$ to $\mathbb{R}^{N-1}\times(\mathbb{R}\backslash\{y_N\})$ given by

$$\widetilde{G}_\Omega(x,y)=\begin{cases}\frac{a_N}{a_{N-1}}\|y'\|^{(3-N)/2}\sum_{n=0}^{\infty}2\nu_n h_{n,y}(x) & (N\ge 4)\\ 2h_{0,y}(x)+4\sum_{n=1}^{\infty}h_{n,y}(x) & (N=3)\end{cases}, \qquad (6)$$

where $h_{n,y}=\sum_{m=1}^{\infty}u_{n,m,y}$ $(n\ge 0)$. We need to establish that $G_\Omega(\cdot,y)$ also has a harmonic extension across a certain portion of the set $\{x_N=y_N\}\backslash\Omega$. To that end we will analyze, in Lemma 9 below, the behaviour of $h_{n,y}$ near the hyperplane $\{x_N=y_N\}$. First we present a useful observation about the Newtonian potential of a measure supported by a hyperplane, where the measure has a harmonic density function with respect to $(N-1)$-dimensional measure on some subregion of the hyperplane.

Lemma 8 *Let h be an integrable harmonic function on a bounded open set $U'\subset\mathbb{R}^{N-1}$, let $c\in\mathbb{R}$ and let v be the Newtonian potential*

$$v(x)=\int_{U'}\frac{h(z')}{\|x-(z',c)\|^{N-2}}\,dz' \qquad (x\in\mathbb{R}^N).$$

Then $\partial^2 v/\partial x_N^2$, which is clearly harmonic on $\mathbb{R}^N\backslash(\overline{U'}\times\{c\})$, has a harmonic extension to $\mathbb{R}^N\backslash(\partial U'\times\{c\})$.

Proof. Let

$$h^* = \begin{cases} h & \text{on } U' \\ 0 & \text{elsewhere on } \mathbb{R}^{N-1} \end{cases}.$$

Then, provided $x_N \neq c$,

$$\begin{aligned}
\frac{\partial^2 v}{\partial x_N^2}(x) &= \int_{\mathbb{R}^{N-1}} \left(\frac{\partial^2}{\partial x_N^2} \frac{1}{\|x-(z',c)\|^{N-2}} \right) h^*(z')dz' \\
&= \int_{\mathbb{R}^{N-1}} \left(-\Delta_{x'} \frac{1}{\|x-(z',c)\|^{N-2}} \right) h^*(z')dz' \\
&= \int_{\mathbb{R}^{N-1}} \left(-\Delta_{z'} \frac{1}{\|x-(z',c)\|^{N-2}} \right) h^*(z')dz' \\
&= (-\Delta h^*) \left(z' \mapsto \|x-(z',c)\|^{2-N} \right),
\end{aligned}$$

where Δh^* denotes the distributional Laplacian on $\mathbb{R}^{N-1}$. From the harmonicity of h on U' we see that the support of the distribution Δh^* is contained in $\partial U'$, whence $\partial^2 v/\partial x_N^2$ has a harmonic extension to $\mathbb{R}^N \backslash (\partial U' \times \{c\})$.
■

For any $y' \in B'$ we define the annular set

$$A_{y'} = \left\{ x' \in \mathbb{R}^{N-1} : \|y'\| < \|x'\| < 2 - \|y'\| \right\}.$$

Lemma 9 *Let $y \in (B' \backslash \{0'\}) \times \mathbb{R}$ and $n \geq 0$. Then the series $\sum_{m=1}^{\infty} u_{n,m,y}$ converges locally uniformly on $A_{y'} \times (\mathbb{R} \backslash \{y_N\})$ to a harmonic function, and this sum $h_{n,y}$ has a harmonic extension to $A_{y'} \times \mathbb{R}$. Further,*

$$|h_{n,y}(x)| \leq \|x'\|^{1-N/2} \binom{n+N-4}{n} \frac{e^{-n|x_N-y_N|/2}}{|x_N-y_N|^2 \left(1-e^{-|x_N-y_N|}\right)} \qquad (x \in A_{y'} \times (\mathbb{R} \backslash \{y_N\})). \tag{7}$$

Proof. By Lemma 7(iii), and then parts (v) and (iii) of Lemma 3,

$$\begin{aligned}
\frac{|u_{n,m,y}(x)|}{j_{\nu_n,m}^2} &\leq \|x'\|^{1-N/2} \binom{n+N-4}{n} \frac{e^{-j_{\nu_n,m}|x_N-y_N|}}{j_{\nu_n,m}^2 \left\{ J_{\nu_n+1}(j_{\nu_n,m}) \right\}^2} \\
&\leq \frac{\pi}{2} \|x'\|^{1-N/2} \binom{n+N-4}{n} \frac{e^{-j_{\nu_n,m}|x_N-y_N|}}{\sqrt{j_{\nu_n,m}^2 - \nu_n^2}} \qquad (8) \\
&\leq \|x'\|^{1-N/2} \binom{n+N-4}{n} \frac{e^{-(m\pi+n)|x_N-y_N|}}{m}. \qquad (9)
\end{aligned}$$

Since

$$\sum_{m=1}^{\infty} \frac{t^m}{m} = \log \frac{1}{1-t} \quad (|t| < 1)$$

and (by the concavity of the function $t \mapsto 1 - e^{-\pi t}$)

$$\frac{1-e^{-\pi t}}{t} \geq \frac{1-e^{-\pi/2}}{1/2} > 1 \quad \left(0 \leq t \leq \frac{1}{2}\right),$$

we see from (9) that

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{|u_{n,m,y}(x)|}{j_{\nu_n,m}^2} &\leq \|x'\|^{1-N/2} \binom{n+N-4}{n} e^{-n|x_N - y_N|} \log \frac{1}{1-e^{-\pi|x_N-y_N|}} \\
&\leq \|y'\|^{1-N/2} \binom{n+N-4}{n} \log \frac{1}{|x_N - y_N|} \qquad (10)
\end{aligned}$$

when $\|x'\| \geq \|y'\|$ and $0 < |x_N - y_N| < 1/2$.

We define the Newtonian potential

$$v_{n,m,y}(x) = \frac{1}{a_N} \int_{A_{y'}} \frac{2u_{n,m,y}(z', y_N)/j_{\nu_n,m}}{\|x - (z', y_N)\|^{N-2}} \, dz'.$$

By Lemma 5 the series

$$\sum_{m=1}^{\infty} \frac{2}{j_{\nu_n,m}} u_{n,m,y}(z', y_N) \quad (z' \in A_{y'})$$

converges uniformly on $\overline{A}_{y'}$ to the sum

$$h(z') = \|z'\|^{\frac{3-N}{2}} P_n^{\left(\frac{N-3}{2}\right)} \left(\frac{\langle z', y' \rangle}{\|z'\| \, \|y'\|} \right) \frac{\|y'\|^{\nu_n}}{2\nu_n} \left(\|z'\|^{-\nu_n} - \|z'\|^{\nu_n} \right)$$

when $N \geq 4$, and to

$$h(z') = \begin{cases} T_n \left(\frac{\langle z', y' \rangle}{\|z'\|\|y'\|} \right) \frac{\|y'\|^n}{2n} \left(\|z'\|^{-n} - \|z'\|^n \right) & (n \geq 1) \\ -\log \|z'\| & (n = 0) \end{cases}$$

when $N = 3$. By Lemma 6 the function h is harmonic on $A_{y'}$. Since

$$\sum_{m=1}^{\infty} v_{n,m,y}(x) = \frac{1}{a_N} \int_{A'_y} \frac{h(z')}{\|x - (z', y_N)\|^{N-2}} dz',$$

we see from Lemma 8 that

$$\frac{\partial^2}{\partial x_N^2} \sum_{m=1}^{\infty} v_{n,m,y} \text{ has a harmonic extension to } A_{y'} \times \mathbb{R}. \qquad (11)$$

We know from Lemma 7(ii) that $j_{\nu_n,m}^{-2} u_{n,m,y} - v_{n,m,y}$ is harmonic on $A_{y'} \times \mathbb{R}$. Since the function

$$x \mapsto \log \frac{1}{|x_N - y_N|}$$

is integrable with respect to surface area measure on spheres contained in $\overline{A}_{y'} \times [y_N - 1/2, y_N + 1/2]$, it follows from (10) and the mean value property that the series of harmonic functions

$$\sum_{m=1}^{\infty} \left(\frac{u_{n,m,y}}{j_{\nu_n,m}^2} - v_{n,m,y} \right)$$

converges locally uniformly on $A_{y'} \times (y_N - 1/2, y_N + 1/2)$. Since it also clearly converges locally uniformly on $A_y' \times (\mathbb{R} \backslash \{y_N\})$, by (9), we conclude that it converges locally uniformly on $A_y' \times \mathbb{R}$ to a harmonic function H. In view of (11) we deduce that there is a harmonic extension to $A_{y'} \times \mathbb{R}$ of the function

$$\frac{\partial^2}{\partial x_N^2} \left(H + \sum_{m=1}^{\infty} v_{n,m,y}(x) \right), \text{ namely, } \frac{\partial^2}{\partial x_N^2} \sum_{m=1}^{\infty} \frac{u_{n,m,y}}{j_{\nu_n,m}^2}.$$

By (9) and standard estimates for derivatives of harmonic functions (Corollary 1.4.3 of [1]), this series equals

$$\sum_{m=1}^{\infty} \frac{1}{j_{\nu_n,m}^2} \frac{\partial^2 u_{n,m,y}}{\partial x_N^2}, \quad \text{that is,} \quad \sum_{m=1}^{\infty} u_{n,m,y},$$

on $A_{y'} \times (\mathbb{R} \backslash \{y_N\})$. Thus the series $\sum_{m=1}^{\infty} u_{n,m,y}$, which converges locally uniformly on $A_{y'} \times (\mathbb{R} \backslash \{y_N\})$, has a harmonic extension to $A_{y'} \times \mathbb{R}$.

Finally, since $t^2 e^{-at} \leq 4(ae)^{-2}$ when $t \geq 0$ and $a > 0$, and $j_{\nu_n,m} \geq (m + 3/4)\pi + \nu_n$, we see from (8) that

$$\begin{aligned} |u_{n,m,y}(x)| &\leq \frac{\pi}{2} \|x'\|^{1-N/2} \binom{n+N-4}{n} \frac{4}{7\pi} j_{\nu_n,m}^2 e^{-j_{\nu_n,m}|x_N - y_N|} \\ &\leq \|x'\|^{1-N/2} \binom{n+N-4}{n} \frac{(4^2 e^{-2})(2/7)}{|x_N - y_N|^2} e^{-j_{\nu_n,m}|x_N - y_N|/2} \\ &\leq \|x'\|^{1-N/2} \binom{n+N-4}{n} \frac{1}{|x_N - y_N|^2} \frac{e^{-n|x_N - y_N|/2}}{\left(e^{|x_N - y_N|}\right)^m}, \end{aligned}$$

and so (7) holds. ■

We recalled at the beginning of this section that $G_\Omega(\cdot, y)$ has a harmonic extension $\widetilde{G}_\Omega(\cdot, y)$ to $\mathbb{R}^{N-1} \times (\mathbb{R} \backslash \{y_N\})$. We will now show that it also has a harmonic extension across part of the set $\{x_N = y_N\} \backslash \Omega$. This is the crucial additional fact that will allow us to establish Theorem 2.

Lemma 10 *Let $y \in (B' \backslash \{0'\}) \times \mathbb{R}$. Then $G_\Omega(\cdot, y)$ has a harmonic extension $\widetilde{G}_\Omega(\cdot, y)$ to $A_{y'} \times \mathbb{R}$. Further, if $b/2 \leq \|y'\| \leq b$ and $\varepsilon \in (0, b)$, where $b \in (0,1)$, then there is a constant $C(N, b, \varepsilon)$ such that*

$$\widetilde{G}_\Omega(\cdot, y) \leq C(N, b, \varepsilon) \quad (b + \varepsilon < \left\|x'\right\| < 2 - b - \varepsilon, |x_N - y_N| < 1). \tag{12}$$

Proof. Let $h_{n,y}$ be as in the statement of Lemma 9. We claim that the functions

$$x \mapsto \sum_{n=0}^{\infty} \nu_n h_{n,y}(x) \quad (N \geq 4) \quad \text{and} \quad x \mapsto \sum_{n=1}^{\infty} h_{n,y}(x) \quad (N = 3) \tag{13}$$

have harmonic extensions to $A_{y'} \times \mathbb{R}$. We will show this when $N \geq 4$, the argument when $N = 3$ being similar and simpler.

Now

$$\begin{aligned} \nu_n \binom{n+N-4}{n} &\leq \left(n + \frac{N-3}{2}\right)(n+N-4)^{N-4} \\ &\leq (n+N-3)^{N-3} \leq ((N-2)n)^{N-3} \quad (n \geq 1). \end{aligned}$$

Thus, using the fact that $\sum_{n=0}^{\infty} n^k e^{-nt} = k!(1 - e^{-t})^{-k-1}$ when $t > 0$ and $k \geq 0$, we see that

$$\begin{aligned} \sum_{n=0}^{\infty} \nu_n \binom{n+N-4}{n} e^{-n|x_N - y_N|/2} &\leq N^N \sum_{n=0}^{\infty} n^{N-3} e^{-n|x_N-y_N|/2} \\ &\leq \frac{N^N (N-3)!}{\left(1 - e^{-|x_N - y_N|/2}\right)^{N-2}}. \end{aligned}$$

Hence, by (7),

$$\begin{aligned} \sum_{n=0}^{\infty} |\nu_n h_{n,y}(x)| &\leq \frac{\|x'\|^{1-N/2}}{|x_N - y_N|^2 (1 - e^{-|x_N - y_N|})} \sum_{n=0}^{\infty} \nu_n \binom{n+N-4}{n} e^{-n|x_N - y_N|/2} \\ &\leq \frac{C(N) \|y'\|^{1-N/2}}{|x_N - y_N|^2 (1 - e^{-|x_N - y_N|/2})^{N-1}} \quad (x \in A_{y'} \times (\mathbb{R} \backslash \{y_N\})). \end{aligned}$$

Let $M_y(x_N)$ denote the expression on the right hand side of the above inequality and let $b \geq \|y'\|$. Since $\log^+ M_y$ is locally integrable on $\mathbb{R}$, we can apply a result of Domar (Theorem 3 and Remark 1 of [2]) to see that the series in (13) converges locally uniformly on $A_{y'} \times \mathbb{R}$ to a harmonic function H_y satisfying

$$|H_y(x)| \leq C(N, b, \varepsilon) \quad (b + \varepsilon < \left\|x'\right\| < 2 - b - \varepsilon, |x_N - y_N| < 1),$$

where $C(N, b, \varepsilon)$ is a positive constant depending at most on N, b, ε.

Finally, in view of (6), $G_\Omega(\cdot, y)$ has a harmonic extension $\widetilde{G}_\Omega(\cdot, y)$ to $A_{y'} \times \mathbb{R}$ satisfying an estimate of the form (12). ■

We now establish the special case of Theorem 2 in which we consider a constant function $\phi = \phi_0$.

Theorem 11 *Let $\phi_0 \in (0,1)$. Then any harmonic function h on the domain*

$$\left\{(x', x_N) : |x_N| < a \quad \text{and} \quad \phi_0 < \left\|x'\right\| < 1\right\}$$

which vanishes on $\partial B' \times (-a, a)$ has a harmonic extension to the domain

$$\left\{(x', x_N) : |x_N| < a \quad \text{and} \quad \phi_0 < \left\|x'\right\| < 2 - \phi_0\right\}.$$

Proof. Let h be as in the statement of the theorem, and let $0 < \varepsilon < \min\{(1-\phi_0)/2, \phi_0/3, a\}$. We may assume, without loss of generality, that $a \in (0, \frac{1}{2})$. Then there is a C^2 function h_0 on Ω which equals h on the set

$$\{(x', x_N) : |x_N| \leq a - \varepsilon \;\; \text{and} \;\; \phi_0 + \varepsilon < \left\|x'\right\| < 1\}$$

and vanishes on

$$\{(x', x_N) : |x_N| \leq a - \varepsilon \;\; \text{and} \;\; \left\|x'\right\| < 2\phi_0/3\}.$$

Let $h_1 = h_0 + v_0$, where

$$v_0(x) = a_N^{-1} \int_{B' \times (-a+\varepsilon, a-\varepsilon)} \widetilde{G}_\Omega(x, y) \Delta h_0(y) dy.$$

Then h_1 is harmonic on $B' \times (-a+\varepsilon, a-\varepsilon)$ and vanishes on $\partial B' \times (-a+\varepsilon, a-\varepsilon)$. By Theorem 1 it has a harmonic extension to $\mathbb{R}^{N-1} \times (-a+\varepsilon, a-\varepsilon)$. Further, the support of $(\Delta h_0)|_{B' \times (-a+\varepsilon, a-\varepsilon)}$ is contained in the set where $2\phi_0/3 \leq \|y'\| \leq \phi_0 + \varepsilon \leq 4\phi_0/3$. Thus we can appeal to Lemma 10, with $b = \phi_0 + \varepsilon$, to see from (12) that v_0 has a harmonic extension to

$$\{(x', x_N) : |x_N| \leq a - \varepsilon \;\; \text{and} \;\; \phi_0 + 2\varepsilon < \left\|x'\right\| < 2 - \phi_0 - 2\varepsilon\}.$$

Hence h_0, and so also h, has a harmonic extension to the above set. Since ε can be arbitrarily small, the theorem is proved. ■

The general case of Theorem 2 may now be deduced as follows. Let (z', z_N) be a point of the domain (1) and choose $\phi_0 \in (\phi(z_N), \|z'\|)$. By the upper semicontinuity of ϕ we can find $\varepsilon \in (0, a - |z_N|)$ such that $\phi(t) < \phi_0$ when $|t - z_N| < \varepsilon$. If h is harmonic on the set (1) and vanishes on $\partial B' \times (-a, a)$, then it is harmonic on

$$\left\{(x', x_N) : |x_N - z_N| < \varepsilon \;\; \text{and} \;\; \phi_0 < \left\|x'\right\| < 1\right\}$$

and vanishes on $\partial B' \times (z_N - \varepsilon, z_N + \varepsilon)$. By Theorem 11 and a translation in the x_N-direction, we see that h has a harmonic extension to the domain

$$\left\{(x', x_N) : |x_N - z_N| < \varepsilon \;\; \text{and} \;\; \phi_0 < \left\|x'\right\| < 2 - \phi_0\right\}.$$

Since ϕ_0 can be arbitrarily close to $\phi(z_N)$, we arrive at the desired conclusion.

References

[1] D. H. Armitage, S. J. Gardiner, Classical potential theory. Springer, London, 2001.

[2] Y. Domar, On the existence of a largest subharmonic minorant of a given function, Ark. Mat. 3 (1957) 429–440.

[3] P. Ebenfelt, D. Khavinson, On point to point reflection of harmonic functions across real-analytic hypersurfaces in $\mathbb{R}^n$, J. Anal. Math. 68 (1996) 145–182.

[4] S. J. Gardiner, H. Render, Harmonic functions which vanish on a cylindrical surface, J. Math. Anal. Appl. 433 (2016), 1870-1882.

[5] D. Khavinson, Holomorphic partial differential equations and classical potential theory. Universidad de La Laguna, Departamento de Análisis Matemático, La Laguna, 1996.

[6] A. Laforgia and M. E. Muldoon, Inequalities and approximations for zeros of Bessel functions of small order, SIAM J. Math. Anal. 14 (1983) 383–388.

[7] H. Lewy, On the reflection laws of second order differential equations in two independent variables, Bull. Amer. Math. Soc. 65 (1959), 37–58.

[8] T. Savina, On non-local reflection for elliptic equations of the second order in $\mathbb{R}^2$ (the Dirichlet condition), Trans. Amer. Math. Soc. 364 (2012), 2443–2460.

[9] E. M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean spaces, Princeton, Univ. Press, 1971.

[10] G. Szegö, Orthogonal Polynomials, Fourth Edition, Amer. Math. Soc., Providence, RI, 1975.

[11] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge Univ. Press, 1922.

[12] J. Wimp, D. Colton, Remarks on the representation of zero by solutions of differential equations. Proc. Amer. Math. Soc. 74 (1979), 232–234.