The second homology of SL_2 of S-integers

Kevin Hutchinson

Abstract. We calculate the structure of the finitely generated groups $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/m]), \mathbb{Z})$ when m is a multiple of 6. Furthermore, we show how to construct homology classes, represented by cycles in the bar resolution, which generate these groups and have prescribed orders. When $n \geq 2$ and m is the product of the first n primes, we combine our results with those of Jun Morita to show that the projection $\mathrm{St}(2, \mathbb{Z}[1/m]) \to \mathrm{SL}_2(\mathbb{Z}[1/m])$ is the universal central extension. Our methods have wider applicability: The main result on the structure of the second homology of certain rings is valid for rings of S-integers with sufficiently many units. For a wide class of rings A, we construct explicit homology classes in $\mathrm{H}_2(\mathrm{SL}_2(A), \mathbb{Z})$, functorially dependent on a pair of units, which correspond to symbols in $K_2(2, A)$.

1. Introduction

We calculate the structure of the finitely generated groups $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/m]), \mathbb{Z})$ – the Schur multiplier of $\mathrm{SL}_2(\mathbb{Z}[1/m])$ – when m is a multiple of 6 (Theorem 6.12 below). Furthermore, we show how to construct explicit homology classes, in the bar resolution, which generate these groups and have prescribed orders (sections 7 and 8). Our methods have wider applicability, however: The main result on the structure of the second homology of certain rings is valid for rings of S-integers with sufficiently many units. The homology classes which we construct make sense over any ring in which 6 is a unit.

For a ring A satisfying some finiteness conditions the homology groups $\mathrm{H}_2(\mathrm{SL}_n(A), \mathbb{Z})$ are naturally isomorphic to the K-theory group $K_2(A)$ when n is sufficiently large. However, $n = 2$ is rarely sufficiently large, even when A is a field.

We review some background results (see Milnor [**8**] for details). For a commutative ring A, the *unstable* K_2-groups of the ring A, $K_2(n, A)$, are defined to be the kernel of a surjective homomorphism $\mathrm{St}(n, A) \to E_n(A)$ where $\mathrm{St}(n, A)$ is the rank $n - 1$ Steinberg group of A and where $E_n(A)$ is the subgroup of $\mathrm{SL}_n(A)$ generated by elementary matrices. There are compatible homomorphisms $\mathrm{St}(n, A) \to \mathrm{St}(n + 1, A)$, $E_n(A) \to E_{n+1}(A)$, and taking direct limits as $n \to \infty$, we obtain a surjective map $\mathrm{St}(A) \to E(A)$ whose kernel is $K_2(A) := \lim K_2(n, A)$. In fact, $K_2(A)$ is central in $\mathrm{St}(A)$ and the extension

$$1 \to K_2(A) \to \mathrm{St}(A) \to E(A) \to 1$$

is the universal central extension of $E(A)$ and hence $\mathrm{H}_2(E(A), \mathbb{Z}) \cong K_2(A)$. Furthermore, for a commutative ring A, $E(A) = \mathrm{SL}(A) = \lim \mathrm{SL}_n(A)$.

1991 *Mathematics Subject Classification.* 19G99, 20G10.

Key words and phrases. K-theory, Group Homology .

When A satisfies some reasonable finiteness conditions these statements remain true when $K_2(A)$, $\mathrm{St}(A)$ and $E(A)$ are replaced with $K_2(n, A)$, $\mathrm{St}(n, A)$ and $E_n(A)$ for all sufficiently large n. In particular, when F is a field with at least 10 elements $\mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z}) \cong K_2(2, F)$.

When F is a global field and when S is a nonempty set of primes of F containing the infinite primes, we let $\mathcal{O}_S$ denote the corresponding ring of S-integers. (For example if $F = \mathbb{Q}$ and $1 < m \in \mathbb{Z}$, we have $\mathbb{Z}[1/m] = \mathcal{O}_S$ where S consists of the primes dividing m and the infinite prime.) Now the groups $\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})$ and $K_2(2, \mathcal{O}_S)$ are finitely-generated abelian groups which satisfy

$$\lim_{S} \mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z}) = \mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z}) \text{ and } \lim_{S} K_2(2, \mathcal{O}_S) = K_2(2, F).$$

It is natural to guess that we might have $\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z}) \cong K_2(2, \mathcal{O}_S)$ when S is sufficiently large in some appropriate sense. The example of $\mathcal{O}_S = \mathbb{Z}$, when $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z}) = 0$ while $K_2(2, \mathbb{Z}) \cong \mathbb{Z}$ shows that some condition on S will be required.

In the current paper, rather than comparing $\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})$ to $K_2(2, \mathcal{O}_S)$ directly, we introduce a convenient proxy for $K_2(2, \mathcal{O}_S)$ which we denote $\tilde{K}_2(2, \mathcal{O}_S)$ (see section 6 below for definitions). There are natural maps

$$\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z}) \to \tilde{K}_2(2, \mathcal{O}_S), \quad K_2(2, \mathcal{O}_S) \to \tilde{K}_2(2, \mathcal{O}_S)$$

and the structure of the group $\tilde{K}_2(2, \mathcal{O}_S)$ is easy to describe (see Lemma 6.3):

$$\tilde{K}_2(2, \mathcal{O}_S) \cong K_2(\mathcal{O}_S)_+ \oplus \mathbb{Z}^r$$

where $K_2(\mathcal{O}_S)_+$ is the subgroup of totally positive elements of $K_2(\mathcal{O}_S)$ and r is the number of real embeddings of F.

Our main theorem (6.10) states that when S is sufficiently large (see the statement for more details) that the natural map $\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z}) \to \tilde{K}_2(2, \mathcal{O}_S)$ is an isomorphism. In the case $F = \mathbb{Q}$, the condition that S be sufficiently large reduces to the requirement that $2, 3 \in S$. In particular, when $6|m$, we obtain isomorphisms

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/m]), \mathbb{Z}) \cong \tilde{K}_2(2, \mathbb{Z}[1/m]) \cong \mathbb{Z} \oplus \left(\oplus_{p|m} \mathbb{F}_p^\times \right).$$

Jun Morita ([**13**]) proved isomorphisms of the form

$$K_2(2, \mathbb{Z}[1/m]) \cong \tilde{K}_2(2, \mathbb{Z}[1/m])$$

for certain integers m (eg. if m is the product of the first n prime numbers). Combining Morita's results with those above we deduce that

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/m]), \mathbb{Z}) \cong K_2(2, \mathbb{Z}[1/m])$$

for such m, and that, consequently, the extension

$$1 \to K_2(2, \mathbb{Z}[1/m]) \to \mathrm{St}(2, \mathbb{Z}[1/m]) \to \mathrm{SL}_2(\mathbb{Z}[1/m]) \to 1$$

is a universal central extension.

The main tool we use to prove Theorem 6.10 is the expression of $\mathrm{SL}_2(\mathcal{O}_{S \cup \{\mathfrak{p}\}})$ as an amalgamated product

$$\mathrm{SL}_2(\mathcal{O}_S) *_{\Gamma_0(\mathcal{O}_S, \mathfrak{p})} H(\mathfrak{p})$$

associated to the action of $\mathrm{SL}_2(\mathcal{O}_{S \cup \{\mathfrak{p}\}})$ on the Serre tree corresponding to the discrete valuation of the prime ideal $\mathfrak{p}$. This decomposition gives a Mayer-Vietoris sequence in homology. Analysis of the terms and the maps in low dimension yields, for S sufficiently large, an exact sequence

$$\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z}) \longrightarrow \mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_{S \cup \{\mathfrak{p}\}}), \mathbb{Z}) \xrightarrow{\delta} \mathrm{H}_1(k(\mathfrak{p}), \mathbb{Z}) \to 0.$$

where the map δ is essentially the tame symbol of K-theory (see Theorem 5.17). This analysis requires, in particular, the deep and beautiful theorem of Vaserstein and Liehl ([**21**] and [**5**]) and the solution of the congruence subgroup problem for SL_2 (Serre, [**14**]).

In the later part of the paper, we tackle an old question in K_2-theory; namely, how to write down natural homology classes in $\mathrm{H}_2(\mathrm{SL}_2(A),\mathbb{Z})$, depending functorially on a pair of units $u,v\in A^\times$, which correspond, under the map $\mathrm{H}_2(\mathrm{SL}_2(A),\mathbb{Z})\to K_2(2,A)$ when it exists, to the symbols $c(u,v)\in K_2(2,A)$. The answer to the corresponding question for $\mathrm{H}_2(\mathrm{SL}_3(A),\mathbb{Z})$ and $K_2(3,A)$ is well-known, namely the homology class (in the bar resolution)

$$\Big([\mathrm{diag}(u,u^{-1},1)|\mathrm{diag}(v,1,v^{-1})]-[\mathrm{diag}(v,1,v^{-1})|\mathrm{diag}(u,u^{-1},1)]\Big)\otimes 1$$

corresponds to the symbol $\{u,v\}\in K_2(3,A)$, at least up to sign. There is no such simple expression in the case of $K_2(2,A)$. The symbols $c(u,v)$ are easily and naturally described in terms of the generators of the Steinberg group, but the corresponding natural homology classes, even in the case of a field, have no known simple construction. Since $K_2(2,\mathbb{Z})$ is infinite cyclic with generator $c(-1,-1)$ while $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}),\mathbb{Z})=0$ it follows that there can be no simple universal expression defined over the ring $\mathbb{Z}$. The homology classes, $C(u,v)$, that we construct in section 7 below are not very elegant (though it seems unlikely that they can be greatly improved on). To begin with, the construction of the representing cycles requires the presence of a unit λ such that λ^2-1 is also a unit, although the resulting homology classes can be shown quite generally to be independent of the choice of λ. Furthermore, the representing cycles consist usually of 32 terms and hence are far from simple.

However, the cycles we construct are explicit and functorial for homomorphisms of rings. We prove (Theorem 7.8) that they map to the symbols $c(u,v)\in K_2(2,A)$ when A is a field. We can thus use them to write down provably non-trivial homology classes in $\mathrm{H}_2(\mathrm{SL}_2(A),\mathbb{Z})$ for more general rings A. In particular, in section 8, we use them to write down explicit elements of the groups $\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S),\mathbb{Z})$ with given order and to construct generators of the groups $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/m]),\mathbb{Z})$ when m is divisible by 6.

2. Preliminaries and notation

2.1. Notation. For a Dedekind Domain A with field of fractions F, $\mathrm{Cl}(A)$ denotes the ideal classgroup of A. If $\mathfrak{p}$ is a nonzero prime ideal of A, $v_\mathfrak{p}:F^\times\to\mathbb{Z}$ denotes the corresponding discrete value. For a global field F and a nonempty set of primes S of F we let $\mathcal{O}_S$ denote the ring of S-integers:

$$\mathcal{O}_S:=\{a\in F^\times\mid v_\mathfrak{p}(a)\geq 0 \text{ for all } \mathfrak{p}\notin S\}.$$

For a finite abelian group M, $M_{(p)}$ denotes the Sylow p-subgroup of M.

For a commutative ring A, we let $\mathrm{R}_A:=\mathbb{Z}[A^\times/(A^\times)^2]$ be the group ring of the group of square classes of units. For $a\in A^\times$, the square class of a will be denoted $\langle a\rangle\in\mathrm{R}_A$. Furthermore, the element $\langle a\rangle-1$ in the augmentation ideal, $\mathcal{I}_A\subset\mathrm{R}_A$, will be denoted $\langle\langle a\rangle\rangle$.

2.2. Elementary matrices. We will have occasion to refer to the following facts:

For a commutative ring A, and any $x\in A$ we define the elementary matrices

$$E_{12}(x):=\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}, E_{21}(x):=\begin{bmatrix}1 & 0\\ x & 1\end{bmatrix}\in\mathrm{SL}_2(A).$$

Let $E_2(A)$ be the subgroup of $\mathrm{SL}_2(A)$ generated by $E_{12}(x)$, $E_{21}(y)$, $x,y\in A$.

The following theorem of Vaserstein and Liehl will be essential below. Its proof relies on the resolution of the congruence subgroup problem for SL_2 (see Serre [**14**]).

Theorem 2.1 (Vaserstein [**21**], Liehl [**5**]). *Let K be a global field and let S be a set of places of K of cardinality at least 2 and containing all archimedean places. Let*

$$\mathcal{O}_S := \{x \in K \mid v(x) \geq 0 \text{ for all } v \notin S\}$$

be the ring of S-integers of K. Let I_1 and I_2 be nonzero ideals of $\mathcal{O}_S$. Let

$$\tilde{\Gamma}(I_1, I_2) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathcal{O}_S) \mid b \in I_1, c \in I_2, a-1, b-1 \in I_1I_2 \right\}$$

Then $\tilde{\Gamma}(I_1, I_2)$ is generated by the elementary matrices

$$E_{12}(x), x \in I_1 \text{ and } E_{21}(y), y \in I_2.$$

Proposition 2.2. *Let A be a commutative ring.*

(1) *$E_2(A) = \mathrm{SL}_2(A)$ if A is a field or a Euclidean domain or if $A = \mathcal{O}_S$ is the ring of S-integers in a global field and $|S| \geq 2$.*
(2) *$E_2(A)$ is perfect if there exists $\lambda_1, \ldots, \lambda_n \in A^\times$ and $b_1, \ldots, b_n \in A$ such that $\sum_{i=1}^n b_i(\lambda_i^2 - 1) = 1$ in A.*
In particular, $E_2(A)$ is perfect if there exists $\lambda \in A^\times$ such that $\lambda^2 - 1 \in A^\times$ also.

Proof. (1) This is standard linear algebra in the case of a Euclidean Domain or a field, and the theorem of Vaserstein-Liehl in the case of S-integers.
(2) For $\lambda \in A^\times$, let

$$D(\lambda) := \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in \mathrm{SL}_2(A).$$

Note that $D(\lambda) \in E_2(A)$ since

$$D(\lambda) = w(\lambda)w(-1) \text{ where } w(\lambda) := \begin{bmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{bmatrix} = E_{12}(\lambda)E_{21}(-\lambda^{-1})E_{12}(\lambda).$$

Then

$$D(\lambda)E_{12}(x)D(\lambda)^{-1} = E_{12}(\lambda^2 x)$$

and hence, for any $b \in A$ we have

$$[D(\lambda), E_{12}(bx)] = D(\lambda)E_{12}(bx)D(\lambda)^{-1}E_{12}(-bx) = E_{12}((\lambda^2-1)bx).$$

Thus

$$E_{12}(x) = E_{12}(\sum_i (\lambda_i^2 - 1)b_i x) = \prod_i E_{12}((\lambda_i^2-1)b_i x) = \prod_i [D(\lambda_i), E_{12}(b_i x)].$$

□

Remark 2.3. On the other hand, the groups $E_2(\mathbb{F}_2) = \mathrm{SL}_2(\mathbb{F}_2)$ and $E_2(\mathbb{F}_3) = \mathrm{SL}_2(\mathbb{F}_3)$ are not perfect. It follows that if the ring A admits a homomorphism to $\mathbb{F}_2$ or $\mathbb{F}_3$ then $E_2(A)$ is not perfect. In particular, the group $E_2(\mathbb{Z})$ is not perfect.

Remark 2.4. In [**18**], R. Swan showed that $E_2(A) \neq \mathrm{SL}_2(A)$ for $A = \mathbb{Z}[\sqrt{-5}]$.
Indeed, when A is the ring of integers in a quadratic imaginary number field then $E_2(A) \neq \mathrm{SL}_2(A)$ except in the five cases that A is a Euclidean Domain (see [**21**]).

2.3. Homology of Groups. For any group G, $F_\bullet(G)$ will denote the (right) bar resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$: i.e. for $n \geq 1$, $F_n(G)$ is the free right $\mathbb{Z}[G]$-module with generators $[g_n|\cdots|g_1]$, $\gamma_i \in G$, and $F_0(G) = \mathbb{Z}[G]$ (regarded as a right $\mathbb{Z}[G]$-module). The boundary homomorphism $d_n : F_n(G) \to F_{n-1}(G)$ is given by

$$d_n([g_n|\cdots|g_1]) = [g_n|\cdots|g_2]g_1 + \sum_{i=1}^{n-1}(-1)^{n-i}[g_{n-1}|\cdots|g_{i+1}g_i|\cdots|g_1] + (-1)^n[g_{n-1}|\cdots|g_1]$$

for $n \geq 2$ and $d_1([g]) := g - 1$.

We let $\bar{F}_\bullet(G)$ denote the complex $\{\bar{F}_n(G)\}_{n\geq 0}$ where

$$\bar{F}_n(G) := F_n(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}.$$

Thus $\mathrm{H}_n(G,\mathbb{Z}) \cong H_n(\bar{F}_\bullet(G))$.

We will require the following standard "centre kills" argument from group homology:

Lemma 2.5. *Let G be a group and let M be a $\mathbb{Z}[G]$-module. Suppose that $g \in Z(G)$ has the property that $g - 1$ acts as an automorphism on M. Then $\mathrm{H}_i(G, M) = 0$ for all $i \geq 0$.*

3. The functor $K_2(2, A)$

In this section, we review some of the theory of the functor $K_2(2, A)$ for commutative rings A.

3.1. Definitions. Let A be a commutative ring.

We let $A^\times$ act by automorphisms on $\mathrm{SL}_2(A)$ as follows: Let

$$M(a) := \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2(A).$$

and define

$$a * X := X^{M(a)} = M(a)^{-1}XM(a)$$

for $a \in A^\times$, $X \in \mathrm{SL}_2(A)$.

In particular, we have

$$a * E_{12}(x) = E_{12}(a^{-1}x) \text{ and } a * E_{21}(x) = E_{21}(ax)$$

for all $a \in A^\times$, $x \in A$.

The rank one Steinberg group $\mathrm{St}(2, A)$ is defined by generators and relations as follows: The generators are the terms

$$x_{12}(t) \text{ and } x_{21}(t), \quad t \in A$$

and the defining relations are

(1)
$$x_{ij}(s)x_{ij}(t) = x_{ij}(s + t)$$
for $i \neq j \in \{1, 2\}$ and all $s, t \in A$, and

(2) For $u \in A^\times$, let
$$w_{ij}(u) := x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$$
for $i \neq j \in \{1, 2\}$. Then
$$w_{ij}(u)x_{ij}(t)w_{ij}(-u) = x_{ji}(-u^{-2}t)$$
for all $u \in A^\times$, $t \in A$.

There is a natural surjective homomorphism $\phi : \mathrm{St}(2,A) \to E_2(A)$ defined by $\phi(x_{ij}(t)) = E_{ij}(t)$ for all t. It is easily verified that the formulae

$$a * x_{12}(t) = x_{12}(a^{-1}t) \text{ and } a * x_{21}(t) = x_{21}(at)$$

define an action of $A^\times$ on $\mathrm{St}(2,A)$ by automorphisms. Clearly the homomorphism ϕ is equivariant with respect to this action.

By definition $K_2(2,A)$ is the kernel of ϕ. It inherits an action of $A^\times$.

For $u \in A^\times$ and for $i \neq j \in \{1,2\}$, we let

$$h_{ij}(u) := w_{ij}(u)w_{ij}(-1).$$

Note that

$$\phi(w_{12}(u)) = \begin{bmatrix} 0 & u \\ -u^{-1} & 0 \end{bmatrix} \text{ and } \phi(h_{12}(u)) = \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}.$$

Note that, from the definitions and defining relation (1), for any $a \in A$ and for any unit u we have

$$x_{ij}(a)^{-1} = x_{ij}(-a) \text{ and } w_{ij}(u)^{-1} = w_{ij}(-u).$$

The defining relation (2) above thus immediately gives the following conjugation formula.

Lemma 3.1. *Let A be a commutative ring. Let $a \in A$ and $u \in A^\times$. For $i \neq j \in \{1,2\}$*

$$x_{ij}(a)^{w_{ij}(-u)} = x_{ji}(-u^{-2}a).$$

Since the right-hand-side is unchanged by $u \to -u$, we deduce:

Corollary 3.2. *Let A be a commutative ring. Let $a \in A$ and $u \in A^\times$. For $i \neq j \in \{1,2\}$*

$$x_{ij}(a)^{w_{ij}(u)^{-1}} = x_{ji}(-u^{-2}a) = x_{ij}(a)^{w_{ij}(u)}.$$

and

$$x_{ji}(a)^{w_{ij}(u)} = x_{ij}(-u^2a).$$

Form the definition of $h_{ij}(u)$, we then obtain:

Corollary 3.3. *Let A be a commutative ring. Let $a \in A$ and $u \in A^\times$. For $i \neq j \in \{1,2\}$*

$$x_{ij}(a)^{h_{ij}(u)} = x_{ij}(u^{-2}a) \text{ and } x_{ij}(a)^{h_{ij}(u)^{-1}} = x_{ij}(u^2a).$$

3.2. Symbols. In particular, for $u, v \in A^\times$ the *symbols*

$$c(u,v) := h_{12}(u)h_{12}(v)h_{12}(uv)^{-1}$$

lie in $K_2(2,A)$.

The elements $c(u,v)$ are central in $\mathrm{St}(2,A)$. We let $C(2,A)$ denote the subgroup of $K_2(2,A)$ generated by these symbols.

Note that for $a, u \in A^\times$ we have

$$a * w_{12}(u) = w_{12}(a^{-1}u) \text{ and } a * w_{21}(u) = w_{21}(au)$$

and hence

$$a * h_{12}(u) = h_{12}(a^{-1}u)h_{12}(a^{-1})^{-1} \text{ and } a * h_{21}(u) = h_{21}(au)h_{21}(a)^{-1}.$$

It follows easily that

$$a * c(u,v) = c(u,a^{-1})^{-1}c(u,a^{-1}v).$$

Thus the abelian group $C(2,A)$ is a module over the group ring $\mathbb{Z}[A^\times]$ with this action.

Lemma 3.4. *Let A be a commutative ring. Then*

$$a^2 * c(u,v) = c(u,v)$$

for all $a, u, v \in A^\times$.
In particular, $C(2,A)$ is naturally an R_A-module.

Proof. We have $h_{ij}(u) = h_{ji}(u)^{-1}$ in $\mathrm{St}(2,A)$. Thus

$$c(u,v) = h_{12}(u)h_{12}(v)h_{12}(uv)^{-1} = h_{21}(u)^{-1}h_{21}(v)^{-1}h_{21}(uv).$$

Thus

$$\begin{aligned}
a * c(u,v) &= h_{21}(a)h_{21}(au)^{-1}h_{21}(a)h_{21}(av)^{-1}h_{21}(auv)h_{21}(a)^{-1}\\
&= h_{21}(a)h_{21}(au)^{-1}h_{21}(a)\cdot\left(h_{21}(u)h_{21}(u)^{-1}\right)\cdot h_{21}(av)^{-1}h_{21}(auv)h_{21}(a)^{-1}\\
&= h_{21}(a)c(u,a)^{-1}c(u,av)h_{21}(a)^{-1}\\
&= c(u,a)^{-1}c(u,av)\\
&= a^{-1} * c(u,v).
\end{aligned}$$

□

The symbols $c(u,v)$ satisfy the following properties (see [**6**], or also [**16**])

Proposition 3.5. *Let A be a commutative ring. Then*

(1) $c(u,v) = 1$ *if* $u = 1$ *or* $v = 1$.
(2) $c(u,v) = c(v^{-1},u)$ *for all* $u, v \in A^\times$.
(3) $c(u,vw)c(v,w) = c(uv,w)c(u,v)$ *for all* $u, v, w \in A^\times$.
(4) $c(u,v) = c(u,-uv)$ *for all* $u, v \in A^\times$
(5) $c(u,v) = c(u,(1-u)v)$ *whenever* $u, 1-u, v \in A^\times$.

Remark 3.6. Combining the result of Lemma 3.4 with Proposition 3.5 (3), we see that the square class $\langle a\rangle \in \mathrm{R}_A$ acts on $C(2,A)$ via

$$\langle a\rangle\, c(u,v) = c(u,a)^{-1}c(u,av) = c(au,v)c(a,v)^{-1}.$$

Futhermore Proposition 3.5 (4) is equivalent to

$$\langle v\rangle\, c(u,-u) = 1 \text{ for all } u, v \in A^\times$$

and Proposition 3.5 (5) is equivalent to

$$\langle v\rangle\, c(u,1-u) = 1 \text{ for all } u, v \in A^\times.$$

We will use the following property of symbols ([**6**]):

Lemma 3.7. *If u, v, w are units in A, then*

$$c(u,v^2w) = c(u,v^2)c(u,w)$$

and

$$c(u,v^2) = c(u,v)c(v,u)^{-1} = c(u^2,v).$$

Furthermore, we have the following theorem of Matsumoto and Moore ([**6**],[**10**]):

Theorem 3.8. *Let F be an infinite field. Then*

(1) *The sequence*

$$1 \to K_2(2,F) \to \mathrm{St}(2,F) \to \mathrm{SL}_2(F) \to 1$$

is the universal central extension of the perfect group $\mathrm{SL}_2(F)$.
In particular, $K_2(2,F) \cong \mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z})$ *naturally.*

(2) $K_2(2,F)$ *has the following presentation: It is generated by the symbols* $c(u,v)$, $u,v \in F^\times$, *subject to the five relations of Proposition 3.5.*

3.3. The stabilization homomorphism $K_2(2,F) \to K_2(F)$. For a field F, the Theorem of Matsumoto also gives a presentation of $K_2(n,F)$ for all $n \geq 3$. In particular, it follows that $K_2(F) = K_2^{\mathrm{M}}(F)$, the second Milnor K-group of the field F. The stabilization map $K_2(2,F) \to K_2(F)$ is surjective and sends the symbols $c(u,v)$ to the symbols $\{u,v\}$ of algebraic K-theory.

Let $\mathrm{GW}(F)$ be the Grothendieck-Witt ring of isometry classes of nondegerate quadratic forms over F. It is generated by the classes $\langle a\rangle$ of 1-dimensional forms and the map $\mathrm{R}_F \to \mathrm{GW}(F)$ sending $\langle a\rangle \to \langle a\rangle$ is a surjection of rings. The fundamental ideal $I(F)$ of $\mathrm{GW}(F)$ is the ideal generated by the elements $\langle\langle a\rangle\rangle := \langle a\rangle - 1$.

There is a natural surjective homomorphism of R_F-modules

$$K_2(2,F) \to I^2(F), \quad c(u,v) \mapsto \langle\langle u\rangle\rangle\langle\langle v\rangle\rangle.$$

Furthermore, by a theorem of Milnor ([**9**]) there is also a surjective map $K_2^{\mathrm{M}}(F) \to I^2(F)/I^3(F)$ sending the symbol $\{u,v\}$ to the class of $\langle\langle u\rangle\rangle\langle\langle v\rangle\rangle$. The kernel of this map is precisely $2K_2^{\mathrm{M}}(F)$.

By a result essentially due to Suslin ([**17**], but see also [**7**]) for an infinite field F, we also have the following description of $K_2(2,F)$:

Theorem 3.9. *Let F be an infinite field. The maps $K_2(2,F) \to K_2(F)$, $K_2(2,F) \to I^2(F)$ induce an isomorphism of* R_F*-modules*

$$K_2(2,F) \to K_2^{\mathrm{M}}(F) \times_{I^2(F)/I^3(F)} I^2(F), \quad c(u,v) \mapsto [u,v] := (\{u,v\}, \langle\langle u\rangle\rangle\langle\langle v\rangle\rangle).$$

Corollary 3.10. *Let F be an infinite field. There is a natural short exact sequence of $GW(F)$-modules*

$$0 \to I^3(F) \to K_2(2,F) \to K_2^{\mathrm{M}}(F) \to 0.$$

3.4. Milnor-Witt K-theory. The homology of the special linear group of a field is related to the Milnor-Witt K-theory of the field (see, for example, [**4**]).

Milnor-Witt K-theory of a field F is a $\mathbb{Z}$-graded algebra $K_\bullet^{\mathrm{MW}}(F)$ generated by symbols $[u]$, $u \in F^\times$ in degree 1 and a symbol η in degree -1, satisfying certain relations (see [**11**] for details). It arises naturally as a ring of operations in stable $\mathbb{A}^1$-homotopy theory.

A deep theorem of Morel asserts:

Theorem 3.11. *[[**12**]] There is a natural isomorphism of graded rings*

$$K_\bullet^{\mathrm{MW}}(F) \cong K_\bullet^{\mathrm{M}}(F) \times_{I^\bullet(F)/I^{\bullet+1}(F)} I^\bullet(F).$$

(Here, when $n < 0$, $K_n^{\mathrm{MW}}(F) := 0$ and $I^n(F) := W(F)$, the Witt ring of the field.)

The theorem of Suslin on the structure of $K_2(2,F)$ quoted above, implies

Proposition 3.12. *There is a natural isomorphism $K_2(2,F) \cong K_2^{\mathrm{MW}}(F)$, sending $c(u,v)$ to* $[u][v]$.

4. The map from $\mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z})$ to $K_2(2,F)$

Let A be a commutative ring for which $E_2(A)=\mathrm{SL}_2(A)$ is a perfect group. Suppose further that the group extension

$$1 \longrightarrow K_2(2,A) \longrightarrow \mathrm{St}(2,A) \xrightarrow{\phi} \mathrm{SL}_2(A) \longrightarrow 1$$

is a central extension.

Let $s:\mathrm{SL}_2(A)\to \mathrm{St}(2,A)$ be a section of ϕ. Then there is a corresponding 2-cocycle f_s : $\mathrm{SL}_2(A)\times\mathrm{SL}_2(A)\to K_2(2,A)$ defined by

$$f_s(X,Y) := s(X)s(Y)s(XY)^{-1}.$$

This yields a cohomology class $f\in \mathrm{H}^2(\mathrm{SL}_2(A),K_2(2,A))$ which is independent of the choice of section s.

However, since $\mathrm{H}_1(\mathrm{SL}_2(A),\mathbb{Z})=0$, the universal coefficient theorem tells us that there is a natural isomorphism

$$\mathrm{H}^2(\mathrm{SL}_2(A),K_2(2,A)) \cong \mathrm{Hom}(\mathrm{H}_2(\mathrm{SL}_2(A),\mathbb{Z}),K_2(2,A))$$

described as follows: Let $z\in \mathrm{H}^2(\mathrm{SL}_2(A),K_2(2,A))$ be represented by the 2-cocycle h. Then h induces a homomorphism

$$\bar{F}_2 \to K_2(2,F),\quad \sum_i n_i[X_i|Y_i] \mapsto \prod_i h(X_i,Y_i)^{n_i}$$

which vanishes on boundaries, and thus in turn induces a homomorphism

$$\bar{h}:\mathrm{H}_2(\mathrm{SL}_2(A),\mathbb{Z})\to K_2(2,F).$$

In particular, the cocycle f_s above induces the homomorphism

$$\mathrm{H}_2(\mathrm{SL}_2(A),\mathbb{Z})\to K_2(2,F),\quad \sum_i n_i[X_i|Y_i] \mapsto \prod_i f_s(X_i,Y_i)^{n_i}$$

This homomorphism is an isomorphism precisely when the central extension is universal. In particular, it is an isomorphism when A is an infinite field, by the theorem of Matsumoto-Moore.

We now specialise to the case of a field F.

For our calculations, we will use the following section $s:\mathrm{SL}_2(F)\to\mathrm{St}(2,F)$:

$$s\left(\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right) := \begin{cases} x_{12}(ab)h_{12}(a), & \text{if } c=0,\\ x_{12}(ac^{-1})w_{12}(-c^{-1})x_{12}(dc^{-1}), & \text{if } c\neq 0.\end{cases}$$

Note that, in particular, we have

$$s(E_{ij}(a)) = x_{ij}(a) \text{ and } s(D(u)) = h_{12}(u)$$

when $i\neq j\in\{1,2\}$, $a\in A$ and $u\in F^\times$.

Furthermore, functoriality of the constructions above guarantee that the induced homomorphism

$$\bar{f}:\mathrm{H}_2(\mathrm{SL}_2(F),\mathbb{Z})\to K_2(2,F)$$

is a map of $\mathbb{Z}[F^\times]$-modules. Recall that this homomorphism is induced by the homomorphism

$$\bar{F}_2(\mathrm{SL}_2(F))\to K_2(2,F),\ [X|Y]\mapsto f_s(X,Y)=s(X)s(Y)s(XY)^{-1}.$$

LEMMA 4.1. *Let F be a field. Let $u, v \in F^\times$ and $a, b \in F$. Let*

$$X = \begin{bmatrix} u & a \\ 0 & u^{-1} \end{bmatrix}, \quad Y = \begin{bmatrix} v & b \\ 0 & v^{-1} \end{bmatrix}$$

Then $f_s(X, Y) = c(u, v)$.

PROOF. We have,

$$s(X) = x_{12}(au)h_{12}(u), \quad s(Y) = x_{12}(bv)h_{12}(v) \text{ and } s(XY) = x_{12}(bu^2v + au)h_{12}(uv).$$

Thus

$$\begin{aligned} f(X,Y) &= x_{12}(au)h_{12}(u)x_{12}(bv)h_{12}(v)h_{12}(uv)^{-1}x_{12}(-bu^2v - au) \\ &= x_{12}(au)x_{12}(bv)^{h_{12}(u)^{-1}}h_{12}(u)h_{12}(v)h_{12}(uv)^{-1}x_{12}(-bu^2v - au) \\ &= x_{12}(au)x_{12}(bu^2v)c(u,v)x_{12}(-bu^2v)x_{12}(-au) \quad \text{by Corollary 3.3} \\ &= c(u,v) \quad \text{since } c(u,v) \text{ is central.} \end{aligned}$$

□

COROLLARY 4.2. *Let F be a field. Let $a, b \in F^\times$. Then*

$$([D(a)|D(b)] - [D(b)|D(a)]) \otimes 1 \in F_2(\mathrm{SL}_2(F)) \otimes \mathbb{Z}$$

is a cycle and the corresponding homology class maps to $c(a^2, b)$ under the natural isomorphism $\mathrm{H}_2(\mathrm{SL}_2(k), \mathbb{Z}) \cong K_2(2, F)$ induced by f_s.

PROOF. The first statement in immediate since $D(a)D(b) = D(ab) = D(b)D(a)$. The image of this cycle is

$$f_s(D(a), D(b)) \cdot f_s(D(b), D(a))^{-1} = c(a, b)c(b, a)^{-1} = c(a^2, b)$$

by Lemma 3.7. □

5. The Mayer-Vietoris sequence

Throughout this section A will denote a Dedekind Domain with field of fractions K.

5.1. The groups $H(I)$. We collect together some basic and well-known facts about certain subgroups of $\mathrm{SL}_2(K)$ (see for example [**14**, p. 520]).

Let I be a fractional ideal of A.

We consider the lattice $\Lambda = \Lambda_I := A \oplus I \subset K \oplus K = K^2$.

Let $H(I)$ denote the the subgroup

$$\{M \in \mathrm{SL}_2(K)\,|M \cdot \Lambda = \Lambda\} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(K) \mid a, d \in A, c \in I, d \in I^{-1} \right\} = \tilde{\Gamma}(I, I^{-1}).$$

Note that, in particular, $H(A) = \mathrm{SL}_2(A)$.

We also note that if J is any nonzero fractional ideal of A, then

$$H(I) = \{M \in \mathrm{SL}_2(K)\,|M \cdot (J\Lambda) = J\Lambda\}$$

where

$$J\Lambda = J \cdot (A \oplus I) = J \oplus IJ.$$

LEMMA 5.1. *Let I be a fractional ideal of the A.*

(1) *Suppose that $I' = aI$ where $0 \neq a \in K$. Then $H(I') = H(I)^{M(a)}$ where*

$$M(a) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2(K).$$

(2) *Suppose I is an integral ideal. Let*

$$A' = \{r \in K \mid v_{\mathfrak{q}}(r) \geq 0 \text{ for all } \mathfrak{q} \nmid I\}.$$

Then there exists $M \in \mathrm{SL}_2(A')$ such that $H(I^2) = \mathrm{SL}_2(A)^M$. In particular, $H(I^2) \cong \mathrm{SL}_2(A)$.

Proof.

(1) This follows from the observation that multiplication by $M(a)$ induces an isomorphism of lattices $A \oplus I' \cong a \cdot (A \oplus I)$, and hence conjugation by $M(a)$ induces an isomorphism of the stabilizers.

(2) We first observe that, since $I^{-1} \cdot \Lambda_I = I^{-1} \oplus I$, $H(I^2)$ is the stabilizer of $I^{-1} \oplus I$.

There exists an integral ideal J of A satisfying: $I + J = A$ and $IJ = xA$ for some nonzero $x \in A$. So $J = xI^{-1}$. Thus multiplication by $M(x)$ induces an isomorphism $I^{-1} \oplus I \cong J \oplus I$.

Choose $a \in I, b \in J$ with $a+b = 1$. Consider the short exact sequence of A-modules

$$0 \longrightarrow xA \xrightarrow{f} J \oplus I \xrightarrow{g} A \longrightarrow 0$$

where $g(y) = (y, -y)$ and $f(y,z) = y + z$. There is a splitting $A \to J \oplus I$ given by $y \mapsto (by, ay)$. This gives an isomorphism of A-modules

$$J \oplus I \cong xA \oplus A, \quad (y,z) \mapsto (ay - bz, y + z);$$

i.e.multiplication by

$$N := \begin{bmatrix} a & -b \\ 1 & 1 \end{bmatrix} \in \mathrm{SL}_2(A)$$

induces an isomorphism of lattices $J \oplus I \cong xA \oplus A$.

Now, multiplication by $M(x)^{-1}$ induces an isomorphism $xA \oplus A \cong A \oplus A$.

Putting all of this together, multiplication by

$$M := M(x)^{-1} N M(x) = \begin{bmatrix} a & -b/x \\ x & 1 \end{bmatrix} \in \mathrm{SL}_2(K)$$

induces an isomorphism of lattices $I^{-1} \oplus I \cong A \oplus A$, and thus conjugation by M induces an isomorphism of stabilizers as required.

Finally, we note that since $xA = IJ$ and $bA = JK$ for some integral ideal K, $(b/x)A = KI^{-1}$ and hence $b/x \in A'$. Thus $M \in \mathrm{SL}_2(A')$ as claimed.

□

Corollary 5.2. *Let I be a fractional ideal of A. Suppose that the class of I in $\mathrm{Cl}(A)$ is a square. Then $H(I) \cong \mathrm{SL}_2(A)$.*

Remark 5.3. In particular, the ideal $\tilde{\mathfrak{p}} := \mathfrak{p}A_{\mathfrak{p}}$ in $A_{\mathfrak{p}}$ is a principal ideal with generator π, say. It follows from Lemma 5.1 that

$$H(\tilde{\mathfrak{p}}) = \mathrm{SL}_2(A_{\mathfrak{p}})^{M(\pi)}.$$

Let $\mathfrak{p}$ be a nonzero prime ideal of A. Let $n \geq 1$ and let $\pi \in A$ satisfy $v_{\mathfrak{p}}(\pi) = 1$. We let $\gamma_{\pi,n} : H(\mathfrak{p}) \to \mathrm{SL}_2(A/\mathfrak{p}^n)$ be the composite

$$H(\mathfrak{p}) \longrightarrow H(\tilde{\mathfrak{p}}) \xrightarrow[\mathrm{conj}_{M(1/\pi)}]{\cong} \mathrm{SL}_2(A_{\mathfrak{p}}) \longrightarrow \mathrm{SL}_2(A_{\mathfrak{p}}/\tilde{\mathfrak{p}}^n) \xrightarrow{\cong} \mathrm{SL}_2(A/\mathfrak{p}^n)$$

Lemma 5.4. *The map $\gamma_{\pi,n}$ is surjective for all n and the kernel of this map is independent of the choice of π.*

Proof. By definition, we have

$$\gamma_{\pi,n}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \bar{a} & \overline{\pi b} \\ \overline{c/\pi} & \bar{d} \end{bmatrix}$$

where

$$\bar{x} := x + \tilde{\mathfrak{p}}^n \in A_{\mathfrak{p}}/\tilde{\mathfrak{p}}^n \cong A/\mathfrak{p}^n.$$

Since $\mathrm{SL}_2(A/\mathfrak{p}^n)$ is generated by elementary matrices, we need only show how to lift these. We begin by observing that $\pi A = \mathfrak{p}J$ where J is an ideal not contained in $\mathfrak{p}$. It follows that $A = \mathfrak{p}^n + J$ for any $n \geq 1$; i.e. the map $J \to A/\mathfrak{p}^n$ is surjective.

Thus, given any $x \in A$ there exists $x' \in J$ with $\bar{x'} = \bar{x}$. Since $x' \in J$ it follows that $x'/\pi \in J \cdot (\mathfrak{p}J)^{-1} = \mathfrak{p}^{-1}$. Hence $E_{12}(x'/\pi) \in H(\mathfrak{p})$ and

$$\gamma_{\pi,n}(E_{12}(x'/\pi)) = E_{12}(\bar{x'}) = E_{12}(\bar{x}).$$

Of course, we also have $E_{21}(\pi x) \in H(\mathfrak{p})$ and $\gamma_{\pi,n}(E_{21}(\pi x)) = E_{21}(\bar{x})$. This proves the surjectivity statement.

For the second part, suppose that $\pi' \in A$ also satisfies $v_{\mathfrak{p}}(\pi') = 1$. Then $\pi' = \pi \cdot u$ for some $u \in A_{\mathfrak{p}}^{\times}$. From the definition, we have

$$\gamma_{\pi',n} = f \circ \gamma_{\pi,n}$$

where f is conjugation by $M(\bar{u}^{-1})$ on $\mathrm{SL}_2(A/\mathfrak{p}^n)$. It follows at once that $\mathrm{Ker}(\gamma_{\pi',n}) = \mathrm{Ker}(\gamma_{\pi,n})$ as claimed. □

We let $\tilde{\Gamma}(A, \mathfrak{p}^n)$ denote the kernel of the $\gamma_{\pi,n}$ (for any choice of π). Thus, for all $n \geq 1$, there is a short exact sequence

$$1 \to \tilde{\Gamma}(A, \mathfrak{p}^n) \to H(\mathfrak{p}) \to \mathrm{SL}_2(A/\mathfrak{p}^n) \to 1.$$

Note that

$$\tilde{\Gamma}(A, \mathfrak{p}^n) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H(\mathfrak{p}) \mid a-1, d-1 \in \mathfrak{p}^n, c \in \mathfrak{p}^{n+1}, b \in \mathfrak{p}^{n-1}\right\}.$$

In particular, for all $n \geq 1$ we have

$$\Gamma(A, \mathfrak{p}^{n+1}) \subset \tilde{\Gamma}(A, \mathfrak{p}^n) \subset \Gamma_0(A, \mathfrak{p}^n) \subset \mathrm{SL}_2(A).$$

For a field F, we will use the notation

$$B(F) := \left\{\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \in \mathrm{SL}_2(F)\right\} \text{ and } B'(F) := \left\{\begin{bmatrix} a & 0 \\ c & a^{-1} \end{bmatrix} \in \mathrm{SL}_2(F)\right\}.$$

Of course, these two subgroups of $\mathrm{SL}_2(F)$ are naturally isomorphic.

We will need the following result below.

Lemma 5.5. *There is a natural short exact sequence*

$$1 \to \tilde{\Gamma}(A, \mathfrak{p}) \to \Gamma_0(A, \mathfrak{p}) \to B'(k(\mathfrak{p})) \to 1.$$

Proof. This is immediate from the fact that the image of $\Gamma_0(A,\mathfrak{p})$ in $\mathrm{SL}_2(A/\mathfrak{p}) = \mathrm{SL}_2(k(\mathfrak{p}))$ under the map $\gamma_{\pi,1}$ is precisely $B'(k(\mathfrak{p}))$. □

5.2. The Mayer-Vietoris sequence. Let $\mathfrak{p}$ be a nonzero prime ideal of A and let $v = v_\mathfrak{p}$ be the associated discrete valuation. We let $k(\mathfrak{p})$ or $k(v)$ denote the residue field $A/\mathfrak{p}$. We will further suppose that the class of $\mathfrak{p}$ has finite order in $\mathrm{Cl}(A)$. Thus $\mathfrak{p}^n = xA$ for some $n \geq 1$ and $x \in A$. (This condition is automatically satisfied when K is a global field.)

Let

$$\Gamma(A,\mathfrak{p}) := \mathrm{Ker}(\mathrm{SL}_2(A) \to \mathrm{SL}_2(k(\pi))) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(A) \ : \ 1-a, 1-d, b, c \in \mathfrak{p} \right\}$$

and let

$$\Gamma_0(A,\mathfrak{p}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(A) \ : \ c \in \mathfrak{p} \right\}.$$

We let $\tilde{\mathfrak{p}}$ denote the extension of $\mathfrak{p}$ to the localization $A_\mathfrak{p}$, which is thus a discrete valuation ring with unique (principal) nonzero prime ideal $\tilde{\mathfrak{p}}$.

The action of $\mathrm{SL}_2(K)$ on the Serre tree associated to the valuation v ([**15**, Chapter II]) yields a decomposition

$$\mathrm{SL}_2(K) = \mathrm{SL}_2(A_\mathfrak{p}) \star_{\Gamma_0(A_\mathfrak{p},\tilde{\mathfrak{p}})} H(\tilde{\mathfrak{p}}) \tag{1}$$

of $\mathrm{SL}_2(K)$ as the sum of $\mathrm{SL}_2(A_\mathfrak{p})$ and $H(\tilde{\mathfrak{p}})$ amalgamated along their intersection
$\mathrm{SL}_2(A_\mathfrak{p}) \cap H(\tilde{\mathfrak{p}}) = \Gamma_0(A_\mathfrak{p},\tilde{\mathfrak{p}})$.

Let

$$A' := \{a \in K \mid v_\mathfrak{q}(a) \geq 0 \text{ for all prime ideals } \mathfrak{q} \neq \mathfrak{p}\}.$$

Note that since $\mathfrak{p}^n = xA$ by assumption, $A' = A[1/x]$.

Since $A[1/x]$ is dense in K in the $\mathfrak{p}$-adic topology, and since

$$\mathrm{SL}_2(A[1/x]) \cap \mathrm{SL}_2(A_\mathfrak{p}) = \mathrm{SL}_2(A), \quad \mathrm{SL}_2(A[1/x]) \cap H(\tilde{\mathfrak{p}}) = H(\mathfrak{p})$$

there is also an induced decomposition

$$\mathrm{SL}_2(A[1/x]) = \mathrm{SL}_2(A) \star_{\Gamma_0(A,\mathfrak{p})} H(\mathfrak{p}).$$

For convenience, in the remainder of this section we will set

$$G := \mathrm{SL}_2(A[1/x]),\ G_1 := \mathrm{SL}_2(A),\ G_2 := H(\mathfrak{p}) \text{ and } \Gamma_0 := \Gamma_0(A,\mathfrak{p}).$$

Thus $G = G_1 \star_{\Gamma_0} G_2$ and this decomposition gives rise a short exact sequence of $\mathbb{Z}[G]$-modules:

$$0 \longrightarrow \mathbb{Z}[G/\Gamma_0] \xrightarrow{\alpha} \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2] \xrightarrow{\beta} \mathbb{Z} \longrightarrow 0.$$

where α is the map

$$\alpha : \mathbb{Z}[G/\Gamma_0] \to \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2],\ g\Gamma_0 \mapsto (gG_1, gG_2)$$

and β is the unique $\mathbb{Z}[G]$-homomorphism

$$\beta : \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2] \to \mathbb{Z},\ (G_1, 0) \mapsto -1, (0, G_2) \mapsto 1.$$

This short exact sequence of $\mathbb{Z}[G]$-modules gives rise to a long exact sequence in homology. Combining this with the isomorphisms of Shapiro's lemma, $\mathrm{H}_r(G,\mathbb{Z}[G/H]) \cong \mathrm{H}_r(H,\mathbb{Z})$, gives us the *Mayer-Vietoris exact sequence* of the amalgamated product:

$$\cdots \xrightarrow{\delta} \mathrm{H}_r(\Gamma_0,\mathbb{Z}) \xrightarrow{\alpha} \mathrm{H}_r(G_1,\mathbb{Z}) \oplus \mathrm{H}_r(G_2,\mathbb{Z}) \xrightarrow{\beta} \mathrm{H}_r(G,\mathbb{Z}) \xrightarrow{\delta} \cdots$$

The maps α and β in this sequence can be described as follows: Let $\iota_1 : \Gamma_0 \to G_1$ and $\iota_2 : \Gamma_0 \to G_2$ be the natural inclusions. Then

$$\alpha(z) = (\iota_1(z), \iota_2(z)) \text{ for all } z \in \mathrm{H}_r(\Gamma_0, \mathbb{Z}).$$

Likewise, let $j_1 : G_1 \to G$ and $j_2 : G_2 \to G$ be the natural inclusions. Then

$$\beta(z_1, z_2) = j_2(z_2) - j_1(z_1) \text{ for all } z_1 \in \mathrm{H}_r(G_1, \mathbb{Z}),\ z_1 \in \mathrm{H}_r(G_1, \mathbb{Z}).$$

The amalgamated product decomposition (1) – i.e. taking the case $A = A_{\mathfrak{p}}$ – also gives rise to a Mayer-Vietoris sequence

$$\cdots \xrightarrow{\delta} \mathrm{H}_r(\Gamma_0(A_{\mathfrak{p}}, \tilde{\mathfrak{p}}), \mathbb{Z}) \xrightarrow{\alpha} \mathrm{H}_r(\mathrm{SL}_2(A_{\mathfrak{p}}), \mathbb{Z}) \oplus \mathrm{H}_r(H(\tilde{\mathfrak{p}}), \mathbb{Z}) \xrightarrow{\beta} \mathrm{H}_r(\mathrm{SL}_2(K), \mathbb{Z}) \xrightarrow{\delta} \cdots$$

5.3. The connecting homomorphism. As above, let $\mathfrak{p}$ be a prime ideal of the Dedekind Domain A and let

$$\delta : \mathrm{H}_2(\mathrm{SL}_2(K), \mathbb{Z}) \to \mathrm{H}_1(\Gamma_0(A_{\mathfrak{p}}, \tilde{\mathfrak{p}}), \mathbb{Z})$$

be the connecting homomorphism in the Mayer-Vietoris sequence associated to the decomposition

$$\mathrm{SL}_2(K) = \mathrm{SL}_2(A_{\mathfrak{p}}) \star_{\Gamma_0(A_{\mathfrak{p}}, \tilde{\mathfrak{p}})} H(\tilde{\mathfrak{p}}) = \mathrm{SL}_2(A_{\mathfrak{p}}) \star_{\Gamma_0(A_{\mathfrak{p}}, \tilde{\mathfrak{p}})} \mathrm{SL}_2(A_{\mathfrak{p}})^{M(\pi)}.$$

Proposition 5.6. *Let $\rho : \Gamma_0(A_{\mathfrak{p}}, \tilde{\mathfrak{p}}) \to k(\tilde{\mathfrak{p}})^\times$ be the (surjective) map*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto a \pmod{\tilde{\mathfrak{p}}}.$$

Then the composite homomorphism, Δ say,

$$K_2(2, K) \cong \mathrm{H}_2(\mathrm{SL}_2(K), \mathbb{Z}) \xrightarrow{\delta} \mathrm{H}_1(\Gamma_0(A_{\mathfrak{p}}, \tilde{\mathfrak{p}}), \mathbb{Z}) \xrightarrow{\rho} \mathrm{H}_1(k(\tilde{\mathfrak{p}})^\times, \mathbb{Z}) \cong k(\tilde{\mathfrak{p}})^\times$$

is the map

$$c(a, b) \mapsto (-1)^{v(a)v(b)} \frac{b^{v(a)}}{a^{v(b)}} \pmod{\tilde{\mathfrak{p}}}.$$

Remark 5.7. In fact, the isomorphisms in the statement of Proposition 5.6 are canonical only up to sign. We have made our choices so that the sign is $+1$; but the choice of sign does not materially affect our main results.

Before proving Proposition 5.6, we require

Lemma 5.8. *Let K be a field with discrete valuation v. Then $K_2(2, K)$ is generated by the set $C_v := \{c(x, u) \mid v(u) = 0, v(x) = 1\}$.*

Proof. Let D be the subgroup of $K_2(2, K)$ generated by C_v. Let $a, b \in K^\times$. We must prove that $c(a, b) \in D$.

Since

$$c(a, b) = c(b^{-1}, a) = c(a^{-1}, b^{-1}) = c(b, a^{-1})$$

we can assume that $v(a), v(b) \geq 0$.

We will prove the result by induction on $n = v(a) + v(b) \geq 0$.

If $n = 0$, then $v(a) = v(b) = 0$ and choosing $\pi \in K^\times$ with $v(\pi) = 1$ we have

$$c(a, b) = c(\pi a, b)^{-1} c(\pi a, b) c(\pi, a) \in D.$$

On the other hand, suppose that $v(a), v(b) > 0$. If $0 < v(b) \leq v(a)$ then $a = bc$ with $0 \leq v(c) < v(a)$ and hence

$$c(a, b) = c(bc, b) = c(-c, b) \in D$$

by the inductive hypothesis. An analogous argument applies to the case $0 < v(a) < v(b)$.

Since $c(a,b) = c(b^{-1},a)$, we can reduce to the case where $v(b) = 0$ and $v(a) \geq 2$. Then let $a = a'\pi$ where $v(\pi) = 1$ and $1 \leq v(a') < v(a)$. We have

$$c(a,b) = c(a'\pi,b) = c(a',\pi b)c(\pi,b)c(a',\pi)^{-1}$$

which lies in D by the induction hypothesis (using the argument for the case $v(a), v(b) > 0$ for the first term). □

Proof of Proposition 5.6. By Lemma 5.8, we must prove that

$$\Delta(c(x,u)) = u \pmod{\tilde{\mathfrak{p}}}$$

whenever $v(u) = 0$, $v(x) = 1$.

We note that it is enough to prove that $\Delta(c(x,u^2)) = u^2 \pmod{\tilde{\mathfrak{p}}}$ whenever $v(u) = 0$, $v(x) = 1$. For if $u \in K$ is not a square, choose an extension v' of v to $K' := K(\sqrt{u})$. Then there is a natural map of Mayer Vietoris exact sequences inducing a commutative square

$$\begin{array}{ccc} \mathrm{H}_2(\mathrm{SL}_2(K),\mathbb{Z}) & \xrightarrow{\delta} & k(v)^\times \\ \downarrow & & \updownarrow i \\ \mathrm{H}_2(\mathrm{SL}_2(K'),\mathbb{Z}) & \xrightarrow{\delta'} & k(v')^\times \end{array}$$

so that $i(\Delta(c(x,u)) = \Delta'(c(x,u)) = \bar{u} \in k(v')^\times$ since u is a square in K', and thus $\Delta(c(x,u)) = \bar{u} \in k(v)^\times$.

Now, by Corollary 4.2, the symbol $c(x,u^2) \in K(2,K)$ corresponds to the homology class represented by the cycle

$$Z := ([D(x)|D(u)] - [D(u)|D(x)]) \otimes 1 \in F_2(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$$

where $G = \mathrm{SL}_2(K)$.

Recall that the Mayer-Vietoris sequence is the long exact homology sequence derived from the short exact sequence of complexes

$$0 \to F_\bullet(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/\Gamma_0] \to F_\bullet(G) \otimes_{\mathbb{Z}[G]} (\mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2]) \to F_\bullet(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \to 0.$$

Now the cycle Z lifts to

$$([D(x)|D(u)] - [D(u)|D(x)]) \otimes (1 \cdot G_1, 0) \in F_2(G) \otimes (\mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2]).$$

Under the boundary map d_2, this is sent to

$$[D(u)] \otimes (D(x) \cdot G_1 - 1 \cdot G_1, 0) \in F_1(G) \otimes (\mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2])$$

since $D(u) \in \Gamma_0 \subset G_1$.

Now let

$$w := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in G_1.$$

Then

$$w \cdot D(x) = w^{M(x)} \in G_2.$$

Thus $(D(x) \cdot G_1 - 1 \cdot G_1, 0)$ is the image of

$$w^{-1} \cdot (w^{M(x)}\Gamma_0 - \Gamma_0) = D(x)\Gamma_0 - w^{-1}\Gamma_0$$

under the map $\alpha : \mathbb{Z}[G/\Gamma_0] \to \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2]$.

Thus the homology class $\delta(Z) \in \mathrm{H}_1(\Gamma_0, \mathbb{Z})$ is represented by the cycle

$$[D(u)] \otimes (D(x)\Gamma_0 - w^{-1}\Gamma_0) = \Big([D(u)]D(x) - [D(u)]w^{-1}\Big) \otimes \Gamma_0 \in F_1(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/\Gamma_0].$$

This, in turn, is the image of

$$\Big([D(u)]D(x) - [D(u)]w^{-1}\Big) \otimes 1 \in F_1(G) \otimes_{\mathbb{Z}[\Gamma_0]} \mathbb{Z}$$

under the natural isomorphism

$$F_\bullet(G) \otimes_{\mathbb{Z}[\Gamma_0]} \mathbb{Z} \cong F_\bullet(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/\Gamma_0].$$

For a group H we let $C_\bullet(H)$ denote the right homogeneous resolution of H. The isomorphism $F_\bullet(H) \to C_\bullet(H)$ of complexes of right $\mathbb{Z}[H]$-modules is given by

$$[h_n| \cdots |h_1] \mapsto (h_n \cdot h_{n-1} \cdots h_1, \ldots, h_1, 1).$$

Thus the cycle $\Big([D(u)]D(x) - [D(u)]w^{-1}\Big) \otimes 1 \in F_1(G) \otimes_{\mathbb{Z}[\Gamma_0]} \mathbb{Z}$ corresponds to the cycle

$$\Big((D(ux), D(u)) - (D(u)w^{-1}, w^{-1})\Big) \otimes 1 \in C_1(G) \otimes_{\mathbb{Z}[\Gamma_0]} \mathbb{Z}.$$

To construct an augmentation-preserving map of $\mathbb{Z}[\Gamma_0]$-resolutions from $C_\bullet(G)$ to $C_\bullet(\Gamma_0)$, we choose any set-theoretic section $s : G/\Gamma_0 \to G$ of the natural surjection $G \to G/\Gamma_0, g \mapsto g\Gamma_0$. For $g \in G$ we let $\bar{g} := s(g\Gamma_0)^{-1}g \in \Gamma_0$. Then the map

$$\tau : C_\bullet(G) \to C_\bullet(\Gamma_0), (g_n, \ldots, g_0) \mapsto (\bar{g}_n, \ldots, \bar{g}_0)$$

is an augmentation preserving map of $\mathbb{Z}[\Gamma_0]$-complexes.

We further specify that we the section s satisfies

$$s(D(u)w^{-1}\Gamma_0) = w^{-1} \text{ and } s(D(x)\Gamma_0) = D(x)$$

for all u with $v(u) = 0$. Then

$$\tau\Big((D(ux), D(u)) - (D(u)w^{-1}, w^{-1})\Big) = (D(u), 1) - (D(u^{-1}), 1) \in C_1(\Gamma_0)$$

since $wD(u)w^{-1} = D(u^{-1})$ in G.

Finally, the homology class

$$\Big((D(u), 1) - (D(u^{-1}), 1)\Big) \otimes 1 \in C_1(\Gamma_0) \otimes_{\mathbb{Z}[\Gamma_0]} \mathbb{Z}$$

corresponds to the element

$$D(u) \cdot D(u^{-1})^{-1} = D(u^2) \in \Gamma_0/[\Gamma_0, \Gamma_0]$$

under the isomorphism $\mathrm{H}_1(\Gamma_0, \mathbb{Z}) \cong \Gamma_0/[\Gamma_0, \Gamma_0]$, and hence maps to $u^2 \pmod{\tilde{\mathfrak{p}}} \in k(\tilde{\mathfrak{p}})^\times$ under the map ρ. □

5.4. The abelianization of some congruence subgroups.

Proposition 5.9. *Let A be a ring of S-integers in a global field K. Suppose that $|S| \geq 2$ and that there exists $\lambda \in A^\times$ such that $\lambda^2 - 1 \in A^\times$ also. Let $\mathfrak{p}$ be a nonzero prime ideal of A.*

Then the map $\rho : \Gamma_0(A, \mathfrak{p}) \to k(\mathfrak{p})^\times$ induces an isomorphism

$$\mathrm{H}_1(\Gamma_0(A, \mathfrak{p}), \mathbb{Z}) \cong k(\mathfrak{p})^\times.$$

Proof. The map ρ induces a short exact sequence

$$1 \longrightarrow \Gamma_1(A,\mathfrak{p}) \longrightarrow \Gamma_0(A,\mathfrak{p}) \xrightarrow{\rho} k(\mathfrak{p})^\times \longrightarrow 1$$

where

$$\Gamma_1(A,\mathfrak{p}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(A) \mid a-1, d-1 \in \mathfrak{p} \right\} = \tilde{\Gamma}(A,\mathfrak{p})$$

in the notation of Theorem 2.1.

Since $k(\mathfrak{p})^\times$ is an abelian group, it follows that

$$[\Gamma_0(A,\mathfrak{p}), \Gamma_0(A,\mathfrak{p})] \subset \Gamma_1(A,\mathfrak{p})$$

On the other hand, by Theorem 2.1, $\Gamma_1(A,\mathfrak{p}) = \tilde{\Gamma}(A,\mathfrak{p})$ is generated by elementary matrices $E_{12}(x), x \in A$, $E_{21}(y), y \in \mathfrak{p}$. However,

$$E_{12}(x) = [D(\lambda), E_{12}(x/(\lambda^2-1))], E_{21}(y) = [D(\lambda), E_{21}(y/(\lambda^2-1)) \in [\Gamma_0(A,\mathfrak{p}), \Gamma_0(A,\mathfrak{p})].$$

So $[\Gamma_0(A,\mathfrak{p}), \Gamma_0(A,\mathfrak{p})] = \Gamma_1(A,\mathfrak{p})$ as required. □

For a field k we let $\mathrm{sl}_2(k)$ denote the 3-dimensional vector space of 2×2 trace zero matrices.

Lemma 5.10. *Let A be a Dedekind domain and let $\mathfrak{p}$ be a maximal ideal. Then for any $m \geq 1$ there are natural isomorphisms of groups*

$$\frac{\tilde{\Gamma}(A,\mathfrak{p}^n)}{\tilde{\Gamma}(A,\mathfrak{p}^{n+1})} \cong \mathrm{sl}_2(k(\mathfrak{p})) \cong \frac{\Gamma(A,\mathfrak{p}^n)}{\Gamma(A,\mathfrak{p}^{n+1})}$$

Proof. From the definitions of $\Gamma(A,\mathfrak{p}^n)$ and $\tilde{\Gamma}(A,\mathfrak{p}^n)$ we have

$$\frac{\tilde{\Gamma}(A,\mathfrak{p}^n)}{\tilde{\Gamma}(A,\mathfrak{p}^{n+1})} \cong \mathrm{Ker}(\mathrm{SL}_2(A/\mathfrak{p}^{n+1}) \to \mathrm{SL}_2(A/\mathfrak{p}^n)) \cong \frac{\Gamma(A,\mathfrak{p}^n)}{\Gamma(A,\mathfrak{p}^{n+1})}.$$

Let $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$. For any $n \geq 1$, the group $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ is a 1-dimensional $k(\mathfrak{p})$-vector spaces with basis $\{\pi^n + \mathfrak{p}^{n+1}\}$.

The required isomorphism

$$\mathrm{sl}_2(k(\mathfrak{p})) \to \mathrm{Ker}(\mathrm{SL}_2(A/\mathfrak{p}^{n+1}) \to \mathrm{SL}_2(A/\mathfrak{p}^n))$$

is then the map

$$\begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix} \mapsto \begin{bmatrix} 1+a\pi^n & b\pi^n \\ c\pi^n & 1+d\pi^n \end{bmatrix}$$

where $a,b,c,d \in A$ map to $\bar{a},\bar{b},\bar{c},\bar{d} \in k(\mathfrak{p})$. □

Corollary 5.11. *Let A be a Dedekind domain and let $\mathfrak{p}$ be a maximal ideal. Suppose that $k(\mathfrak{p})$ is a finite field with q elements. Then*

$$[\tilde{\Gamma}(A,\mathfrak{p}) : \tilde{\Gamma}(A,\mathfrak{p}^n)] = q^{3(n-1)} = [\Gamma(A,\mathfrak{p}) : \Gamma(A,\mathfrak{p}^n)]$$

for all $n \geq 1$.

Lemma 5.12. *Suppose that I and J are comaximal ideals in A; i.e. $I+J = A$. Then the composite map $\Gamma(A,I) \to \mathrm{SL}_2(A) \to \mathrm{SL}_2(A/J)$ is surjective.*

Proof. By the Chinese Remainder Theorem the map $A/IJ \to (A/I)\times(A/J)$, $a \mapsto (a+I, a+J)$ is an isomorphism of rings. It follows that the map

$$\mathrm{SL}_2(A/IJ) \to \mathrm{SL}_2(A/I)\times\mathrm{SL}_2(A/J), \quad X \mod IJ \mapsto (X \mod I, X \mod J)$$

is an isomorphism of groups and hence that

$$\mathrm{SL}_2(A) \to \mathrm{SL}_2(A/I)\times\mathrm{SL}_2(A/J), \quad X \mapsto (X \mod I, X \mod J)$$

is a surjective group homomorphism. This implies the statement of the Lemma. □

Lemma 5.13. *Suppose that $k(\mathfrak{p})$ is a finite field with q elements. We have*

$$[\mathrm{SL}_2(A) : \Gamma(A,\mathfrak{p})] = q(q^2-1) = [\mathrm{SL}_2(A) : \tilde{\Gamma}(A,\mathfrak{p})]$$

Proof. The first equality follows from the isomorphism

$$\frac{\mathrm{SL}_2(A)}{\Gamma(A,\mathfrak{p})} \cong \mathrm{SL}_2(k(\mathfrak{p})).$$

For the second inequality, denote by C the image of the map

$$\tilde{\Gamma}(A,\mathfrak{p}) \to \mathrm{SL}_2(A) \to \mathrm{SL}_2(A/\mathfrak{p}^2).$$

Then C fits into a short exact sequence

$$1 \to W \to C \to T \to 1$$

where

$$T = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in \mathrm{SL}_2(A/\mathfrak{p}^2) \mid a-1, d-1 \in \mathfrak{p} \right\} \cong k(\mathfrak{p})$$

and

$$W = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \in \mathrm{SL}_2(A/\mathfrak{p}^2) \right\} \cong A/\mathfrak{p}^2.$$

It follows that $|C| = [\tilde{\Gamma}(A,\mathfrak{p}) : \Gamma(A,\mathfrak{p}^2)] = q^3$. Since $[\mathrm{SL}_2(A) : \Gamma(A,\mathfrak{p}^2)] = \left|\mathrm{SL}_2(A/\mathfrak{p}^2)\right| = q^4(q^2-1)$, the second equality follows. □

Proposition 5.14. *Let A be a ring of S-integers in a global field K where $|S| \geq 2$. Let $\mathfrak{p}$ be a nonzero prime ideal and let $p > 0$ be the characteristic of the residue field $k(\mathfrak{p})$.*

Suppose that $\mathfrak{p}^n = xA$ for some $n \geq 1$ and $x \in A$. Suppose further that there exist $\lambda \in A[1/x]^\times$ such that $\lambda^2 - 1 \in A[1/x]^\times$ also. Then $\mathrm{H}_1(\Gamma(A,\mathfrak{p}),\mathbb{Z})$ and $\mathrm{H}_1(\tilde{\Gamma}(A,\mathfrak{p}),\mathbb{Z})$ are finite abelian p-groups.

Proof. Let Γ denote either $\Gamma(A,\mathfrak{p})$ or $\Gamma(A,\mathfrak{p})$. By Proposition 2 of [**14**] the commutator subgroup $[\Gamma,\Gamma]$ contains a principal congruence subgroup $\Gamma(A,I)$ for some ideal I of A. There exists a nonzero ideal J of A such that I factors as $\mathfrak{p}^m J$ where $J \not\subset \mathfrak{p}$ and $m \geq 1$. Since $\Gamma(A,\mathfrak{p}^{m+1}J) \subset \Gamma(A,\mathfrak{p}^m J)$, we can suppose without loss that $m \geq 2$.

By definition, $\Gamma(A,I)$ is the kernel of the natural map $\mathrm{SL}_2(A) \to \mathrm{SL}_2(A/I)$. This map is surjective since $\mathrm{SL}_2(A/I) = E_2(A/I)$.

Since $J \not\subset \mathfrak{p}$, it follows that $A = \mathfrak{p}^m + J$ and hence, by Lemma 5.12, the map $\Gamma(A,\mathfrak{p}^m) \to \mathrm{SL}_2(A/J)$ is surjective. Since $\Gamma(A,\mathfrak{p}^m) \subset \Gamma$, it follows that the map $\Gamma \to \mathrm{SL}_2(A/J)$ is surjective.

However, since $\mathrm{SL}_2(A/J) = E_2(A/J)$ and since $x + J$ is a unit in A/J the hypotheses of the proposition ensure that all elementary matrices are commutators and hence that $\mathrm{SL}_2(A/J)$ is a perfect group. It then follows that the natural map $[\Gamma,\Gamma] \to \mathrm{SL}_2(A/J)$ is surjective.

Thus $[\Gamma,\Gamma]/\Gamma(A,I)$ surjects onto $\mathrm{SL}_2(A/J)$ and hence

$$|\mathrm{SL}_2(A/J)| \text{ divides } [[\Gamma,\Gamma]:\Gamma(A,I)]].$$

It follows that

$$[\mathrm{SL}_2(A):[\Gamma,\Gamma]]\,|\mathrm{SL}_2(A/\mathfrak{p}^m)| = (q^2-1)q^{3m-2}.$$

Since $[\mathrm{SL}_2(A):\Gamma] = q(q^2-1)$ by Lemma 5.13 it follows that $|\Gamma/[\Gamma,\Gamma]|$ divides q^{3m-3}, and so is a power of p as claimed. □

5.5. The second homology of congruence subgroups.

Lemma 5.15. *Let k be a finite field of characteristic p and let M be an $\mathrm{SL}_2(k)$-module. Then, for all $i \geq 0$, the natural map*

$$\mathrm{H}_i(B(k), M)_{(p)} \to \mathrm{H}_i(\mathrm{SL}_2(k), M)_{(p)}$$

is an isomorphism.

Proof. As in the proof of [**3**, Cor 3.10.2]. □

Proposition 5.16. *Let A be a ring of S-integers in a global field K where $|S| \geq 2$. Let $\mathfrak{p}$ be a nonzero prime ideal and let $p > 0$ be the characteristic of the residue field $k(\mathfrak{p})$. Suppose that $\mathfrak{p}^m = xA$ for some $m \geq 1$, $x \in A$. Suppose further that there exist $\lambda \in A[1/x]^\times$ such that $\lambda^2 - 1 \in A[1/x]^\times$ also.*

Then the natural maps

$$\iota_1 : \mathrm{H}_2(\Gamma_0(A,\mathfrak{p}),\mathbb{Z}) \to \mathrm{H}_2(\mathrm{SL}_2(A),\mathbb{Z})$$

and

$$\iota_2 : \mathrm{H}_2(\Gamma_0(A,\mathfrak{p}),\mathbb{Z}) \to \mathrm{H}_2(H(\mathfrak{p}),\mathbb{Z})$$

are surjective.

Proof. Let $k = k(\mathfrak{p})$. There is a commutative diagram of group extensions

$$\begin{tikzcd}
1 \arrow[r] & \Gamma(A,\mathfrak{p}) \arrow[r] \arrow[d, "\mathrm{id}"] & \Gamma_0(A,\mathfrak{p}) \arrow[r] \arrow[d, "\iota_1"] & B(k) \arrow[r] \arrow[d, "\iota"] & 1 \\
1 \arrow[r] & \Gamma(A,\mathfrak{p}) \arrow[r] & \mathrm{SL}_2(A) \arrow[r] & \mathrm{SL}_2(k) \arrow[r] & 1
\end{tikzcd}$$

and (using Lemma 5.5)

$$\begin{tikzcd}
1 \arrow[r] & \tilde{\Gamma}(A,\mathfrak{p}) \arrow[r] \arrow[d, "\mathrm{id}"] & \Gamma_0(A,\mathfrak{p}) \arrow[r] \arrow[d, "\iota_2"] & B'(k) \arrow[r] \arrow[d, "\iota"] & 1 \\
1 \arrow[r] & \tilde{\Gamma}(A,\mathfrak{p}) \arrow[r] & H(\mathfrak{p}) \arrow[r, "\gamma_{\pi,1}"] & \mathrm{SL}_2(k) \arrow[r] & 1.
\end{tikzcd}$$

We give the argument for ι_1. The analogous argument for ι_2 is achieved by replacing $B(k)$ with $B'(k)$.

The top group extension gives rise to a spectral sequence

$$E^2_{i,j}(\Gamma_0(A,\mathfrak{p})) = \mathrm{H}_i(B(k), \mathrm{H}_j(\Gamma(A,\mathfrak{p}),\mathbb{Z})) \Rightarrow \mathrm{H}_{i+j}(\Gamma_0(A,\mathfrak{p}),\mathbb{Z})$$

and the lower one gives rise to the spectral sequence

$$E^2_{i,j}(\mathrm{SL}_2(A)) = \mathrm{H}_i(\mathrm{SL}_2(k), \mathrm{H}_j(\Gamma(A,\mathfrak{p}),\mathbb{Z})) \Rightarrow \mathrm{H}_{i+j}(\mathrm{SL}_2(A),\mathbb{Z}).$$

The map of extensions induces a natural map of spectral sequences compatible with the map ι_1 on abutments.

For $H = \Gamma_0(A, \mathfrak{p})$ or $\mathrm{SL}_2(A)$, the image of the edge homomorphism $E^\infty_{0,j}(H) \to \mathrm{H}_j(H, \mathbb{Z})$ is equal to the image of $\mathrm{H}_j(\Gamma(A, \mathfrak{p}), \mathbb{Z}) \to \mathrm{H}_j(H, \mathbb{Z})$. Thus, comparing the E^∞-terms of total degree 2, we obtain a commutative diagram of the form

$$\begin{tikzcd}
\mathrm{H}_2(\Gamma(A,\mathfrak{p}),\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}_2(\Gamma_0(A,\mathfrak{p}),\mathbb{Z}) \arrow[r] \arrow[d, "\iota_1"] & C(\Gamma_0(A,\mathfrak{p})) \arrow[r] \arrow[d] & 0 \\
\mathrm{H}_2(\Gamma(A,\mathfrak{p}),\mathbb{Z}) \arrow[r] & \mathrm{H}_2(\mathrm{SL}_2(A),\mathbb{Z}) \arrow[r] & C(\mathrm{SL}_2(A)) \arrow[r] & 0
\end{tikzcd}$$

where, for $H = \Gamma_0(A, \mathfrak{p})$ or $\mathrm{SL}_2(A)$, $C(H)$ is a group fitting into an exact sequence

$$0 \to E^\infty_{1,1}(H) \to C(H) \to E^\infty_{2,0}(H) \to 0.$$

Since $\mathrm{H}_1(\Gamma(A, \mathfrak{p}), \mathbb{Z})$ is a finite abelian p-group, it follows from Lemma 5.15 that the natural maps

$$E^2_{i,1}(\Gamma_0(A, \mathfrak{p})) = \mathrm{H}_i(B(k), \mathrm{H}_1(\Gamma(A, \mathfrak{p}), \mathbb{Z})) \to \mathrm{H}_i(\mathrm{SL}_2(k), \mathrm{H}_1(\Gamma(A, \mathfrak{p}), \mathbb{Z})) = E^2_{i,1}(\mathrm{SL}_2(A))$$

are all isomorphisms.

Since, furthermore, all these groups $E^2_{i,1}$ are finite abelian p-groups, it follows that the differentials

$$d^2_{i,0} : E^2_{i,0}(H) \to E^2_{i-1,1}(H)$$

factor through $E^2_{i,0}(H)_{(p)}$, for $H = \Gamma_0(A, \pi)$ or $\mathrm{SL}_2(A)$.

By Lemma 5.15 again, we have natural isomorphisms

$$E^2_{i,0}(\Gamma_0(A, \mathfrak{p}))_{(p)} = \mathrm{H}_i(B(k), \mathbb{Z})_{(p)} \cong \mathrm{H}_i(\mathrm{SL}_2(k), \mathbb{Z})_{(p)} = E^2_{i,0}(\mathrm{SL}_2(A))_{(p)}.$$

Thus, we have

$$E^\infty_{1,1}(\Gamma_0(A, \mathfrak{p})) \cong E^\infty_{1,1}(\mathrm{SL}_2(A))$$

since

$$E^\infty_{1,1}(H) = \mathrm{Coker}(d^2 : E^2_{3,0}(H)_{(p)} \to E^2_{1,1}(H))$$

when $H = \Gamma_0(A, \mathfrak{p})$ or $\mathrm{SL}_2(A)$.

Finally, for $H = \Gamma_0(A, \mathfrak{p})$ or $\mathrm{SL}_2(A)$, we have

$$E^\infty_{2,0}(H) = \mathrm{Ker}(d^2 : E^2_{2,0}(H) \to E^2_{0,1}(H)).$$

But straightforward calculations (see, for example, [**3**, Section 3]) show that

$$E^2_{2,0}(\Gamma_0(A, \mathfrak{p})) = \mathrm{H}_2(B(k), \mathbb{Z}) = \mathrm{H}_2(B(k), \mathbb{Z})_{(p)} = \mathrm{H}_2(\mathrm{SL}_2(k), \mathbb{Z})_{(p)} = \mathrm{H}_2(\mathrm{SL}_2(k), \mathbb{Z}) = E^2_{2,0}(\mathrm{SL}_2(A)).$$

Thus

$$E^\infty_{2,0}(\Gamma_0(A, \mathfrak{p})) \cong E^\infty_{2,0}(\mathrm{SL}_2(A)).$$

Hence the map $C(\Gamma_0(A, \mathfrak{p})) \to C(\mathrm{SL}_2(A))$ is an isomorphism, and the result follows. □

5.6. An exact sequence for the second homology of SL_2 of S-integers. Let K be a global field and let $S \subset T$ be nonempty sets of primes of K containing the infinite primes. Then there is a natural short exact sequence

$$0 \longrightarrow K_2(\mathcal{O}_S) \longrightarrow K_2(\mathcal{O}_T) \xrightarrow{\sum \tau_{\mathfrak{p}}} \oplus_{\mathfrak{p} \in T \setminus S} k(\mathfrak{p})^\times \longrightarrow 0.$$

In this section, we demonstrate an analogous exact sequence for $\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})$, at least when S is sufficiently large.

Theorem 5.17. *Let A be a ring of S-integers in a global field K where $|S| \geq 2$. Let $\mathfrak{p}$ be a nonzero prime ideal and let $p > 0$ be the characteristic of the residue field $k(\mathfrak{p})$.*

Suppose also that there exists $\lambda \in A^\times$ such that $\lambda^2 - 1 \in A^\times$ also.

Let $x \in A$ and $m \geq 1$ such that $\mathfrak{p}^m = xA$.

Then there is a natural exact sequence

$$\mathrm{H}_2(\mathrm{SL}_2(A), \mathbb{Z}) \longrightarrow \mathrm{H}_2(\mathrm{SL}_2(A[1/x]), \mathbb{Z}) \xrightarrow{\delta} \mathrm{H}_1(k(\mathfrak{p})^\times, \mathbb{Z}) \longrightarrow 0$$

Here the map δ fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{H}_2(\mathrm{SL}_2(A[1/x]), \mathbb{Z}) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}_1(k(\mathfrak{p})^\times, \mathbb{Z}) \arrow[d, "\cong"] \\
\mathrm{H}_2(\mathrm{SL}(K), \mathbb{Z}) = K_2^{\mathrm{M}}(K) \arrow[r, "\tau_{\mathfrak{p}}"] & k(\mathfrak{p})^\times
\end{tikzcd}$$

where $\tau_{\mathfrak{p}} : K_2^{\mathrm{M}}(K) \to k(\mathfrak{p})^\times$ is the tame symbol

$$\tau_{\mathfrak{p}}(\{x, y\}) = (-1)^{v(x)v(y)} x^{-v(y)} y^{v(x)} \pmod{\mathfrak{p}} \in k(\mathfrak{p})^\times.$$

Proof. By Proposition 5.16 the map $\iota_2 : \mathrm{H}_2(\Gamma_0(A, \mathfrak{p}), \mathbb{Z}) \to \mathrm{H}_2(H(\mathfrak{p}), \mathbb{Z})$ is surjective. Thus the Mayer-Vietoris sequence yields the exact sequence

$$\mathrm{H}_2(\mathrm{SL}_2(A), \mathbb{Z}) \longrightarrow \mathrm{H}_2(\mathrm{SL}_2(A[1/x]), \mathbb{Z}) \xrightarrow{\delta} \mathrm{H}_1(\Gamma_0(A, \mathfrak{p}), \mathbb{Z}) \longrightarrow 0.$$

The remaining statements of the theorem follow from Proposition 5.6 and Proposition 5.9. □

Remark 5.18. In this proof, the hypothesis that $\lambda, \lambda^2 - 1 \in A^\times$ is needed so that Proposition 5.9 is validly applied.

The first part of the proof only requires the weaker condition that $\lambda, \lambda^2 - 1 \in A[1/x]^\times$. For example, taking $A = \mathbb{Z}[1/3]$, $\mathfrak{p} = 2A$, $x = 2$ and $\lambda = 3$, we obtain an exact sequence

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/3]), \mathbb{Z}) \to \mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/6]), \mathbb{Z}) \to \mathrm{H}_1(\Gamma_0(\mathbb{Z}[1/3], 2), \mathbb{Z}) \to 0.$$

In this sequence,

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/3]), \mathbb{Z}) \cong \mathbb{Z} \text{ and } \mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/6]), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$$

by the calculations of Adem-Naffah, [**1**], and Tuan-Ellis, [**19**].

Thus

$$\mathrm{H}_1(\Gamma_0(\mathbb{Z}[1/3], 2), \mathbb{Z}) \neq 0$$

while $\mathrm{H}_1(k(2)^\times, \mathbb{Z}) = \mathrm{H}_1(\mathbb{F}_2^\times, \mathbb{Z}) = 0$, so that the conclusion of Proposition 5.9 is false in the case $A = \mathbb{Z}[1/3]$ and $\mathfrak{p} = 2A$.

Remark 5.19. Theorem 5.17 is not valid for more general Dedekind Domains A, even when there is a unit λ such that $\lambda^2 - 1$ is also a unit.

For example, let k be an infinite field and let $K = k(t)$, $A = k[t]$, $\mathfrak{p} = tA$, $x = t$. It is shown in [**2**, Theorem 4.1] that the cokernel of the natural map

$$\mathrm{H}_2(\mathrm{SL}_2(A), \mathbb{Z}) \to \mathrm{H}_2(\mathrm{SL}_2(A[1/t]), \mathbb{Z})$$

is isomorphic to $K_1^{\mathrm{MW}}(k)$, the first Milnor-Witt K-group of the residue field k. It seems reasonable to suppose that this statement should be true for a larger class of Dedekind Domains.

Note that there is a natural surjective map $K_1^{\mathrm{MW}}(k) \to K_1^{\mathrm{M}}(k) \cong k^\times$ which is an isomorphism when the field k is finite. However, in general, the kernel of this homomorphism is $I^2(k)$ (see section 3.4 above).

Corollary 5.20. *Let K be a global field. Let S be a set of primes of K containing the infinite primes. Suppose that $|S| \geq 2$ and that $\mathcal{O}_S$ contains a unit λ such that $\lambda^2 - 1$ is also a unit.*

Let T be any set of primes containing S. Then there is a natural exact sequence

$$\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z}) \to \mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_T), \mathbb{Z}) \to \oplus_{\mathfrak{p} \in T \setminus S} k(\mathfrak{p})^\times \to 0.$$

Proof. We proceed by induction on $|T \setminus S|$.

The case $|T \setminus S| = 1$ is just Theorem 5.17.

The inductive step follows immediately by applying the snake lemma to the commutative diagram with exact rows

$$\begin{tikzcd}
 & \mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_{T'}), \mathbb{Z}) \arrow[r] \arrow[d, two heads] & \mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_T), \mathbb{Z}) \arrow[r] \arrow[d] & k(\mathfrak{q})^\times \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \oplus_{\mathfrak{p} \in T'} k(\mathfrak{p})^\times \arrow[r] & \oplus_{\mathfrak{p} \in T} k(\mathfrak{p})^\times \arrow[r] & k(\mathfrak{q})^\times \arrow[r] & 0
\end{tikzcd}$$

where $\mathfrak{q}$ is any element in $T \setminus S$ and $T' = T \setminus \{\mathfrak{q}\}$. □

Corollary 5.21. *Let K be a global field. Let S be a set of primes of K containing the infinite primes. Suppose that $|S| \geq 2$ and that $\mathcal{O}_S$ contains a unit λ such that $\lambda^2 - 1$ is also a unit.*

Then there is a natural exact sequence

$$\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z}) \to \mathrm{H}_2(\mathrm{SL}_2(K), \mathbb{Z}) \to \oplus_{\mathfrak{p} \notin S} k(\mathfrak{p})^\times \to 0.$$

Proof. Since

$$K = \lim_{S \subset T} \mathcal{O}_T$$

this follows from Corollary 5.20 by taking (co)limits. □

6. The main theorem

Let K be a global field. In this section, we use the results above to prove our main theorem which identifies $\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})$ with a certain subgroup of $K_2(2, K)$, which we now describe.

For a prime $\mathfrak{p}$ of K, we denote by $T_\mathfrak{p}$ the composite

$$K_2(2, K) \twoheadrightarrow K_2^{\mathrm{M}}(K) \xrightarrow{\tau_\mathfrak{p}} k(\mathfrak{p})^\times$$

where $\tau_\mathfrak{p}$ is the tame symbol, as above.

When S is a nonempty set of primes of K containing the infinite primes, we set

$$\tilde{K}_2(2, \mathcal{O}_S) := \mathrm{Ker}(K_2(2, K) \to \oplus_{\mathfrak{p} \notin S} k(\mathfrak{p})^\times).$$

We begin by noting that this group is closely related to $K_2(\mathcal{O}_S)$:

Lemma 6.1. *For any global field K and for any nonempty set S of primes which contains the infinite primes there is a natural exact sequence*

$$0 \to I^3(K) \to \tilde{K}_2(2, \mathcal{O}_S) \to K_2(\mathcal{O}_S) \to 0.$$

In particular, $\tilde{K}_2(2, \mathcal{O}_S) \cong K_2(\mathcal{O}_S)$ if K is of positive characteristic or is a totally imaginary number field.

Proof. Apply the snake lemma and Corollary 3.10 (1) to the map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \tilde{K}_2(2, O_S) \arrow[r] \arrow[d] & K_2(2, K) \arrow[r] \arrow[d] & \oplus_{\mathfrak{p} \notin S} k(\mathfrak{p})^\times \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & K_2(O_S) \arrow[r] & K_2^{\mathrm{M}}(K) \arrow[r] & \oplus_{\mathfrak{p} \notin S} k(\mathfrak{p})^\times \arrow[r] & 0. \end{tikzcd}$$

The second statement follows from the fact that, for a global field K, $I^3(K) \cong \mathbb{Z}^{r(K)}$ where $r(K)$ is the number of distinct real embeddings of K. □

Example 6.2. Consider the global field $K = \mathbb{Q}$.

For any set S of prime numbers, we will set

$$\mathbb{Z}_S := \mathbb{Z}[\{1/p\}_{p \in S}] = O_{S \cup \{\infty\}}.$$

The kernel of the surjective map

$$K_2(2, \mathbb{Q}) \to \oplus_p \mathbb{F}_p^\times$$

is an infinite cyclic direct summand with generator $c(-1, -1)$.

It follows that for any set S of prime numbers

$$\tilde{K}_2(2, \mathbb{Z}_S) \cong \mathbb{Z} \oplus \left(\oplus_{p \in S} \mathbb{F}_p^\times\right).$$

More generally, we have the following description of the groups $\tilde{K}_2(2, O_S)$:

For a global field K, let Ω be the set of real embeddings of K. For $\sigma \in \Omega$, there is a corresponding homomorphism

$$T_\sigma : K_2(K) \to \mu_2, \quad \{a, b\} \mapsto \begin{cases} -1, & \text{if } \mathrm{sgn}(\sigma(a)), \mathrm{sgn}(\sigma(b)) < 0 \\ 1, & \text{otherwise} \end{cases}$$

Let

$$K_2(K)_+ := \mathrm{Ker}(\oplus_{\sigma \in \Omega} T_\sigma : K_2(K) \to \mu_2^\Omega)$$

and let $K_2(O_S)_+ = K_2(O_S) \cap K_2(K)_+$.

Lemma 6.3. *Let K be a global field. Let S be a nonempty set of primes of K including the infinite primes. Then*

$$\tilde{K}_2(2, O_S) \cong K_2(O_S)_+ \oplus \mathbb{Z}^\Omega.$$

Proof. By classical quadratic form theory, the group $I^n(\mathbb{R})$ is infinite cyclic with generator $\langle\langle -1 \rangle\rangle^n = (-2)^{n-1} \langle\langle -1 \rangle\rangle$.

It is shown in [**2**] that for a global field K the natural surjective map

$$K_2(2, K) \to I^2(\mathbb{R})^\Omega \cong \mathbb{Z}^\Omega, \quad c(u, v) \mapsto (\langle\langle \mathrm{sgn}(\sigma(u)) \rangle\rangle \langle\langle \mathrm{sgn}(\sigma(u)) \rangle\rangle)_{\sigma \in \Omega}$$

has kernel isomorphic to $K_2(K)_+$, where this isomorphism is realised by restricting the natural map $K_2(2, K) \to K_2(K)$. Furthermore, the composite map $I^3(K) \to K_2(2, K) \to I^2(\mathbb{R})^\Omega$ induces an isomorphism

$$I^3(K) \cong I^3(\mathbb{R})^\Omega = 2 \cdot (I^2(\mathbb{R})^\Omega).$$

Since $I^3(K) \subset \tilde{K}_2(2, O_S)$, the image of the map

$$\tilde{K}_2(2, O_S) \to I^2(\mathbb{R})^\Omega \cong \mathbb{Z}^\Omega$$

contains a full sublattice.

On the other hand, the kernel of this map is isomorphic – via the map $K_2(2, K) \to K_2(K)$ – to $K_2(O_S) \cap K_2(K)_+$. □

It is natural to ask, of course, about the relation between $\tilde{K}_2(2, O_S)$ and $K_2(2, O_S)$.

It is a theorem of van der Kallen ([**20**]) that when K is a global field and when S contains all infinite places and $|S| \geq 2$ then the stabilization map

$$K_2(2, O_S) \to K_2(O_S)$$

is always surjective.

We deduce:

Lemma 6.4. *Let K be a global field and let S be a nonempty set of primes of K containing the infinite primes. Then the image of the natural map $K_2(2, O_S) \to K_2(2, K) \cong K_2(2, K)$ lies in $\tilde{K}_2(2, O_S)$.*

Furthermore, when $|S| \geq 2$, and when there exist units $u_\sigma \in O_S^\times$, $\sigma \in \Omega$ satisfying

$$\mathrm{sgn}(\tau(u_\sigma)) = (-1)^{\delta_{\sigma,\tau}},$$

the resulting natural map $K_2(2, O_S) \to \tilde{K}_2(2, O_S)$ is surjective; i.e. the image of the map $K_2(2, O_S) \to K_2(2, K)$ is precisely $\tilde{K}_2(2, O_S)$.

Proof. The diagram

$$\begin{tikzcd}
 & K_2(2, O_S) \arrow[r] \arrow[d] & K_2(2, K) \arrow[r] \arrow[d] & \oplus_{\mathfrak{p}\notin S} k(\mathfrak{p})^\times \arrow[d, "\mathrm{id}"] & \\
0 \arrow[r] & K_2(O_S) \arrow[r] & K_2^{\mathrm{M}}(K) \arrow[r] & \oplus_{\mathfrak{p}\notin S} k(\mathfrak{p})^\times \arrow[r] & 0
\end{tikzcd}$$

commutes.

Our hypothesis on units ensures that the map

$$\tilde{K}_2(2, O_S) \to K_2(2, K) \to I^2(\mathbb{R})^\Omega$$

is surjective.

Since we also have

$$K_2(O_S)_+ \subset K_2(O_S) \subset \tilde{K}_2(2, O_S)$$

by the result of van der Kallen, the second statement follows.

□

One would expect that the resulting map $K_2(2, O_S) \to \tilde{K}_2(2, O_S)$ is very often an isomorphism. It seems to be difficult, however, to prove this in any given instance. In the case $K = \mathbb{Q}$, Jun Morita, [**13**, Theorems 2,3] has proved:

Theorem 6.5. *Let S be any of the following sets of primes numbers:*

$S = \{p_1, \ldots, p_n\}$, the set of the first n successive prime numbers, or S is one of $\{2, 5\}$, $\{2, 3, 7\}$, $\{2, 3, 11\}$, $\{2, 3, 5, 11\}$, $\{2, 3, 13\}$, $\{2, 3, 7, 13\}$, $\{2, 3, 17\}$, $\{2, 3, 5, 19\}$.

Then $K_2(2, \mathbb{Z}_S)$ is central in $\mathrm{St}(2, \mathbb{Z}_S)$ and the natural map

$$K_2(2, \mathbb{Z}_S) \to \tilde{K}_2(2, \mathbb{Z}_S) \cong \mathbb{Z} \oplus \left(\oplus_{p\in S} \mathbb{F}_p^\times\right)$$

is an isomorphism.

Lemma 6.6. *Let K be a global field and let S be a nonempty set of primes of K containing the infinite primes. Then the image of the natural map*

$$\mathrm{H}_2(\mathrm{SL}_2(O_S), \mathbb{Z}) \to \mathrm{H}_2(\mathrm{SL}_2(K), \mathbb{Z}) \xrightarrow{\cong} K_2(2, K)$$

lies in $\tilde{K}_2(2, O_S)$.

Proof. The diagram

$$\begin{tikzcd}
\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S),\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{H}_2(\mathrm{SL}(\mathcal{O}_S),\mathbb{Z}) \arrow[d] \\
\mathrm{H}_2(\mathrm{SL}_2(K),\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_2(\mathrm{SL}(K),\mathbb{Z}) \arrow[d, "\cong"] \\
K_2(2,K) \arrow[r] & K_2^{\mathrm{M}}(K)
\end{tikzcd}$$

commutes. But $\mathrm{H}_2(\mathrm{SL}(\mathcal{O}_S),\mathbb{Z}) \cong K_2(\mathcal{O}_S)$ and the natural map $K_2(\mathcal{O}_S) \to K_2(K) = K_2^{\mathrm{M}}(K)$ induces an isomorphism

$$K_2(\mathcal{O}_S) \cong \mathrm{Ker}(K_2^{\mathrm{M}}(k) \to \oplus_{\mathfrak{p}\notin S} k(\mathfrak{p})^\times).$$

□

If K is a global field and and if S is a nonempty set of primes containing the infinite primes we will let

$$\mathcal{K}_S := \mathrm{Ker}(\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S),\mathbb{Z}) \to \mathrm{H}_2(\mathrm{SL}_2(K),\mathbb{Z})).$$

Note that

$$\mathcal{K}_S := \mathrm{Ker}(\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S),\mathbb{Z}) \to \tilde{K}_2(2,\mathcal{O}_S))$$

since $\tilde{K}_2(2,\mathcal{O}_S) \subset K_2(2,K) \cong \mathrm{H}_2(\mathrm{SL}_2(K),\mathbb{Z})$.

Remark 6.7. In general, the kernels $\mathcal{K}_S$ can be arbitrarily large, even in the case $K = \mathbb{Q}$: The calculations of Adem-Naffah, [**1**], show that the ranks of the groups $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/p]),\mathbb{Z})$ grow with linearly p when p is a prime number. On the other hand, the rank of $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Q}),\mathbb{Z})$ is 1.

Lemma 6.8. *Let K be a global field. Let S be a set of primes of K containing the infinite primes. Suppose that $|S| \geq 2$ and that $\mathcal{O}_S$ contains a unit λ such that $\lambda^2 - 1$ is also a unit. Then*

(1) *The natural map*
$$\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S),\mathbb{Z}) \to \tilde{K}_2(2,\mathcal{O}_S)$$
is surjective.
(2) *If $T \supset S$, then the natural map $\mathcal{K}_S \to \mathcal{K}_T$ is surjective.*

Proof.

(1) By Corollary 5.21, we have a commutative diagram with exact rows

$$\begin{tikzcd}
 & \mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S),\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{H}_2(\mathrm{SL}_2(K),\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & \oplus_{\mathfrak{p}\notin S} k(\mathfrak{p})^\times \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \tilde{K}_2(2,\mathcal{O}_S) \arrow[r] & K_2(2,K) \arrow[r] & \oplus_{\mathfrak{p}\notin S} k(\mathfrak{p})^\times \arrow[r] & 0.
\end{tikzcd}$$

(2) Apply the snake lemma to the diagram

$$\begin{tikzcd}
 & \mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S),\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_T),\mathbb{Z}) \arrow[r] \arrow[d] & \oplus_{\mathfrak{p}\in T\setminus S} k(\mathfrak{p})^\times \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \tilde{K}_2(2,\mathcal{O}_S) \arrow[r] & \tilde{K}_2(2,\mathcal{O}_T) \arrow[r] & \oplus_{\mathfrak{p}\in T\setminus S} k(\mathfrak{p})^\times \arrow[r] & 0.
\end{tikzcd}$$

□

REMARK 6.9. Note, on the other hand, that the map

$$0 = \mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z}) \to \tilde{K}_2(2, \mathbb{Z}) \cong K_2(2, \mathbb{Z}) \cong \mathbb{Z}$$

cannot be surjective.

THEOREM 6.10. *Let K be a global field.*

(1) *There exists a finite set S of primes of K satisfying*
   (a) *S contains all infinite primes and $|S| \geq 2$.*
   (b) *There exists a unit λ of $\mathcal{O}_S$ such that $\lambda^2 - 1$ is also a unit.*
   (c) *The natural map $\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z}) \to \tilde{K}_2(2, \mathcal{O}_S)$ is an isomorphism.*
(2) *If T is any set of primes containing S then $\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_T), \mathbb{Z}) \cong \tilde{K}_2(2, \mathcal{O}_T)$; i.e. there is a natural short exact sequence*

$$0 \longrightarrow \mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_T), \mathbb{Z}) \longrightarrow K_2(2, K) \xrightarrow{\sum T_{\mathfrak{p}}} \oplus_{\mathfrak{p} \notin T} k(\mathfrak{p})^\times \longrightarrow 0.$$

PROOF.

(1) Let S_0 be any set of primes satisfying (a) and (b). Since $\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})$ is a finitely-generated abelian group, so also is $\mathcal{K}_{S_0}$. Since $\mathrm{H}_2(\mathrm{SL}_2(K), \mathbb{Z}) = \lim_T \mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_T), \mathbb{Z})$, the limit being taken over finite sets T of primes, it follows that there is a finite set of primes S containing S_0 for which

$$\mathcal{K}_{S_0} = \mathrm{Ker}(\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_{S_0}), \mathbb{Z}) \to \mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})).$$

   By Lemma 6.8 it follows that $\mathcal{K}_S = 0$ and hence that $\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z}) \cong \tilde{K}_2(2, \mathcal{O}_S)$ as required.
(2) This is immediate from Lemma 6.8.

□

LEMMA 6.11. *Let $K = \mathbb{Q}$ and let $S = \{2, 3, \infty\}$. Then S satisfies conditions (a)-(c) of Theorem 6.10 (1).*

PROOF. The set $S = \{2, 3, \infty\}$ clearly satisfies conditions (a) and (b).
Observe that

$$\mathcal{O}_S = \mathbb{Z}_{\{2,3\}} = \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right] = \mathbb{Z}\left[\frac{1}{6}\right].$$

By Lemma 6.8, the natural map

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/6]), \mathbb{Z}) \to \tilde{K}_2(2, \mathbb{Z}[1/6]) \cong \mathbb{Z} \oplus \mathbb{F}_3^\times$$

(see Example 6.2) is surjective.
On the other hand, the calculations of Tuan and Ellis, [**19**], show that

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/6]), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

It follows that the natural map above is an isomorphism. □

In view of Theorem6.10 (2) and Example 6.2, we immediately deduce:

THEOREM 6.12. *Let T be any set of prime numbers containing $2, 3$. Then there is an isomorphism*

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}_T), \mathbb{Z}) \cong \mathbb{Z} \oplus \left(\oplus_{p \in T} \mathbb{F}_p^\times\right).$$

In particular, if $m \in \mathbb{Z}$ and if $6|m$ then

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/m]), \mathbb{Z}) \cong \mathbb{Z} \oplus \left(\oplus_{p|m} \mathbb{F}_p^\times\right).$$

Combining this with Morita's Theorem (6.5) we deduce:

Proposition 6.13. *Let S be any of the following sets of primes numbers:*
$S = \{p_1, \ldots, p_n\}$, the set of the first n successive prime numbers, or S is one of $\{2,3,7\}$, $\{2,3,11\}$, $\{2,3,5,11\}$, $\{2,3,13\}$, $\{2,3,7,13\}$, $\{2,3,17\}$, $\{2,3,5,19\}$.
Then the natural map

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}_S), \mathbb{Z}) \to K_2(2, \mathbb{Z}_S)$$

is an isomorphism and

$$1 \to K_2(2, \mathbb{Z}_S) \to \mathrm{St}(2, \mathbb{Z}_S) \to \mathrm{SL}_2(\mathbb{Z}_S) \to 1$$

is the universal central extension of $\mathrm{SL}_2(\mathbb{Z}_S)$.

Proof. Since $K_2(2, \mathbb{Z}_S)$ is central in $\mathrm{St}(2, \mathbb{Z}_S)$, there is a natural homomorphism $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}_S), \mathbb{Z}) \to K_2(2, \mathbb{Z}_S)$ through which the map $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}_S), \mathbb{Z}) \to K_2(2, \mathbb{Q})$ factors.
Since $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}_S), \mathbb{Z})$ and $K_2(2, \mathbb{Z}_S)$ are both isomorphic to $\tilde{K}_2(2, \mathbb{Z}_S) \subset K_2(2, \mathbb{Q})$, the result follows immediately. □

7. Some 2-dimensional homology classes

In this section we construct explicit cycles in the bar resolution of $\mathrm{SL}_2(A)$ which represent homology classes in $\mathrm{H}_2(\mathrm{SL}_2(A), \mathbb{Z})$. We show that these classes map to the symbols $c(u, v) \in K_2(2, A)$, when A is a field.

7.1. The homology classes $C(a, b)$. Let A be a commutative ring and let $a \in A^\times$. We define the following elements of $\mathrm{SL}_2(A)$:

$$w := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad G_a := \begin{bmatrix} 0 & -1 \\ 1 & a + a^{-1} \end{bmatrix}, \quad H_a := E_{21}(a) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}.$$

Note that

$$wG_a = \begin{bmatrix} 1 & a + a^{-1} \\ 0 & 1 \end{bmatrix} = E_{12}(a + a^{-1}).$$

We also define

$$R_a := H_a G_a H_a^{-1} = H_a G_a H_{-a} = \begin{bmatrix} a & -1 \\ 0 & a^{-1} \end{bmatrix}.$$

Thus, by definition,

$$H_a G_a = R_a H_a = \begin{bmatrix} 0 & -1 \\ 1 & a^{-1} \end{bmatrix}.$$

Let

$$\Theta_a := [H_a | G_a] - [R_a | H_a] + [w^{-1} | wG_a] \in \bar{F}_2(\mathrm{SL}_2(A)) = \bar{F}_2.$$

Then

$$d_2(\Theta_a) = [w^{-1}] + [wG_a] - [R_a] \in \bar{F}_1.$$

Now let $a, b \in A^\times$. Then

$$d_2(\Theta_{ab} - \Theta_a - \Theta_b + \Theta_1) = ([R_a] + [R_b] - [R_{ab}]) + ([wG_{ab}] - [wG_a] - [wG_b] + [wG_1] - [R_1]).$$

Now

$$[R_a] + [R_b] = [R_a R_b] + d_2([R_a | R_b])$$

and

$$[R_{ab}] = [R_a R_b] + [(R_a R_b)^{-1}] - d_2\left([R_a R_b | (R_a R_b)^{-1} (R_{ab})]\right).$$

Hence

$$[R_a]+[R_b]-[R_{ab}] = -[(R_aR_b)^{-1}R_{ab}] + d_2\left([R_a|R_b]+[R_aR_b|(R_aR_b)^{-1}(R_{ab})]\right)$$

where

$$(R_aR_b)^{-1}R_{ab} = \begin{bmatrix} 1 & (ab)^{-1}(a+b^{-1}-1) \\ 0 & 1 \end{bmatrix} = E_{12}((ab)^{-1}(a+b^{-1}-1)).$$

Putting this together, we deduce

$$\begin{aligned} & d_2(\Theta_{ab}-\Theta_a-\Theta_b+\Theta_1-[R_a|R_b]-[R_aR_b|(R_aR_b)^{-1}(R_{ab})]) \\ = \;& [wG_{ab}]-[wG_a]-[wG_b]+[wG_1]-[R_1]+[(R_aR_b)^{-1}(R_{ab})] \\ = \;& [E_{12}(ab+(ab)^{-1})]-[E_{12}(a+a^{-1})]-[E_{12}(b+b^{-1})] \\ & +[E_{12}(2)]-[E_{12}(-1)]+[E_{12}((ab)^{-1}(a+b^{-1}-1))]. \end{aligned}$$

Now suppose that there exists $\lambda \in A^\times$ such that $\lambda^2-1 \in A^\times$. Let

$$D(\lambda) := \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in \mathrm{SL}_2(A).$$

Recall that for any $x \in A$

$$D(\lambda)E_{12}(x)D(\lambda)^{-1} = E_{12}(\lambda^2 x)$$

and hence for any $x \in A$ we have

$$\begin{aligned} E_{12}(x) &= D(\lambda)E_{12}(x')D(\lambda)^{-1}E_{12}(x')^{-1} \\ &= D(\lambda)E_{12}(x')(E_{12}(x')D(\lambda))^{-1} = [D(\lambda),E_{12}(x')]. \end{aligned}$$

where $x' := x/(\lambda^2-1)$.

Now if G is any group and if $g,h \in G$ then

$$D_2([(gh)(hg)^{-1}|hg]-[g|h]+[h|g]) = [(gh)(hg)^{-1}] = [[g,h]]\,.$$

Thus, we define

$$\Psi_x = \Psi_{x,\lambda} := [E_{12}(x)|E_{12}(x')D(\lambda)] - [D(\lambda)|E_{12}(x')] + [E_{12}(x')|D(\lambda)] \in \bar{F}_2.$$

By the preceeding remarks, we have $d_2(\Psi_{x,\lambda}) = [E_{12}(x)] \in \bar{F}_1$ for any $x \in A$.

From all of these calculations we deduce:

Proposition 7.1. *Let A be a commutative ring. Suppose that there exists $\lambda \in A^\times$ such that $\lambda^2-1 \in A^\times$. Let $a,b \in A^\times$. Then*

$$\begin{aligned} F(a,b)_\lambda : \;&= [R_a|R_b]+[R_aR_b|(R_aR_b)^{-1}(R_{ab})]+\Theta_a+\Theta_b-\Theta_{ab}-\Theta_1 \\ &+ \Psi_{ab+(ab)^{-1}}-\Psi_{a+a^{-1}}-\Psi_{b+b^{-1}}+\Psi_2-\Psi_{-1}+\Psi_{(ab)^{-1}(a+b^{-1}-1)} \in \bar{F}_2 \end{aligned}$$

is a cycle, representing a homology class $C(a,b)_\lambda \in \mathrm{H}_2(\mathrm{SL}_2(A),\mathbb{Z})$.

Remark 7.2. The cycles $F(a,b)_\lambda$ are clearly functorial in the sense that if $\psi : A \to B$ is a homomorphism of commutative rings and if $a,b,\lambda,\lambda^2-1 \in A^\times$ then

$$\psi_*(F(a,b)_\lambda) = F(\psi(a),\psi(b))_{\psi(\lambda)} \in \bar{F}_2(\mathrm{SL}_2(B)).$$

Remark 7.3. More generally, suppose that $\Lambda = (\lambda_1,\ldots,\lambda_n,b_1,\ldots,b_n) \in (A^\times)^n \times (A^n)$ satisfies

$$\sum_{i=1}^n (\lambda_i^2-1)b_i = 1$$

Then for any $x \in A$

$$E_{12}(x) = \prod_i [D(\lambda_i), E_{12}(b_i x)].$$

by the proof of Proposition 2.2.

Since

$$[\prod_{i=1}^{n} c_i] = \sum_{i=1}^{n} [c_i] - d_2\left(\sum_{k=1}^{n-1} [c_1 \cdots c_k | c_{k+1}]\right)$$

in $\bar{F}_1(A)$, we can easily write down an element $\Psi_{x,\Lambda} \in \bar{F}_2(A)$ satisfying $d_2(\Psi_{x,\Lambda}) = [E_{12}(x)]$ and thus construct cycles $F(a,b)_\Lambda$.

Remark 7.4. Specialising to the case $a = b = -1$ we obtain:

$$F(-1,-1)_\lambda = [R_{-1}|R_{-1}] + [E_{12}(2)|E_{12}(-3)] + \Psi_{-3} - \Psi_{-1} + 2(\Theta_{-1} - \Theta_1 + \Psi_2 - \Psi_{-2}).$$

As we will see, when A is a field with at least four elements, the homology class $C(a,b)_\lambda$ does not depend on the choice of λ. In fact, this is the case for many commutative rings. For example, we have:

Lemma 7.5. *Let A be a commutative ring. Suppose there exists $n \in \mathbb{Z}$ such that $n, n^4 - 1 \in A^\times$. Then, for any $a, b \in A^\times$, the homology class $C(a,b)_\lambda \in \mathrm{H}_2(\mathrm{SL}_2(A), \mathbb{Z})$ is independent of the choice of λ.*

Proof. Suppose that $\lambda, \mu \in A^\times$ satisfy the condition $\lambda^2 - 1, \mu^2 - 1 \in A^\times$.

Let $a, b \in A^\times$. Note that $F(a,b)_\lambda - F(a,b)_\mu$ is a sum or difference of terms of the form $\Psi_{x,\lambda} - \Psi_{x,\mu}$, $x \in A$. We will show that each such term is a boundary.

We begin by observing that, for any $x \in A$, the elements $\Psi_{x,\lambda}$ and $\Psi_{x,\mu}$ lie in $\bar{F}_2(B)$ where

$$B := \left\{ \begin{bmatrix} u & y \\ 0 & u^{-1} \end{bmatrix} \in \mathrm{SL}_2(A) \mid u \in A^\times \right\}$$

is the subgroup of upper-triangular matrices in $\mathrm{SL}_2(A)$.

Note that

$$d_2(\Psi_{x,\lambda} - \Psi_{x,\mu}) = [E_{12}(x)] - [E_{12}(x)] = 0$$

so that $\Psi_{x,\lambda} - \Psi_{x,\mu}$ represents a homology class in $\mathrm{H}_2(B, \mathbb{Z})$. We will show that it represents the trivial class.

Let $T := \{D(u) \mid u \in A^\times\}$ be the group of diagonal matrices and let $\pi : B \to T$ be the natural surjective homomorphism sending $D(u)E_{12}(z)$ to $D(u)$. Then

$$U := \mathrm{Ker}(\pi) = \{E_{12}(y) \mid y \in A\}$$

is the group of unipotent matrices.

We have $T \cong A^\times$ via $D(u) \leftrightarrow u$ and $U \cong A$, via $E_{12}(x) \leftrightarrow x$.

Note that

$$\begin{aligned} \pi(\Psi_{x,\lambda}) &= \pi\left([E_{12}(x)|E_{12}(x')D(\lambda)] - [D(\lambda)|E_{12}(x')]\right) \\ &= [I|D(\lambda)] - [D(\lambda)|I] + [I|D(\lambda)] \in \bar{F}_2(T). \end{aligned}$$

Since

$$d_3([X|I|I] = [I|I] - [I|X] \text{ and } d_3([I|I|X]) = [X|I] - [I|I]$$

it follows easily that $\pi(\Psi_{x,\lambda} - \Psi_{x,\mu}) \in d_3(\bar{F}_3(T))$. Thus $\pi(\Psi_{x,\lambda} - \Psi_{x,\mu})$ represents the trivial homology class in $\mathrm{H}_2(T, \mathbb{Z})$.

To conclude, we will show that our hypotheses are enough to ensure that π induces an isomorphism $\mathrm{H}_2(B,\mathbb{Z}) \cong \mathrm{H}_2(T,\mathbb{Z})$.

We consider the Hochschild-Serre spectral sequence associated to the short exact sequence

$$1 \to U \to B \to T \to 1.$$

This has the form

$$E^2_{i,j} = \mathrm{H}_i(T, \mathrm{H}_j(U,\mathbb{Z})) \Rightarrow \mathrm{H}_{i+j}(B,\mathbb{Z})$$

$D(u) \in T$ acts by conjugation on $U \cong A$ as multiplication by u^2. Thus the induced action of $D(u)$ on $\mathrm{H}_2(U,\mathbb{Z}) \cong U \bigwedge_{\mathbb{Z}} U \cong A \bigwedge_{\mathbb{Z}} A$ is multiplication by u^4.

In particular, $D(n)$ acts as multiplication by n^2 on $\mathrm{H}_1(U,\mathbb{Z})$, and as multiplication by n^4 on $\mathrm{H}_2(U,\mathbb{Z})$.

Since T is abelian, and since n^2-1, n^4-1 are units in A, it follows from the "centre kills" argument that $\mathrm{H}_i(T,\mathrm{H}_j(U,\mathbb{Z})) = 0$ for $1 \leq j \leq 2$.

Thus, from the spectral sequence, the map π induces an isomorphism $\mathrm{H}_n(B,\mathbb{Z}) \cong \mathrm{H}_n(T,\mathbb{Z})$ for $n \leq 2$. □

Remark 7.6. Since $2^4 - 1 = 3 \cdot 5$, the condition of the Lemma 7.5 is satisfied by any ring in which $2, 3$ and 5 are units.

7.2. A variation. We describe a slightly more compact 2-cycle $\tilde{F}(a,b)_\lambda$ in the case where a^2-1, b^2-1 and $(ab)^2-1$ are all units in A.

Suppose that $a \in A$ is a unit such that $a^2 - 1 \in A^\times$ also. Let

$$\tilde{H}_a = \begin{bmatrix} \frac{1}{1-a} & \frac{a}{1-a} \\ \frac{a}{1+a} & \frac{1}{1+a} \end{bmatrix} \in \mathrm{SL}_2(A).$$

Then

$$\tilde{H}_a G_a \tilde{H}_a^{-1} = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} = D(a).$$

Thus if we let

$$\tilde{\Theta}_a := [\tilde{H}_a | G_a] - [D(a)|\tilde{H}_a] + [w^{-1}|wG_a] \in \bar{F}_2$$

then

$$d_2(\tilde{\Theta}_a) = [w^{-1}] + [wG_a] - [D(a)].$$

If $a^2-1, b^2-1, (ab)^2-1 \in A^\times$ then

$$\begin{aligned} d_2(\tilde{\Theta}_a + \tilde{\Theta}_b - \tilde{\Theta}_{ab} - \Theta_1) &= [D(ab)] - [D(a)] - [D(b)] \\ &\quad + [E_{12}(a+a^{-1})] + [E_{12}(b+b^{-1})] - [E_{12}(ab+(ab)^{-1})] + [E_{12}(-1)] - [E_{12}(2)] \\ &= d_2(-[D(a)|D(b)] + \Psi_{a+a^{-1}} + \Psi_{b+b^{-1}} - \Psi_{ab+(ab)^{-1}} + \Psi_{-1} - \Psi_2). \end{aligned}$$

We deduce:

Proposition 7.7. *If $a, b, \lambda, a^2-1, b^2-1, (ab)^2-1, \lambda^2-1 \in A^\times$ then*

$$\tilde{F}(a,b)_\lambda := [D(a)|D(b)] + \tilde{\Theta}_a + \tilde{\Theta}_b - \tilde{\Theta}_{ab} - \Theta_1 + \Psi_{ab+(ab)^{-1}} - \Psi_{a+a^{-1}} - \Psi_{b+b^{-1}} + \Psi_2 - \Psi_{-1}$$

is a cycle representing a homology class $\tilde{C}(a,b)_\lambda \in \mathrm{H}_2(\mathrm{SL}_2(A),\mathbb{Z})$.

7.3. Symbols as homology classes. In this section, the map of sets $s : \mathrm{SL}_2(F) \to \mathrm{St}(2, F)$ and the homomorphism $\bar{f} : \mathrm{H}_2(\mathrm{SL}_2(F), \mathbb{Z}) \to K_2(2, F)$ are those described in section 4 above.

THEOREM 7.8. *Let F be a field with at least four elements. Let $\lambda \in F^\times \setminus \{\pm 1\}$.*

(1) *Let $a, b \in F^\times$. Then*

$$\bar{f}(C(a,b)_\lambda) = c(a,b).$$

(2) *Suppose further that $a, b, ab \notin \{\pm 1\}$. Then*

$$\bar{f}(\tilde{C}(a,b)_\lambda) = c(a,b).$$

PROOF. We begin by noting that, by Lemma 4.1, we have

$$\bar{f}(\Psi_x) = 1 \text{ for all } x \in F \text{ and } \bar{f}([R_aR_b|(R_aR_b)^{-1}(R_{ab})]) = 1$$

since $c(1, v) = c(u, 1) = 1$ in $K_2(2, F)$.
Also, by Lemma 4.1,

$$\bar{f}([R_a|R_b]) = \bar{f}([D(a)|D(b)] = c(a,b).$$

(1) For any $u \in F^\times$

$$\begin{aligned}
\bar{f}([H_u|G_u]) &= s(H_u)s(G_u)s(H_uG_u)^{-1} \\
&= x_{21}(u) \cdot w_{12}(-1)x_{12}(u + u^{-1}) \cdot x_{12}(-u^{-1})w_{12}(1) \\
&= x_{21}(u) \cdot (w_{12}(-1)x_{12}(u)w_{12}(1)) \\
&= x_{21}(u)x_{12}(u)^{w_{12}(1)} \\
&= x_{21}(u)x_{21}(-u) = 1 \quad \text{by Lemma 3.1.}
\end{aligned}$$

and

$$\begin{aligned}
\bar{f}([w^{-1}|wG_u]) &= s(w^{-1})s(wG_u)s(G_u)^{-1} \\
&= w_{12}(-1)x_{12}(u + u^{-1}) \cdot \left(w_{12}(-1)x_{12}(u + u^{-1})\right)^{-1} = 1.
\end{aligned}$$

Furthermore

$$\begin{aligned}
\bar{f}([R_u|H_u]) &= s(R_u)s(H_u)s(R_uH_u)^{-1} \\
&= x_{12}(-u)h_{12}(u)x_{21}(u)x_{12}(-u^{-1})w_{12}(1) \\
&= h_{12}(u)x_{12}(-u^{-1})x_{21}(u)x_{12}(-u^{-1})w_{12}(1) \text{ since } x_{12}(-u)^{h_{12}(u)} = x_{12}(-u^{-1}) \\
&= h_{12}(u)w_{12}(-u^{-1})w_{12}(1).
\end{aligned}$$

Now

$$w_{12}(-u^{-1})w_{12}(1) = w_{12}(-u^{-1})w_{12}(-1)w_{12}(-1)^{-1}w_{12}(-1)^{-1} = h_{12}(-u^{-1})h_{12}(-1)^{-1}$$

and hence

$$\bar{f}([R_u|H_u]) = c(u, -u^{-1}) = c(-u, u) = 1.$$

Thus

$$\bar{f}(\Theta_u) = 1$$

for all units u.

Putting all of this together gives $\bar{f}(F(a,b)_\lambda) = c(a,b)$ as required.

(2) We must show that $\bar{f}(\tilde{\Theta}_a) = 1$ whenever $a, a^2 - 1 \in F^\times$.
As above, we have $\bar{f}([w^{-1}|wG_a]) = 1$.
Now,

$$s(D(a)) = h_{12}(a), \quad s(\tilde{H}_a) = x_{12}\left(\frac{1+a}{a(1-a)}\right)w_{12}\left(\frac{1+a}{-a}\right)x_{12}(a^{-1}),$$
$$\text{and} \quad s(D(a)\tilde{H}_a) = x_{12}\left(\frac{a(1+a)}{1-a}\right)w_{12}(-(1+a))x_{12}(a^{-1}).$$

Thus

$$\begin{aligned}
\bar{f}([D(a)|\tilde{H}_a]) &= s(D(a))s(\tilde{H}_a)s(D(a)\tilde{H}_a)^{-1} \\
&= h_{12}(a)x_{12}\left(\frac{1+a}{a(1-a)}\right)w_{12}\left(\frac{1+a}{-a}\right)x_{12}(a^{-1})x_{12}(-a^{-1})w_{12}(1+a)x_{12}\left(\frac{a(1+a)}{a-1}\right) \\
&= h_{12}(a)x_{12}\left(\frac{1+a}{a(1-a)}\right)w_{12}\left(\frac{1+a}{-a}\right)w_{12}(1+a)x_{12}\left(\frac{a(1+a)}{a-1}\right) \\
&= h_{12}(a)w_{12}\left(\frac{1+a}{-a}\right)x_{21}\left(\frac{-a}{1-a^2}\right)w_{12}(1+a)x_{12}\left(\frac{a(1+a)}{a-1}\right)
\end{aligned}$$

using

$$x_{12}\left(\frac{1+a}{a(1-a)}\right)^{w_{12}\left(\frac{1+a}{-a}\right)} = x_{21}\left(\frac{-a}{1-a^2}\right).$$

Since, by Lemma 3.1,

$$x_{21}\left(\frac{-a}{1-a^2}\right)^{w_{12}(1+a)} = x_{12}\left(\frac{a(1+a)}{1-a}\right),$$

this gives

$$\begin{aligned}
\bar{f}([D(a)|\tilde{H}_a]) &= h_{12}(a)w_{12}\left(\frac{1+a}{-a}\right)w_{12}(1+a)x_{12}\left(\frac{a(1+a)}{1-a}\right)x_{12}\left(\frac{a(1+a)}{a-1}\right) \\
&= h_{12}(a)w_{12}\left(\frac{1+a}{-a}\right)w_{12}(1+a) \\
&= h_{12}(a)h_{12}\left(\frac{1+a}{-a}\right)h_{12}(-(1+a))^{-1} \\
&= c(a, -(1+a)a^{-1}) = c(a, 1+a).
\end{aligned}$$

Now

$$s(\tilde{H}_aG_a) = s\left(\begin{bmatrix} \frac{a}{1-a} & \frac{a^2}{1-a} \\ \frac{1}{1+a} & \frac{a^{-1}}{1+a} \end{bmatrix}\right) = x_{12}\left(\frac{a(1+a)}{1-a}\right)w_{12}(-(1+a))x_{12}(a^{-1}).$$

So

$$\begin{aligned}
\bar f([\tilde H_a|G_a]) &= s(\tilde H_a)s(G_a)s(\tilde H_a G_a)^{-1}\\
&= x_{12}\left(\frac{1+a}{a(1-a)}\right)w_{12}\left(\frac{1+a}{-a}\right)x_{12}(a^{-1})w_{12}(-1)x_{12}(a+a^{-1})x_{12}(-a^{-1})w_{12}(1+a)x_{12}\left(\frac{a(1+a)}{a-1}\right)\\
&= x_{12}\left(\frac{1+a}{a(1-a)}\right)w_{12}\left(\frac{1+a}{-a}\right)x_{12}(a^{-1})w_{12}(-1)x_{12}(a)w_{12}(1+a)x_{12}\left(\frac{a(1+a)}{a-1}\right)\\
&= w_{12}\left(\frac{1+a}{-a}\right)x_{21}\left(\frac{-a}{1-a^2}\right)w_{12}(-1)x_{21}(-a^{-1})x_{12}(a)w_{12}(1+a)x_{12}\left(\frac{a(1+a)}{a-1}\right)
\end{aligned}$$

using

$$x_{12}\left(\frac{1+a}{a(1-a)}\right)^{w_{12}\left(\frac{1+a}{-a}\right)} = x_{21}\left(\frac{-a}{1-a^2}\right) \text{ and } x_{12}(a^{-1})^{w_{12}(-1)} = x_{21}(-a^{-1}).$$

Since $x_{12}(-a)w_{12}(a) = x_{21}(-a^{-1})x_{12}(a)$, we thus have

$$\begin{aligned}
\bar f([\tilde H_a|G_a]) &= w_{12}\left(\frac{1+a}{-a}\right)x_{21}\left(\frac{-a}{1-a^2}\right)w_{12}(-1)x_{12}(-a)w_{12}(a)w_{12}(1+a)x_{12}\left(\frac{a(1+a)}{a-1}\right)\\
&= w_{12}\left(\frac{1+a}{-a}\right)w_{12}(-1)x_{12}\left(\frac{a}{1-a^2}\right)x_{12}(-a)w_{12}(a)w_{12}(1+a)x_{12}\left(\frac{a(1+a)}{a-1}\right).
\end{aligned}$$

Since

$$x_{12}\left(\frac{a}{1-a^2}\right)x_{12}(-a) = x_{12}\left(\frac{a}{1-a^2}-a\right) = x_{12}\left(\frac{a^3}{1-a^2}\right),$$

we obtain

$$\bar f([\tilde H_a|G_a]) = w_{12}\left(\frac{1+a}{-a}\right)w_{12}(-1)x_{12}\left(\frac{a^3}{1-a^2}\right)w_{12}(a)w_{12}(1+a)x_{12}\left(\frac{a(1+a)}{a-1}\right).$$

The conjugation rules of Corollary 3.2 give

$$x_{12}\left(\frac{a^3}{1-a^2}\right)w_{12}(a)w_{12}(1+a) = w_{12}(a)x_{12}\left(\frac{-a}{1-a^2}\right)w_{12}(1+a) = w_{12}(a)w_{12}(1+a)x_{12}\left(\frac{a(1+a)}{1-a}\right).$$

Therefore

$$\begin{aligned}
\bar f([\tilde H_a|G_a]) &= w_{12}\left(\frac{1+a}{-a}\right)w_{12}(-1)w_{12}(a)w_{12}(1+a)x_{12}\left(\frac{a(1+a)}{1-a}\right)x_{12}\left(\frac{a(1+a)}{a-1}\right)\\
&= w_{12}\left(\frac{1+a}{-a}\right)w_{12}(-1)w_{12}(a)w_{12}(1+a)\\
&= h_{12}\left(\frac{1+a}{-a}\right)h_{12}(a)h_{12}(-(1+a))^{-1}\\
&= c(-(1+a)a^{-1},a) = c(1+a,a).
\end{aligned}$$

Putting this together, we get

$$\begin{aligned}
&\bar f(\tilde\Theta_a) = \bar f([\tilde H_a|G_a])\cdot \bar f([D(a)|\tilde H_a])^{-1} = c(1+a,a)c(a,1+a)^{-1}\\
&= c(a^2,1+a) = c((-a)^2,1+a) = c(-a,1+a)c(1+a,-a) = 1.
\end{aligned}$$

□

8. Applications: generators for $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/m]),\mathbb{Z})$

The general principle is the following:

Lemma 8.1. *Let $m = q_1q_2\cdots q_n$ where $q_1,\ldots,q_n$ are distinct primes. Suppose that the positive integers $u_2,\ldots,u_n \in \mathbb{Z}[1/m]^\times$ satisfy the conditions*

(1) *u_i is a primitive root modulo q_i for $i \geq 2$,*
(2) *When $i \neq j \in \{2,\ldots,n\}$,*

$$q_i^{v_{q_j}(u_i)} \cong 1 \pmod{q_j}.$$

Then there is a direct sum decomposition

$$\tilde{K}_2(2,\mathbb{Z}[1/m]) \cong \mathbb{Z}\oplus\mathbb{Z}/(q_2-1)\oplus\cdots\oplus\mathbb{Z}/(q_n-1)$$

with the property that infinite cyclic factor is generated by $c(-1,-1)$ and the factor $\mathbb{Z}/(q_i-1)$ is generated by $c(u_i,q_i)$.

Proof. The isomorphism

$$\tilde{K}_2(2,\mathbb{Z}[1/m]) \cong \mathbb{Z}\oplus\left(\oplus_{i=2}^n\mathbb{F}_{q_i}^\times\right)$$

is induced by the map

$$\sigma:\tilde{K}_2(2,\mathbb{Z}[1/m])\to\mathbb{Z},\quad c(a,b)\mapsto\begin{cases}1, & a<0 \text{ and } b<0\\ 0, & \text{otherwise}\end{cases}$$

and the tame symbols $T_{p_i}: K(2,\mathbb{Q})\to\mathbb{F}_{p_i}^\times$.

Now

$$T_{p_i}(c(u_i,q_i)) = \tau_{q_i}(\{u_i,q_i\}) = u_i \pmod{q_i} = w_i$$

while for $j\neq i$

$$T_{q_j}(c(u_i,q_i)) = q_i^{v_{q_j}(u_i)} \pmod{q_j} \equiv 1 \pmod{q_j}.$$

□

Remark 8.2. It is not known whether there must exist units satisfying condition (1) in general, but exceptions, if they exist, are rare.

If units u_i are found satisfying condition (1), then it can always be arranged for condition (2) to hold; namely, multiply u_i by a high power of m_i where $m_i = (\prod_{k=1}^n q_k)/q_i$.

Combining Lemma 8.1 with Theorems 6.12 and 7.8, we deduce:

Corollary 8.3. *Let $m = q_1\cdots q_n$ be distinct primes satisfying $q_1<q_2<\cdots<q_n$ and $q_1 = 2, q_2 = 3$. Let $u_2,\ldots,u_n$ be as in Lemma 8.1. There is a direct sum decomposition*

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/m]),\mathbb{Z}) \cong \mathbb{Z}\oplus(\oplus_{i=2}^n\mathbb{Z}/(q_i-1)\mathbb{Z})$$

where the first summand corresponds to the subgroup of $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/m]),\mathbb{Z})$ generated by the homology class $C(-1,-1)$, and the summand $\mathbb{Z}/(q_i-1)\mathbb{Z}$ corresponds to the subgroup generated by the homology class $C(u_i,q_i)$.

Example 8.4. In the case $m = 6$, we can take $u_2 = 2$. We deduce that the cyclic factors of

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/6]),\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}/2$$

are generated by the homology classes $C(-1,-1)$ and $C(2,3)$.

Example 8.5. In the case $m = 30$, then the units $u_2 = 2$, $u_3 = 2$ satisfy the necessary congruences. Thus the cyclic factors of

$$\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}[1/30]), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$$

are generated by the homology classes $C(-1,-1)$, $C(2,3)$ and $C(2,5)$.

Example 8.6. By Theorem 6.12, we have

$$\mathrm{H}_2(\mathbb{Z}[1/42], \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{F}_3^\times \oplus \mathbb{F}_7^\times \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6.$$

The first factor is generated by the homology class $C(-1,-1)$. Furthermore, $u_2 = 2 = u_3$ satisfy the congruences of Lemma 8.1. It follows that the homology classes $C(2,3)$ and $C(2,7)$ generate the second and third cyclic factors.

Example 8.7. Let ω be a primitive cube root of unity and let p be a rational prime which is congruent to 1 modulo 3. Let $\mathcal{O} = \mathbb{Z}[\omega, \frac{1}{3p}]$.

Observe that $\omega \in \mathcal{O}^\times$ and $\omega^2 - 1 = \sqrt{-3}\omega \in \mathcal{O}^\times$ also. Then $p\mathbb{Z}[\omega] = \mathfrak{p}_1\mathfrak{p}_2$ where $k(\mathfrak{p}_i) \cong \mathbb{F}_p$ for $i = 1, 2$. Since $K_2(\mathbb{Z}[\omega]) = 0$, we have

$$\begin{aligned} K_2(\mathcal{O}) \cong \tilde{K}_2(2, \mathcal{O}) &\cong k(\mathfrak{p}_1)^\times \oplus k(\mathfrak{p}_2)^\times \oplus k(\mathfrak{q})^\times \\ &\cong \mathbb{F}_p^\times \oplus \mathbb{F}_p^\times \oplus \mathbb{F}_3^\times \end{aligned}$$

where $\mathfrak{q} = \sqrt{-3}\mathbb{Z}[\omega]$.

By Lemma 6.8 and Theorem 7.8 the natural map

$$\mathrm{H}_2(\mathrm{SL}_2(\mathcal{O}), \mathbb{Z}) \to K_2(\mathcal{O})$$

is surjective and the homology class $C(-\omega, p)$ maps, via the tame symbol, to the element $-\bar{\omega} \in k(\mathfrak{p}_i)^\times$ of order 6, while the class $C(3, p)$ maps to $\bar{3} \in k(\mathfrak{p}_i)^\times \cong \mathbb{F}_p^\times$.

References

[1] Alejandro Adem and Nadim Naffah. On the cohomology of $\mathrm{SL}_2(\mathbf{Z}[1/p])$. In *Geometry and cohomology in group theory (Durham, 1994)*, volume 252 of *London Math. Soc. Lecture Note Ser.*, pages 1–9. Cambridge Univ. Press, Cambridge, 1998. 21, 25

[2] Kevin Hutchinson. Low-dimensional homology of SL_2 of Laurent polynomials. *arXiv:1404.5896*. 21, 23

[3] Kevin Hutchinson. A Bloch-Wigner complex for SL_2. *J. K-Theory*, 12(1):15–68, 2013. 19, 20

[4] Kevin Hutchinson and Liqun Tao. Homology stability for the special linear group of a field and Milnor-Witt K-theory. *Doc. Math.*, (Extra Vol.):267–315, 2010. 8

[5] Bernhard Liehl. On the group SL_2 over orders of arithmetic type. *J. Reine Angew. Math.*, 323:153–171, 1981. 3, 4

[6] Hideya Matsumoto. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. *Ann. Sci. École Norm. Sup. (4)*, 2:1–62, 1969. 7

[7] A. Mazzoleni. A new proof of a theorem of Suslin. *K-Theory*, 35(3-4):199–211 (2006), 2005. 8

[8] J. Milnor. *Introduction to Algebraic K-Theory*. Annals of Mathematics Studies No. 72. Princeton University Press, 1971. 1

[9] John Milnor. Algebraic K-theory and quadratic forms. *Invent. Math.*, 9:318–344, 1969/1970. 8

[10] Calvin C. Moore. Group extensions of p-adic and adelic linear groups. *Inst. Hautes Études Sci. Publ. Math.*, (35):157–222, 1968. 7

[11] Fabien Morel. An introduction to $\mathbb{A}^1$-homotopy theory. In *Contemporary developments in algebraic K-theory*, ICTP Lect. Notes, XV, pages 357–441 (electronic). Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004. 8

[12] Fabien Morel. Sur les puissances de l'idéal fondamental de l'anneau de Witt. *Comment. Math. Helv.*, 79(4):689–703, 2004. 8

[13] Jun Morita. On the group structure of rank one K_2 of some $\mathbf{Z}_S$. *Bull. Soc. Math. Belg. Sér. A*, 42(3):561–575, 1990. Algebra, groups and geometry. 2, 24

[14] Jean-Pierre Serre. Le problème des groupes de congruence pour SL2. *Ann. of Math. (2)*, 92:489–527, 1970. 3, 10, 18

[15] Jean-Pierre Serre. *Trees*. Springer-Verlag, Berlin, 1980. Translated from the French by John Stillwell. 13

[16] Robert Steinberg. *Lectures on Chevalley groups*. Yale University, New Haven, Conn., 1968. Notes prepared by John Faulkner and Robert Wilson. 7

[17] A. A. Suslin. Torsion in K_2 of fields. *K-Theory*, 1(1):5–29, 1987. 8

[18] Richard G. Swan. Generators and relations for certain special linear groups. *Bull. Amer. Math. Soc.*, 74:576–581, 1968. 4

[19] Bui Anh Tuan and Graham Ellis. The homology of $SL_2(\mathbb{Z}[1/m])$ for small m. *J. Algebra*, 408:102–108, 2014. 21, 26

[20] Wilberd van der Kallen. Stability for K_2 of Dedekind rings of arithmetic type. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 217–248. Springer, Berlin, 1981. 24

[21] L. N. Vaseršteĭn. The group SL_2 over Dedekind rings of arithmetic type. *Mat. Sb. (N.S.)*, 89(131):313–322, 351, 1972. 3, 4

School of Mathematical Sciences, University College Dublin

E-mail address: `kevin.hutchinson@ucd.ie`