ON THE THIRD HOMOLOGY OF SL_2 AND WEAK HOMOTOPY INVARIANCE

KEVIN HUTCHINSON AND MATTHIAS WENDT

Abstract. The goal of the paper is to achieve - in the special case of the linear group SL_2 - some understanding of the relation between group homology and its $\mathbb{A}^1$-invariant replacement. We discuss some of the general properties of the $\mathbb{A}^1$-invariant group homology, such as stabilization sequences and Grothendieck-Witt module structures. Together with very precise knowledge about refined Bloch groups, these methods allow us to deduce that in general there is a rather large difference between group homology and its $\mathbb{A}^1$-invariant version. In other words, weak homotopy invariance fails for SL_2 over many families of non-algebraically closed fields.

Contents

1. Introduction

In this paper, we investigate the difference between group homology and its $\mathbb{A}^1$-invariant version. It is well-known that K-theory and hence also the homology of the infinite general linear group GL is $\mathbb{A}^1$-invariant, i.e. for a regular ring R, the map $GL(R) \to GL(R[T])$ given by inclusion of constants induces an isomorphism in homology. This, however, is only a stable phenomenon: examples [KM97] show that H_1 of SL_2 is not $\mathbb{A}^1$-invariant because there exist many non-elementary matrices in $SL_2(R[T])$ if R is not a field. One can nevertheless force group homology of a linear group G to be $\mathbb{A}^1$-invariant by considering the homology of the polynomial singular resolution $BG(k[\Delta^\bullet])$. There is a natural change-of-topology morphism $BG(k) \to BG(k[\Delta^\bullet])$, and we will investigate some of the properties of this morphism in the paper.

The main result is the following failure of weak homotopy invariance (see 2.2 below). We refer to Theorem 7.4 and Theorem 7.6 for precise formulations.

Date: October 16, 2013.

1991 *Mathematics Subject Classification.* 20G10 (14F42).

Key words and phrases. weak homotopy invariance, group homology.

Theorem 1. *(1) For k an infinite finitely-generated field and ℓ an odd prime such that $[k(\zeta_\ell):k]$ is even, the kernel of the change-of-topology morphism*

$$H_3(SL_2(k),\mathbb{Z}/\ell)\to H_3(BSL_2(k[\Delta^\bullet]),\mathbb{Z}/\ell)$$

is not finitely generated. In particular, weak homotopy invariance with finite coefficients can fail for fields which are not algebraically closed.

(2) For k a field complete with respect to a discrete valuation, with finite residue field $\overline{k}$ of order $q=p^f$ (p odd), the change-of-topology morphism

$$H_3(SL_2(k),\mathbb{Z}[1/2])\to H_3(BSL_2(k[\Delta^\bullet]),\mathbb{Z}[1/2])$$

factors through $K_3^{\mathrm{ind}}(k)\otimes\mathbb{Z}[1/2]$, and its kernel is isomorphic to the pre-Bloch group $\mathcal{P}(\overline{k})\otimes\mathbb{Z}[1/2]$ which is cyclic of order $(q+1)'$ (where n' denotes the odd part of the positive integer n).

There are several ingredients coming together. On the side of $\mathbb{A}^1$-invariant group homology, we can use $\mathbb{A}^1$-homotopy theory to produce stabilization results and Grothendieck-Witt module structures. On the other hand, the computations in [Hut11a, Hut11b, Hut13] allow to understand very explicitly the structure of $H_3(SL_2(k))$. In particular, the above theorem follows from a comparison of the $k^\times/(k^\times)^2$-module structures on both sides: the module structure on the $\mathbb{A}^1$-invariant group homology factors through an action of the Grothendieck-Witt ring of k but the corresponding statement for $H_3(SL_2(k))$ is false for fields with non-trivial valuations. In the case of number fields there is a further contrast between the two sides: the $\mathbb{A}^1$-invariant group homology is finitely generated, while the residue maps of [Hut11a] show that the group homology itself is not finitely generated.

There are some negative consequences of our findings. As mentioned before, the results strengthen our understanding that homotopy invariance of algebraic K-theory is a strictly stable phenomenon and tends to fail badly in unstable settings. Our results also have consequences for "unstable K-theories". On the one hand, one could use a Quillen-style definition via homotopy groups of the plus construction of the classifying space: $\pi_\bullet BSL_n(k)^+$. On the other hand, one could use the Karobi-Villamayor-style definition using the simplicial polynomial resolution: $\pi_\bullet BSL_n(k[\Delta^\bullet])$. It is well-known (and primarily a consequence of homotopy invariance of algebraic K-theory) that these definitions agree in the limit $n\to\infty$. Our results show that they do not agree for $n=2$. In particular, there will never be a unique natural definition of unstable K-theory. In a similar spirit, the failure of weak homotopy invariance also implies that it is not possible to use $\mathbb{A}^1$-homotopy theory to prove results on the homology of linear groups over arbitrary fields. However, our results leave open the possibility that group homology coincides with its $\mathbb{A}^1$-invariant version over separably closed fields: the change-of-topology morphism is injective for quadratically closed fields, cf. Corollary 4.3. Weak homotopy invariance with finite coefficients over separably closed fields is a necessary ingredient in Morel's approach to the Friedlander-Milnor conjecture [Mor11].

We would like to mention some questions that could be pursued in further research: first, the relation between homotopy invariance and weak homotopy invariance deserves further study. The behaviour of the spectral sequence associated to the bisimplicial set $BSL_2(k[\Delta^\bullet])$ plays a major role, and relates disparate phenomena like counterexamples to homotopy invariance, scissors congruence groups and surjective stabilization for symplectic groups. Second, it would be interesting to see what happens to (weak) homotopy invariance for homology groups beyond the metastable range as well as for higher rank groups.

Structure of the paper: We review the definition and basic properties of the $\mathbb{A}^1$-invariant version of group homology in Section 2. In Section 3 we prove some stabilization results which are need in Section 4 and Section 7. In Section 4 we review the relationship between $H_3(SL_2(k),\mathbb{Z})$ and $K_3^{\mathrm{ind}}(k)$ and apply these results to the change-of-topology morphism in the case of quadratically closed fields of characteristic 0. The Grothendieck-Witt module structures on $\mathbb{A}^1$-invariant homology of SL_2 are investigated in Section 5. Some details on (refined) Bloch groups and their module structures are provided in Section 6. In Section 7, we use the results of the preceding sections to prove our main results: that the kernel of the change-of-topology morphism is often very large. In Section 8 we conclude with some remarks on the cokernel of the change-of-topology morphism.

Acknowledgements: We would like to thank Aravind Asok for some discussions on the computations in [AF12a], and Jens Hornbostel and Marco Schlichting for some discussions on finiteness properties of symplectic K-theory of number fields.

2. Group homology made $\mathbb{A}^1$-invariant: definition

In this section, we recall the construction which enforces $\mathbb{A}^1$-invariance in group homology, i.e. replaces group homology by something representable in $\mathbb{A}^1$-homotopy theory. The crucial definition is the singular resolution of a linear algebraic group, cf. [Jar83]:

Definition 2.1. *Let k be a field. There is a standard simplicial k-algebra $k[\Delta^\bullet]$ with n-simplices given by*

$$k[\Delta^n] = k[X_0,\ldots,X_n]/(\sum X_i - 1)$$

and face and degeneracy maps given by

$$d_i(X_j) = \begin{cases} X_j & j < i \\ 0 & j = i \\ X_{j-1} & j > i \end{cases}, \qquad s_i(X_j) = \begin{cases} X_j & j < i \\ X_i + X_{i+1} & j = i \\ X_{j+1} & j > i \end{cases}.$$

To a linear algebraic group G, we can associate a simplicial group $G(k[\Delta^\bullet])$ which can be considered as a topologized version of the discrete group $G(k)$.

Recall that the classifying space of the simplicial group $G(k[\Delta^\bullet])$ is defined to be the diagonal of the bisimplicial set $BG(k[\Delta^\bullet])$. One simplicial direction is given by the usual classifying space construction, the other one is given by the simplicial algebra $k[\Delta^\bullet]$ above.

Definition 2.2. *We call the homology of $BG(k[\Delta^\bullet])$ the* group homology made $\mathbb{A}^1$-invariant. *There is a natural inclusion $G(k) \hookrightarrow G(k[\Delta^\bullet])$ which identifies $G(k)$ with the set of zero-simplices $G(k[X_0]/(X_0-1))$ in $G(k[\Delta^\bullet])$. We refer to this as the natural change-of-topology morphism. We say that the group G has* weak homotopy invariance in degree n over the field k with M-coefficients, *if the change-of-topology map induces an isomorphism*

$$H_n(G(k),M) \xrightarrow{\cong} H_n(BG(k[\Delta^\bullet]),M).$$

The change-of-topology morphism and in particular its effect on third homology is the centre of interest in the present work. We will only consider constant coefficients.

Remark 2.3. *Recall that in $\mathbb{A}^1$-homotopy theory, one can associate a singular resolution $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(X)$ to any simplicial sheaf X on the site Sm_k of smooth schemes over a field k, essentially by setting*

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(X)(U) = \mathrm{Hom}(U \times \Delta^\bullet, X).$$

Because a field is a local henselian ring, the simplicial group $G(k[\Delta^\bullet])$ above is the simplicial group of sections of (the Nisnevich sheafification of) $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)$ over

$\operatorname{Spec} k$. *The corresponding classifying space is the simplicial set of sections of the corresponding simplicial (pre-)sheaf* $B\operatorname{Sing}_\bullet^{\mathbb{A}^1}(G)$ *over* $\operatorname{Spec} k$.

The singular resolution of algebraic groups is almost fibrant in $\mathbb{A}^1$-homotopy theory. We collect this statement in the following proposition:

Proposition 2.4. *Let G be an isotropic reductive group of rank ≥ 2 or SL_2 over a perfect base field k. Then the resolution $\operatorname{Sing}_\bullet^{\mathbb{A}^1}(G)$ has the affine Brown-Gersten property in the Nisnevich topology. In particular, there are isomorphisms*

$$\pi_i\left(G(k[\Delta^\bullet])\right) \to \pi_i^{\mathbb{A}^1}(G)(\operatorname{Spec} k).$$

Proof. This proposition is a consequence of the theory of the affine Brown-Gersten property and $\mathbb{A}^1$-invariance of unstable K_1-functors. For the affine Brown-Gersten property, cf. [Mor12, Appendix A] resp. [Wen10, Section 3]. The $\mathbb{A}^1$-invariance of unstable K_1-functors has been established by Stavrova [Sta11]. The part of Proposition 2.4 dealing with isotropic reductive groups of rank ≥ 2 has been formulated in [VW12, Proposition 4.1]. The SL_2-case of Proposition 2.4 has been proven by Moser, cf. [Mos].

Note that in the central theorem on the affine Brown-Gersten property [Mor12, Appendix A], the base field is assumed to be perfect. However, the isomorphism

$$\pi_i\left(G(k[\Delta^\bullet])\right) \to \pi_i^{\mathbb{A}^1}(G)(\operatorname{Spec} k).$$

applies to all finitely generated field extension of the base field. In particular, there is no perfectness assumption necessary for the special groups Sp_{2n} which are defined over $\mathbb{Z}$. ☐

For the rest of the paper, we will mostly need this in the case of SL_2 and the symplectic groups Sp_{2n}. The result makes it possible to deduce statements about $H_\bullet(BG(k[\Delta^\bullet]))$ from computations in $\mathbb{A}^1$-homotopy theory.

3. GROUP HOMOLOGY MADE $\mathbb{A}^1$-INVARIANT: STABILIZATION

In this section, we will develop some stabilization results for the $\mathbb{A}^1$-invariant homology of symplectic groups. These will be helpful in understanding the change-of-topology morphism.

3.1. Fibre sequences in $\mathbb{A}^1$-homotopy theory.

We first discuss results producing fibre sequences in $\mathbb{A}^1$-homotopy theory as well as the corresponding exact stabilization sequences. The results are due to Morel [Mor12, Theorem 8.1, Proposition 8.12] and Wendt [Wen11].

Proposition 3.1. *Let k be an infinite field, and let G be a smooth split reductive group. Then the classifying space $B\operatorname{Sing}_\bullet^{\mathbb{A}^1}(G)$ of rationally trivial G-torsors is $\mathbb{A}^1$-local. For every rationally trivial G-torsor $p: E \to B$ there is a fibre sequence of simplicial sheaves*

$$\operatorname{Sing}_\bullet^{\mathbb{A}^1}(G) \to \operatorname{Sing}_\bullet^{\mathbb{A}^1}(E) \to \operatorname{Sing}_\bullet^{\mathbb{A}^1}(B),$$

which is $\mathbb{A}^1$-local. In particular, there is a long exact sequence

$$\cdots \to \pi_i^{\mathbb{A}^1}(G)(\operatorname{Spec} k) \to \pi_i^{\mathbb{A}^1}(E)(\operatorname{Spec} k) \to \pi_i^{\mathbb{A}^1}(B)(\operatorname{Spec} k) \to \cdots$$

Proof. The locality of the classifying space is established in [Mor12, Theorem 8.1] and [Mor11]. The fibre sequence statement is [Wen11, Corollary 1.1]. The long exact sequence is a direct consequence of that. ☐

The next result recalls the computation of low-dimensional homotopy groups of spheres due to Morel, cf. [Mor12, Theorem 6.40, Theorem 8.9].

Proposition 3.2. *The space* $\mathbb{A}^n \setminus \{0\}$ *is* $\mathbb{A}^1$-$(n-2)$*-connected and*

$$\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})(\operatorname{Spec} k) \cong K_n^{MW}(k).$$

The space $\mathbb{A}^n \setminus \{0\}$ *has the affine Brown-Gersten property, in particular, the simplicial set* $\operatorname{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})$ *is* $(n-2)$*-connected and*

$$\pi_{n-1}(\operatorname{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})(\operatorname{Spec} k)) \cong K_n^{MW}(k).$$

Denote by $q_{2n} = \sum_{i=1}^n X_iY_i$ the split quadratic form in $2n$ variables and by $Q_{2n-1} = V(q_{2n})$ the $(2n-1)$-dimensional split affine quadric. It is well-known that the projection morphism

$$Q_{2n-1} \to \mathbb{A}^n \setminus \{0\} : (X_1, \dots, X_n, Y_1, \dots, Y_n) \mapsto (Y_1, \dots, Y_n)$$

is Zariski locally trivial with fibres $\mathbb{A}^n$, hence it is an $\mathbb{A}^1$-weak equivalence. The results stated in Proposition 3.2 above apply verbatim to Q_{2n-1}. While $\mathbb{A}^n \setminus \{0\}$ is the space appearing in the $\mathbb{A}^1$-homotopy discussions in [Mor12, Theorem 8.9], the latter is the space naturally relevant for stabilization theorems.

3.2. **A relative Hurewicz argument.**

The next result contains the central argument which yields the stabilization results.

Theorem 3.3. *Let* k *be an infinite field,* $H \hookrightarrow G$ *an inclusion of linear algebraic groups over* k*.*

Assume that H *is isotropic, that the* H*-torsor* $G \to G/H$ *is rationally trivial and that* $\operatorname{Sing}_\bullet^{\mathbb{A}^1} G/H(k)$ *is* n*-connected. Then the induced morphisms*

$$H_i(BH(k[\Delta^\bullet]), \mathbb{Z}) \to H_i(BG(k[\Delta^\bullet]), \mathbb{Z})$$

are isomorphisms for all $i = 0, \dots, n$*. Moreover, there is an exact sequence*

$$H_{n+2}(BH(k[\Delta^\bullet]), \mathbb{Z}) \to H_{n+2}(BG(k[\Delta^\bullet]), \mathbb{Z}) \to H_{n+2}^{rel} \to$$
$$\to H_{n+1}(BH(k[\Delta^\bullet]), \mathbb{Z}) \to H_{n+1}(BG(k[\Delta^\bullet]), \mathbb{Z}) \to 0,$$

where H_{n+2}^{rel} *is obtained from* $\pi_{n+1}(\operatorname{Sing}_\bullet^{\mathbb{A}^1} G/H(k))$ *by factoring out the action of* $\pi_1(BH(k[\Delta^\bullet]))$*.*

Proof. For any closed subgroup H of G, the quotient G/H exists and is a quasi-projective variety. Note that our assumption is that the H-torsor $G \to G/H$ is locally trivial in the Nisnevich topology.

We choose the base points of $\operatorname{Sing}_\bullet^{\mathbb{A}^1} H$ and $\operatorname{Sing}_\bullet^{\mathbb{A}^1} G$ to be the identity, and the base point of $\operatorname{Sing}_\bullet^{\mathbb{A}^1} G/H$ to be the image of the identity in $\operatorname{Sing}_\bullet^{\mathbb{A}^1} G$. The fibre over this point is H, and from Proposition 3.1 we obtain a fibre sequence of simplicial sets

$$H(k[\Delta^\bullet]) \to G(k[\Delta^\bullet]) \to \operatorname{Sing}_\bullet^{\mathbb{A}^1}(G/H)(k).$$

From the standard theory of classifying spaces for simplicial groups, we obtain another fibre sequence of simplicial sets

$$\operatorname{Sing}_\bullet^{\mathbb{A}^1}(G/H)(k) \to BH(k[\Delta^\bullet]) \to BG(k[\Delta^\bullet]).$$

The associated long exact homotopy sequence for this fibre sequence is

$$\cdots \to \pi_n(BH(k[\Delta^\bullet])) \to \pi_n(BG(k[\Delta^\bullet])) \to$$
$$\to \pi_{n-1}(\operatorname{Sing}_\bullet^{\mathbb{A}^1} G/H(k)) = \pi_n(BG(k[\Delta^\bullet]), BH(k[\Delta^\bullet)) \to$$
$$\to \pi_{n-1}(BH(k[\Delta^\bullet])) \to \pi_{n-1}(BG(k[\Delta^\bullet])) \to \cdots$$

The assumption that $\operatorname{Sing}_\bullet^{\mathbb{A}^1} G/H(k)$ is n-connected is equivalent to the assumption that the pair $(BG(k[\Delta^\bullet]), BH(k[\Delta^\bullet]))$ is $(n+1)$-connected.

Now we consider the long exact relative homology sequence for the pair

$$(BG(k[\Delta^\bullet]), BH(k[\Delta^\bullet])).$$

By construction, the spaces $BG(k[\Delta^\bullet])$ and $BH(k[\Delta^\bullet])$ are connected, and we saw above that the pair $(BG(k[\Delta^\bullet]), BH(k[\Delta^\bullet]))$ is $(n+1)$-connected. By the relative Hurewicz theorem, we find that $\widetilde{H}_i(BG(k[\Delta^\bullet]), BH(k[\Delta^\bullet])) = 0$ for $i < n+2$ and $H_{n+2}(B\operatorname{Sing}_\bullet^{\mathbb{A}^1} G(k), B\operatorname{Sing}_\bullet^{\mathbb{A}^1} H(k))$ is obtained from

$$\pi_{n+2}(B\operatorname{Sing}_\bullet^{\mathbb{A}^1} G(k), B\operatorname{Sing}_\bullet^{\mathbb{A}^1} H(k)) = \pi_{n+1}(\operatorname{Sing}_\bullet^{\mathbb{A}^1} G/H(k))$$

by factoring out the action of $\pi_1(B\operatorname{Sing}_\bullet^{\mathbb{A}^1} H(k))$.

From this, all the claims in the theorem follow by invoking the relative Hurewicz theorem [GJ99, Corollary III.3.12]. □

3.3. **Stabilization theorems.** We first provide an analogue of the stabilization results of Hutchinson and Tao [HT10] for the homology of SL_n made $\mathbb{A}^1$-invariant:

Proposition 3.4. *Let k be an infinite field. Then the homomorphisms*

$$H_i(BSL_{n-1}(k[\Delta^\bullet]), \mathbb{Z}) \to H_i(BSL_n(k[\Delta^\bullet]), \mathbb{Z})$$

induced by the standard inclusion $SL_{n-1}(k) \hookrightarrow SL_n(k)$ are isomorphisms for $i \leq n-2$. There is an exact sequence

$$H_n(BSL_{n-1}(k[\Delta^\bullet]), \mathbb{Z}) \to H_n(BSL_n(k[\Delta^\bullet]), \mathbb{Z}) \to K_n^{MW}(k) \to$$

$$\to H_{n-1}(BSL_{n-1}(k[\Delta^\bullet]), \mathbb{Z}) \to H_{n-1}(BSL_n(k[\Delta^\bullet]), \mathbb{Z}) \to 0.$$

Proof. The special linear groups are special in the sense of Serre, so any SL_n-torsor is already Zariski-locally trivial. In particular, we can apply Theorem 3.3. The quotient SL_n/SL_{n-1} is classically identified with $\mathbb{A}^n \setminus \{0\}$, and by Proposition 3.2, the space $\operatorname{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})$ is $(n-2)$-connected. This implies the claim about the isomorphisms. For the exact sequence, we still need to identify

$$H_n^{rel} = \pi_{n-1}(\operatorname{Sing}_\bullet^{\mathbb{A}^1}(SL_n/SL_{n-1})(k))/\pi_1(BSL_{n-1}(k[\Delta^\bullet])).$$

But $BSL_{n-1}(k[\Delta^\bullet])$ is simply-connected because the group $SL_{n-1}(k[\Delta^\bullet])$ is connected. Therefore, using Proposition 3.2, we find that H_n^{rel} can be identified with $K_n^{MW}(k)$. □

As an aside, we state a restricted $\mathbb{A}^1$-homotopy version of the stabilization result for GL_n due to Nesterenko and Suslin, cf. [NS90].

Proposition 3.5. *Let k be an infinite field. Then the homomorphisms*

$$H_i(BGL_{n-1}(k[\Delta^\bullet]), \mathbb{Z}) \to H_i(BGL_n(k[\Delta^\bullet]), \mathbb{Z})$$

induced by the standard inclusion $GL_{n-1} \hookrightarrow GL_n$ are isomorphisms for $i \leq n-2$. There is an exact sequence

$$H_n(BGL_{n-1}(k[\Delta^\bullet]), \mathbb{Z}) \to H_n(BGL_n(k[\Delta^\bullet]), \mathbb{Z}) \to K_n^{M}(k) \to$$

$$\to H_{n-1}(BGL_{n-1}(k[\Delta^\bullet]), \mathbb{Z}) \to H_{n-1}(BGL_n(k[\Delta^\bullet]), \mathbb{Z}) \to 0,$$

Proof. The argument is the same as for the SL_n-case, with one exception. We have

$$\pi_1(B\operatorname{Sing}_\bullet^{\mathbb{A}^1} GL_n(k)) = H_1(GL_n(k), \mathbb{Z}) = k^\times.$$

The operation of $k^\times$ on $\mathbb{A}^n \setminus \{0\}$ is the one induced by the embedding of $k^\times \hookrightarrow GL_n(k) : \lambda \mapsto \operatorname{diag}(\lambda, 1, \ldots, 1)$. It is proven in [Mor12, Lemma 3.10] that the abelian group of homomorphisms $K_n^{MW} \to K_n^{MW}$ is isomorphic to $K_0^{MW}(k) =$

$GW(k)$, the Grothendieck-Witt ring of k. The action is then given by a homomorphism $k^\times \to GW(k)$. Moreover, the Brouwer degree of the map $\mathbb{A}^n \setminus \{0\} \to \mathbb{A}^n \setminus \{0\}$ induced by $\lambda \in k^\times$ is exactly the image of λ under the canonical morphism

$$\mathbb{G}_m \to \mathbb{G}_m/2 \to K_0^{MW}(k) = GW(k).$$

A unit $\lambda \in k^\times$ therefore acts via multiplication with the class $[\lambda] \in K_0^{MW}(k)$. The corresponding quotient of $K_n^{MW}(k)$ modulo this action is $K_n^M(k)$. The action of $k^\times$ on $K_n^{MW}(k)$ has also been described in [BM99]. ☐

Next, we consider the symplectic groups. This is the main result we will use in the later development.

Proposition 3.6. *Let k be an infinite field. Then the homomorphisms*

$$H_i(BSp_{2n-2}(k[\Delta^\bullet]), \mathbb{Z}) \to H_i(BSp_{2n}(k[\Delta^\bullet]), \mathbb{Z})$$

induced by the standard inclusion $Sp_{2n-2}(k) \hookrightarrow Sp_{2n}(k)$ are isomorphisms for $i \leq 2n-2$. There is an exact sequence

$$H_{2n}(BSp_{2n-2}(k[\Delta^\bullet]), \mathbb{Z}) \to H_{2n}(BSp_{2n}(k[\Delta^\bullet]), \mathbb{Z}) \to K_{2n}^{MW}(k) \to$$

$$\to H_{2n-1}(BSp_{2n-2}(k[\Delta^\bullet]), \mathbb{Z}) \to H_{2n-1}(BSp_{2n}(k[\Delta^\bullet]), \mathbb{Z}) \to 0.$$

Proof. The symplectic groups are also special in the sense of Serre, so any Sp_{2n}-torsor is already Zariski-locally trivial. In particular, we can apply Theorem 3.3. The quotient Sp_{2n}/Sp_{2n-2} is classically identified with $Q_{4n-1} \simeq \mathbb{A}^{2n} \setminus \{0\}$, and by Proposition 3.2, the space $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{A}^{2n} \setminus \{0\})$ is $(2n-2)$-connected. This implies the claim about the isomorphisms. For the exact sequence, we still need to identify

$$H_{2n}^{rel} = \pi_{n-1}(\mathrm{Sing}_\bullet^{\mathbb{A}^1}(Sp_{2n}/Sp_{2n-2})(k))/\pi_1(BSp_{2n-2}(k[\Delta^\bullet])).$$

But $BSp_{2n-2}(k[\Delta^\bullet])$ is simply-connected because the group $Sp_{2n-2}(k[\Delta^\bullet])$ is connected. Therefore, using Proposition 3.2, we find that H_{2n}^{rel} can be identified with $K_{2n}^{MW}(k)$. ☐

Finally, we mention an $\mathbb{A}^1$-homotopy version of the stabilization results of Cathelineau for the orthogonal groups, cf. [Cat07].

Proposition 3.7. *Let k be an infinite field (of characteristic $\neq 2$). We consider spin resp. special orthogonal groups for hyperbolic forms, which we denote by Spin_n resp. SO_n.*

- *The homomorphisms*

  $$H_i(B\,\mathrm{Spin}_{2n-1}(k[\Delta^\bullet]), \mathbb{Z}) \to H_i(B\,\mathrm{Spin}_{2n}(k[\Delta^\bullet]), \mathbb{Z})$$

  induced by the standard inclusion $\mathrm{Spin}_{2n-2} \hookrightarrow \mathrm{Spin}_{2n}$ are isomorphisms for $i \leq n-2$. The same holds for the groups SO_{2n}.
- *There is an exact sequence*

  $$H_n(B\,\mathrm{Spin}_{2n-1}(k[\Delta^\bullet]), \mathbb{Z}) \to H_n(B\,\mathrm{Spin}_{2n}(k[\Delta^\bullet]), \mathbb{Z}) \to K_n^{MW}(k) \to$$

  $$\to H_{n-1}(B\,\mathrm{Spin}_{2n-1}(k[\Delta^\bullet]), \mathbb{Z}) \to H_{n-1}(B\,\mathrm{Spin}_{2n}(k[\Delta^\bullet]), \mathbb{Z}) \to 0.$$
- *The quotient of $K_n^{MW}(k)$ by the action of*

  $$\pi_1(B\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} SO(k)) \cong k^\times/(k^\times)^2$$

  is isomorphic to $K_n^M(k)$. Therefore, the exact sequences above hold with Spin_n replaced by SO_n and $K_n^{MW}(k)$ replaced by $K_n^M(k)$.

Proof. We have

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1} \mathrm{Spin}_{2n} / \mathrm{Spin}_{2n-1} \simeq \mathbb{A}^n \setminus \{0\}.$$

Again, the spin groups are perfect, which implies that there is no action to be divided out. The more general results of [Wen11] apply to show the existence of an $\mathbb{A}^1$-fibre sequence even without rational triviality assumptions. For the special orthogonal groups, the action of $k^\times/(k^\times)^2$ is again the one induced by the canonical morphism $k^\times/(k^\times)^2 \to K_0^{MW}(k)$, cf. Proposition 3.5. □

Remark 3.8. *The existing methods are not yet sufficient to produce stabilization results for the other inclusion* $\mathrm{Spin}_{2n} \hookrightarrow \mathrm{Spin}_{2n+1}$. *The quotient there is an even-dimensional quadric which is not yet known to possess the affine Brown-Gersten property.*

3.4. $H_{2n}(Sp_{2n}(k),\mathbb{Z})$ **surjects onto** $K_{2n}^{MW}(k)$**.** For fields k of characteristic not equal to 2, it is shown in [HT10, Lemma 3.5, Theorem 3.9] that there is a natural homomorphism of graded $\mathbb{Z}[k^\times]$-algebras

$$\sigma_n = T_n \circ \epsilon_n : H_n(SL_n(k),\mathbb{Z}) \to K_n^{MW}(k), \quad n \geq 0.$$

Here the algebra structure on $(H_n(SL_n(k),\mathbb{Z}))_{n\geq 0}$ comes from the external product

$$H_n(SL_n(k),\mathbb{Z}) \otimes H_m(SL_m(k),\mathbb{Z}) \to H_{n+m}(SL_{n+m}(k),\mathbb{Z}))$$

induced by the block matrix homomorphism $SL_n(k) \times SL_n(k) \to SL_{n+m}(k)$.

Thus, for any $n \geq 0$, the inclusion $Sp_{2n}(k) \to SL_{2n}(k)$ induces a natural map $H_{2n}(Sp_{2n}(k),\mathbb{Z}) \to K_{2n}^{MW}(k)$.

Lemma 3.9. *For any field k of characteristic not equal to 2, the natural map $H_{2n}(Sp_{2n}(k),\mathbb{Z}) \to K_{2n}^{MW}(k)$ is surjective.*

Proof. Since $Sp_2(k) = SL_2(k)$, the map $H_2(Sp_2(k),\mathbb{Z}) \to K_2^{MW}(k)$ is an isomorphism, cf. [HT10, Theorem 3.10]. There are natural group homomorphisms $Sp_{2n}(k) \times Sp_{2m}(k) \to Sp_{2n+2m}(k)$, and hence, for any n a homomorphism

$$\underbrace{Sp_2(k) \times \cdots \times Sp_2(k)}_{n} \to Sp_{2n}(k).$$

These maps induce a commutative diagram

$$\begin{tikzcd}
H_2(Sp_2(k),\mathbb{Z}) \otimes \cdots \otimes H_2(Sp_2(k),\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H_{2n}(Sp_{2n}(k),\mathbb{Z}) \arrow[d] \\
H_2(SL_2(k),\mathbb{Z}) \otimes \cdots \otimes H_2(SL_2(k),\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H_{2n}(SL_{2n}(k),\mathbb{Z}) \arrow[d] \\
K_2^{MW}(k) \otimes \cdots \otimes K_2^{MW}(k) \arrow[r] & K_{2n}^{MW}(k)
\end{tikzcd}$$

Finally, since $K_n^{MW}(k)$ is additively generated by products $[a_1]\cdots[a_n]$, $[a_i] \in K_1^{MW}(k)$, the lowest horizontal arrow in this diagram is a surjection. □

3.5. **Injective stabilization for symplectic groups.** We will provide an improvement of the stabilization result Proposition 3.6 for the symplectic groups, using the surjectivity statement of Lemma 3.9. The main ingredient is the following comparison between stabilization morphisms for discrete and simplicial groups.

Proposition 3.10. *Let k be a field of characteristic 0.*

(1) There is a commutative diagram

$$\begin{tikzcd}
H_n(GL_n(k),\mathbb{Z}) \arrow[rr] \arrow[dr, "\epsilon_n"'] & & H_n(BGL_n(k[\Delta^\bullet]),\mathbb{Z}) \arrow[dl] \\
& K_n^M(k), &
\end{tikzcd}$$

where the top morphism is the change of topology, the left descending morphism is the one of [NS90] *and the right descending morphism is the one from the stabilization sequence of Proposition 3.5.*

(2) *There is a commutative diagram*

$$\begin{tikzcd}
H_n(SL_n(k),\mathbb{Z}) \arrow[rr] \arrow[dr, "T_n\circ\epsilon_n"'] & & H_n(BSL_n(k[\Delta^\bullet]),\mathbb{Z}) \arrow[dl] \\
& K_n^{MW}(k), &
\end{tikzcd}$$

where the top morphism is the change of topology, the left descending morphism is the one of [HT10] *and the right descending morphism is the one from the stabilization sequence of Proposition 3.4.*

(3) *There is a commutative diagram*

$$\begin{tikzcd}
H_{2n}(Sp_{2n}(k),\mathbb{Z}) \arrow[rr] \arrow[dr, "T_n\circ\epsilon_n\circ\iota"'] & & H_{2n}(BSp_{2n}(k[\Delta^\bullet]),\mathbb{Z}) \arrow[dl] \\
& K_{2n}^{MW}(k), &
\end{tikzcd}$$

where the top morphism is the change of topology, the left descending morphism is the one of Lemma 3.9 and the right descending morphism is the one from the stabilization sequence of Proposition 3.6.

Proof. We first show that (2) implies (3). Consider the following commutative diagram:

$$\begin{tikzcd}
H_{2n}(Sp_{2n}(k),\mathbb{Z}) \arrow[r] \arrow[d] & H_{2n}(BSp_{2n}(k[\Delta^\bullet]),\mathbb{Z}) \arrow[r] \arrow[d] & K_{2n}^{MW}(k) \arrow[d, "="] \\
H_{2n}(SL_{2n}(k),\mathbb{Z}) \arrow[r] & H_{2n}(BSL_{2n}(k[\Delta^\bullet]),\mathbb{Z}) \arrow[r] & K_{2n}^{MW}(k)
\end{tikzcd}$$

The right square is part of a commutative ladder of stabilization sequences arising as in [AF12a, Section 3] from the inclusion $Sp_{2n} \hookrightarrow SL_{2n}$ and the subsequent isomorphism of quotients $SL_{2n}/SL_{2n-1} \cong Sp_{2n}/Sp_{2n-2}$. The left square is simply induced by the respective inclusions of groups and change-of-topology maps. Our goal is to show that the top composition $H_{2n}(Sp_{2n}(k),\mathbb{Z}) \to K_{2n}^{MW}(k)$ is the map from Lemma 3.9. But this is a consequence of the commutativity and (1).

To show (2), we use the stabilization results above, in a manner analogous to the arguments in [AF12b, Section 3, Lemma 3.4]. There is a case distinction, based on the parity of n. In the case $n = 2i+1$, we consider the following diagram:

$$\begin{tikzcd}
& K_{2i+2}^{MW}(k) \arrow[d, "\alpha"] \arrow[dr, "0"] & \\
H_{2i+1}(SL_{2i+1}(k),\mathbb{Z}) \arrow[r, "\iota_{2i+1}"'] \arrow[d, "\cong"'] & H_{2i+1}(BSL_{2i+1}(k[\Delta^\bullet]),\mathbb{Z}) \arrow[r, "\sigma"'] \arrow[d] & K_{2i+1}^{MW}(k) \\
H_{2i+1}(SL_{2i+2}(k),\mathbb{Z}) \arrow[r, "\iota_{2i+2}"'] & H_{2i+1}(BSL_{2i+2}(k[\Delta^\bullet]),\mathbb{Z}) \arrow[ur, dotted] &
\end{tikzcd}$$

The left vertical map is the stabilization map of [HT10], which is an isomorphism by [HT10, Corollary 6.12]. The middle column is a part of the stabilization exact sequence of Proposition 3.4. The morphism σ is the map we want to compare with $T_n \circ \epsilon_n$. Note that the map α is the one from the stabilization sequence for the inclusion $SL_{2i+1} \hookrightarrow SL_{2i+2}$, while the morphism σ is the one induced from the stabilization sequence for the inclusion $SL_{2i} \hookrightarrow SL_{2i+1}$. The composition $\sigma \circ \alpha : K^{MW}_{2i+2}(k) \to K^{MW}_{2i+1}(k)$ is then the same as the corresponding composition in the stabilization sequences for $\mathbb{A}^1$-homotopy groups of [Wen11, Theorem 6.8]. This morphism has been identified as 0 in [AF12b, Lemma 3.3], hence σ extends through $H_{2i+1}(BSL_{2i+2}(k[\Delta^\bullet]), \mathbb{Z})$ as claimed. The question reduces to commutativity of the same diagram for $n = \infty$, i.e. the stable case.

In the case $n = 2i$, we consider the following diagram:

$$\begin{tikzcd}
 & K^{MW}_{2i+1}(k) \arrow[dd, "\alpha"] \arrow[dr, dotted, "\beta"] & 0 \arrow[d] \\
I^{2i+1}(k) \arrow[rr, dotted, "="] \arrow[d, "\gamma"] & & I^{2i+1}(k) \arrow[d] \\
H_{2i}(SL_{2i}(k), \mathbb{Z}) \arrow[r, "\iota_{2i}"'] \arrow[d] & H_{2i}(BSL_{2i}(k[\Delta^\bullet]), \mathbb{Z}) \arrow[r, "\sigma"'] \arrow[d] & K^{MW}_{2i}(k) \arrow[d] \\
H_{2i}(SL_{2i+1}(k), \mathbb{Z}) \arrow[r, "\iota_{2i+1}"'] \arrow[d] & H_{2i}(BSL_{2i+1}(k[\Delta^\bullet]), \mathbb{Z}) \arrow[r, dotted] \arrow[d] & K^{M}_{2i}(k) \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

The right vertical column is one of the standard exact sequences for Milnor-Witt K-theory, cf. [Mor04]. The left vertical column is the exact sequence from the Hutchinson-Tao stabilization theorem [HT10]. The middle column is the stabilization exact sequence of Proposition 3.4. The diagram without the dotted arrows is commutative, the only square is obviously commutative.

Note that the map α is the one from the stabilization sequence for the inclusion $SL_{2i} \hookrightarrow SL_{2i+1}$, while the morphism σ is the one induced from the stabilization sequence for the inclusion $SL_{2i-1} \hookrightarrow SL_{2i}$. The composition $\sigma \circ \alpha : K^{MW}_{2i+1}(k) \to K^{MW}_{2i}(k)$ is then the same as the corresponding composition in the stabilization sequences for $\mathbb{A}^1$-homotopy groups of [Wen11, Theorem 6.8]. This morphism has been identified as η in [AF12b, Section 3, Lemma 3.3], hence α factors through β as claimed.

The composition $\sigma \circ \iota_{2i} \circ \gamma$ has been identified in [HT10, Corollary 6.13] as the canonical inclusion $I^{2i+1}(k) \hookrightarrow K^{MW}_{2i}(k)$ arising from $\eta : K^{MW}_{2i+1}(k) \to K^{MW}_{2i}(k)$. Hence, we have the dotted equality arrow in the diagram. This means that on the subgroup $I^{2i+1} \hookrightarrow H_{2i}(SL_{2i}(k), \mathbb{Z})$, the two maps - change of topology plus stabilization and $T_{2i} \circ \epsilon_{2i}$ agree. To show that they agree on all of $H_{2i}(SL_{2i}(k), \mathbb{Z})$, we need to consider the bottom row. The dotted arrow exists, since $K^{MW}_{2i+1}(k) \to K^{MW}_{2i}(k)$ has been identified with η.

There is another map $H_{2i}(SL_{2i+1}(k), \mathbb{Z}) \to K^{M}_{2i}(k)$ induced from $T_{2i} \circ \epsilon_{2i} : H_{2i}(SL_{2i}(k), \mathbb{Z}) \to K^{MW}_{2i}(k)$ modulo I^{2i+1}. It now suffices to identify this map with the bottom composition in the big diagram.

To show that the bottom composition agrees with $T_{2i} \circ \epsilon_{2i}$ modulo I^{2i+1}, it suffices to check this after stabilization to SL_∞. We have a commutative diagram

$$\begin{tikzcd}
H_{2i}(SL_{2i+1}(k),\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H_{2i}(BSL_{2i+1}(k[\Delta^\bullet]),\mathbb{Z}) \arrow[d, "\cong"'] \\
H_{2i}(SL_\infty(k),\mathbb{Z}) \arrow[r, "\cong"'] & H_{2i}(BSL_\infty(k[\Delta^\bullet]),\mathbb{Z})
\end{tikzcd}$$

The left isomorphism is a case of [HT10, Theorem 1.1], the right isomorphism is a case of Proposition 3.4, and the bottom isomorphism is a consequence of homotopy invariance of algebraic K-theory. After stabilization, we see that both maps $H_{2i}(SL_\infty(k),\mathbb{Z}) \to K^M_{2i}(k)$ and $H_{2i}(BSL_\infty(k[\Delta^\bullet]),\mathbb{Z}) \to K^M_{2i}(k)$ factor through Suslin's homomorphism $H_{2i}(BGL_\infty(k[\Delta^\bullet]),\mathbb{Z}) \to K^M_{2i}(k)$ and the respective inclusion.

Finally, (1) is proved in the same way as (2). The result becomes easier as there is no case distinction, we always have a diagram as in (2), case $n = 2i+1$. The argument given there also applies to reduce the claim to the case $n = \infty$.

Now we have reduced all the claims to the commutativity of the diagram

$$\begin{tikzcd}
H_n(GL_n(k),\mathbb{Z}) \arrow[rr] \arrow[dr] & & H_n(BGL_n(k[\Delta^\bullet]),\mathbb{Z}) \arrow[dl] \\
& K^M_n(k), &
\end{tikzcd}$$

where the morphism on the left is Suslin's homomorphism, the top morphism is the change-of-topology and the right morphism is obtained from the $\mathbb{A}^1$-stabilization sequence. This compatibility follows from Suslin's characterization of the homomorphism $H_n(GL_n(k),\mathbb{Z}) \to K^M_n(k)$, cf. [Sus84] or [BM99, Theorem 1.3], and the computations in [AF12b, Section 3, Lemma 3.10].

The result is proved. ☐

The following is now an obvious consequence of Proposition 3.6, Lemma 3.9 and Proposition 3.10.

Corollary 3.11. *Let k be a field of characteristic 0.*

(1) In the stabilization sequence for the special linear groups, cf. Proposition 3.4, the morphism

$$H_{2n}(BSL_{2n}(k[\Delta^\bullet]),\mathbb{Z}) \to K^{MW}_{2n}(k)$$

is surjective. Hence, the standard inclusion $SL_{2n-1} \hookrightarrow SL_{2n}$ induces an isomorphism

$$H_{2n-1}(BSL_{2n-1}(k[\Delta^\bullet]),\mathbb{Z}) \xrightarrow{\cong} H_{2n-1}(BSL_{2n}(k[\Delta^\bullet]),\mathbb{Z}) \cong H_{2n-1}(BSL_\infty(k)).$$

(2) In the stabilization sequence for the symplectic groups, cf. Proposition 3.6, the morphism

$$H_{2n}(BSp_{2n}(k[\Delta^\bullet]),\mathbb{Z}) \to K^{MW}_{2n}(k)$$

is surjective. Hence, the standard inclusion $Sp_{2n-2} \hookrightarrow Sp_{2n}$ induces an isomorphism

$$H_{2n-1}(BSp_{2n-2}(k[\Delta^\bullet]),\mathbb{Z}) \xrightarrow{\cong} H_{2n-1}(BSp_{2n}(k[\Delta^\bullet]),\mathbb{Z}) \cong H_{2n-1}(BSp_\infty(k)).$$

Remark 3.12. *We recall the computation of the homotopy group $\pi_2^{\mathbb{A}^1}(SL_2)$ from* [AF12a, Theorem 3]. *For an infinite perfect field of characteristic $\neq 2$, there is an exact sequence*

$$0 \to S_4''(k) \to \pi_2^{\mathbb{A}^1}(SL_2)(\operatorname{Spec} k) \to KSp_3(k) \to 0.$$

The group $S_4''(k)$ sits in an exact sequence

$$I^5(k) \to S_4''(k) \to S_4'(k) \to 0,$$

where $I^5(k)$ is the fifth power of the fundamental ideal of the Witt ring $W(k)$ and there is a surjection $K_4^M(k)/12 \twoheadrightarrow S_4'(k)$.

Comparing with our stabilization result above, we find that the group S_4'' lies in the kernel of the Hurewicz homomorphism $\pi_3^{\mathbb{A}^1}(BSL_2)(k) \to H_3(BSL_2(k[\Delta^\bullet]),\mathbb{Z})$, at least if the field k has characteristic 0.

3.6. **Remarks on join operations.** A natural way to generalize Proposition 3.10 to infinite fields of characteristic $\neq 2$ would be to define an external product structure

$$H_n(BSL_n(k[\Delta^\bullet]),\mathbb{Z}) \otimes H_m(BSL_m(k[\Delta^\bullet]),\mathbb{Z}) \to H_{n+m}(BSL_{n+m}(k[\Delta^\bullet]),\mathbb{Z})$$

such that we get a commutative diagram

$$\begin{tikzcd}
H_n(BSL_n(k[\Delta^\bullet]),\mathbb{Z}) \otimes H_m(BSL_m(k[\Delta^\bullet]),\mathbb{Z}) \arrow[r] \arrow[d] & K_n^{MW}(k) \otimes K_m^{MW}(k) \arrow[d] \\
H_{n+m}(BSL_{n+m}(k[\Delta^\bullet]),\mathbb{Z}) \arrow[r] & K_{n+m}^{MW}(k)
\end{tikzcd}$$

where all morphism $H_i(BSL_i(k[\Delta^\bullet]),\mathbb{Z}) \to K_i^{MW}(k)$ are induced from the stabilization sequence Proposition 3.4 and the vertical map on the right is the product structure of Milnor-Witt K-theory.

Of course, the natural morphism $BSL_n \times BSL_m \to BSL_{n+m}$ induced by

$$SL_n \times SL_m \to SL_{n+m} : (A,B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

provides the obvious candidate. Moreover, it has the nice property that the change-of-topology morphism $H_i(SL_n(k)) \to H_i(BSL_n(k[\Delta^\bullet]))$ maps the external product of homology of the discrete groups

$$H_n(SL_n(k),\mathbb{Z}) \otimes H_m(SL_m(k),\mathbb{Z}) \to H_{n+m}(SL_{n+m}(k),\mathbb{Z})$$

to the product above.

Checking compatibility with the product structure in Milnor-Witt K-theory is more involved. For now, we can check if such a compatibility is at least true on the level of homotopy groups of $BSL_n(k[\Delta^\bullet])$. In that case, the morphism $\pi_n(BSL_n(k[\Delta^\bullet])) \to K_n^{MW}(k)$ is given by the natural projection $SL_n \to \mathbb{A}^n \setminus \{0\}$. Moreover, the product structure described above is induced from an $\mathbb{A}^1$-version of James' intrinsic join construction [Jam58]

$$SL_n(k[\Delta^\bullet]) * SL_m(k[\Delta^\bullet]) \to SL_{n+m}(k[\Delta^\bullet]).$$

An argument as in [Jam58] shows that this intrinsic join is compatible with the morphisms in the stabilization sequence where the product structure on spheres is also induced from the usual join. The latter then gives rise to the product of Milnor-Witt K-theory. This way, we find that the stabilization morphisms on $\mathbb{A}^1$-homotopy groups maps the join product on $\mathbb{A}^1$-homotopy groups of the groups SL_n to the product in Milnor Witt K-theory.

This indicates that the stabilization morphism on homology would also map the external product on $\mathbb{A}^1$-invariant group homology into the product of Milnor-Witt K-theory. However, a precise argument for this is still missing. The description of the effect of the stabilization morphism on homology is more difficult than on homotopy.

4. The third homology of SL_2 and indecomposable K_3

Suslin has shown in [Sus90, Corollary 5.2] that for any field k, the Hurewicz homomorphism $K_3(k) \to H_3(GL_\infty(k))$ induces an isomorphism

$$H_3(SL_\infty(k)) \cong \frac{K_3(k)}{\{-1\} \cdot K_2(k)}.$$

It follows that there is a natural induced surjective map

$$H_3(SL_\infty(k)) \to K_3^{\mathrm{ind}}(k) := \frac{K_3(k)}{K_3^M(k)}.$$

Let γ denote the induced composite map

$$H_3(SL_2(k)) \to H_3(SL_\infty(k)) \to K_3^{\mathrm{ind}}(k).$$

Lemma 4.1. *Let k be an infinite field.*

(1) γ is surjective.

(2) γ induces an isomorphism

$$H_3(SL_2(k), \mathbb{Z}[1/2]) \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} \mathbb{Z} \cong K_3^{\mathrm{ind}}(k) \otimes \mathbb{Z}[1/2]$$

(3) If $k^\times = (k^\times)^2$ then γ induces an isomorphism

$$H_3(SL_2(k), \mathbb{Z}) \cong K_3^{\mathrm{ind}}(k).$$

Proof. (1) This is [HT09, Lemma 5.1]

(2) This is [Mir08, Proposition 6.4 (ii)].

(3) This is [Mir08, Proposition 6.4 (iii)].

□

We also note that since the map γ factors through the stabilization homomorphism we have

Lemma 4.2. *Let k be a field of characteristic 0. Then there is a natural surjective homomorphism $\gamma' : H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}) \to K_3^{\mathrm{ind}}(k)$ giving rise to a commutative triangle*

$$\begin{tikzcd}
H_3(SL_2(k), \mathbb{Z}) \arrow[d] \arrow[dr, "\gamma"] & \\
H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}) \arrow[r, "\gamma'"] & K_3^{\mathrm{ind}}(k).
\end{tikzcd}$$

Proof. There is a natural commutative square

$$\begin{tikzcd}
H_3(SL_2(k), \mathbb{Z}) \arrow[r] \arrow[d] & H_3(SL_\infty(k), \mathbb{Z}) \arrow[d, "\cong"] \\
H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}) \arrow[r] & H_3(BSL_\infty(k[\Delta^\bullet]), \mathbb{Z}).
\end{tikzcd}$$

In this diagram, the right-hand vertical map is an isomorphism by Corollary 3.11. □

Combining Lemma 4.1 (3) and Lemma 4.2, we immediately deduce:

Corollary 4.3. *Let k be a field of characteristic 0 satisfying $k^\times = (k^\times)^2$. Then the change-of-topology morphism*

$$H_3(SL_2(k), \mathbb{Z}) \to H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z})$$

is injective. The image of this map is isomorphic to $K_3^{\mathrm{ind}}(k)$ and is a direct summand of $H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z})$.

5. Actions of multiplicative groups and Grothendieck-Witt rings

In this section, we will discuss the comparison of natural $\mathbb{Z}[k^\times/(k^\times)^2]$-module structures on the homology groups $H_\bullet(SL_2(k),\mathbb{Z})$ and $H_\bullet(BSL_2(k[\Delta^\bullet]),\mathbb{Z})$. The two main statements are that the change-of-topology morphism is equivariant for these additional module structures, and that on the $\mathbb{A}^1$-homotopy side, the module structure on $H_\bullet(BSL_2(k[\Delta^\bullet]),\mathbb{Z})$ descends to $GW(k)$-module structure. This is achieved by analysing the action of units on the $\mathbb{A}^1$-homotopy type of $\mathbb{P}^1$.

5.1. **Review of Grothendieck-Witt rings.** We first recall definition and notation for Grothendieck-Witt rings, collated from various sources [KK82], [HT10] and [Mor12, Section 3].

For a field k, we have the ring $\mathbb{Z}[k^\times/(k^\times)^2]$, which is the integral group ring over the group $k^\times/(k^\times)^2$ of square classes of k. For $a \in k^\times$ we will let $\langle a\rangle \in \mathbb{Z}[k^\times/(k^\times)^2]$ denote the square class of a. We let $\mathcal{I}_k$ denote the augmentation ideal with generators $\langle\langle a\rangle\rangle := \langle a\rangle - 1$.

The Grothendieck-Witt ring $GW(k)$ of the field k is the group completion of the set of isometry classes of nondegenerate symmetric bilinear forms over k. The addition and multiplication operations are given by orthogonal sum and tensor product of symmetric bilinear forms, respectively. For $a \in k$, the 1-dimensional form $(x,y) \mapsto axy$ is denoted by $\langle a\rangle \in GW(k)$. The dimension function provides an augmentation $\dim : GW(k) \to \mathbb{Z}$, and the augmentation ideal $I(k)$ is called the *fundamental ideal.* It is generated by the Pfister forms $\langle\langle a\rangle\rangle := \langle a\rangle - 1$.

There is a homomorphism of augmented rings

$$\mathbb{Z}[k^\times/(k^\times)^2] \to GW(k) : \langle a\rangle \mapsto \langle a\rangle\,.$$

This homomorphism is surjective, and its kernel is the ideal, $\mathcal{J}_k$, generated by the 'Steinberg elements' $\langle\langle a\rangle\rangle\,\langle\langle 1-a\rangle\rangle \in \mathcal{I}_k^2$.

Recall from [Mor12, Definition 3.1] that the Milnor-Witt K-theory $K^{MW}_\bullet(k)$ of the field k is defined to be the graded associative ring generated by symbols $[a], a \in k^\times$ of degree 1 and η of degree -1 with the following relations:

1. $[a][1-a] = 0$ for each $a \in k^\times \setminus \{1\}$,
2. $[ab] = [a] + [b] + \eta[a][b]$ for each $a, b \in k^\times$,
3. $[a]\eta = \eta[a]$ for each $a \in k^\times$,
4. $\eta h = 0$ for $h = \eta[-1] + 2$.

There is an augmentation $K^{MW}_0(k) \to K^{MW}_0(k)/\eta \cong \mathbb{Z}$. The morphism

$$GW(k) \to K^{MW}_0(k) : \langle a\rangle \mapsto 1 + \eta[a]$$

induces an isomorphism of augmented rings.

5.2. Module structures and equivariance.

Definition 5.1. *The standard exact sequence of algebraic groups*

$$1 \hookrightarrow SL_2 \hookrightarrow GL_2 \xrightarrow{\det} \mathbb{G}_m \to 1$$

induces an action of $\mathbb{G}_m$ on SL_2. For any k-algebra R, we can describe the action of $k^\times$ on $SL_2(R)$ as

$$k^\times \times SL_2(R) \to SL_2(R) : (u, A) \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \cdot A \cdot \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

We note that for a ring homomorphism $R \to S$, the induced homomorphism $SL_2(R) \to SL_2(S)$ is obviously $k^\times$-equivariant.

Lemma 5.2. *(i) The projection*

$$SL_2 \to \mathbb{P}^1 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [c:d].$$

induces an action of $k^\times$ on $\mathbb{P}^1(k)$. This action is given by

$$k^\times \times \mathbb{P}^1(k) \to \mathbb{P}^1(k) : (u, [x,y]) \mapsto [ux : y].$$

(ii) The conjugation action above induces an action of $k^\times$ on the simplicial sets $SL_2(k[\Delta^\bullet])$ and $BSL_2(k[\Delta^\bullet])$.

Proof. (i) is obvious.

(ii) The action of $k^\times$ on $SL_2(k[\Delta^\bullet])$ follows from the $k^\times$-equivariance of maps induced from ring homomomorphisms. This action is an action by group automorphisms (as opposed to just automorphisms as a space), hence it induces an action of $k^\times$ on $BSL_2(k[\Delta^\bullet])$. □

From the above definition, the following is immediate:

Lemma 5.3. *The following morphisms are $k^\times$-equivariant, for the module structures defined above:*

(i) the Hurewicz homomorphism $\pi_\bullet(BSL_2(k[\Delta^\bullet])) \to H_\bullet(BSL_2(k[\Delta^\bullet]))$,
(ii) the identification $\pi_\bullet(BSL_2(k[\Delta^\bullet])) \cong \pi_\bullet^{\mathbb{A}^1}(BSL_2)(k)$,
(iii) the loop-space isomorphism $\pi_{\bullet+1}^{\mathbb{A}^1}(BSL_2) \cong \pi_\bullet^{\mathbb{A}^1}(SL_2)$,
(iv) the projection $\pi_\bullet^{\mathbb{A}^1}(SL_2) \to \pi_\bullet^{\mathbb{A}^1}(\mathbb{P}^1)$ (which happens to be an isomorphism for $\bullet \geq 2$).

Proposition 5.4. *The $k^\times$-action on $SL_2(k[\Delta^\bullet])$ induces a $GW(k)$-module structure on $\pi_3(BSL_2(k[\Delta^\bullet]))$ and hence on $H_3(BSL_2(k[\Delta^\bullet]))$.*

Proof. By [Caz08, Theorem 2] or [Mor12, Theorem 7.36], we have

$$[(\mathbb{P}^1,\infty),(\mathbb{P}^1,\infty)]_\bullet \cong GW(k) \times_{k^\times/(k^\times)^2} k^\times$$

where the left-hand side denotes pointed $\mathbb{A}^1$-homotopy classes of morphisms $\mathbb{P}^1 \to \mathbb{P}^1$. Moreover, the endomorphism $[x:y] \mapsto [ux:y]$ corresponds to the pair $(\langle u \rangle, u)$ on the right-hand side. From [Mor12, Theorem 6.13], we find

$$[(SL_2,1),(SL_2,1)]_\bullet \cong (K_2^{MW}(k))_{-2} \cong K_0^{MW}(k) \cong GW(k),$$

where the left-most term denotes pointed $\mathbb{A}^1$-homotopy classes of maps $SL_2 \to SL_2$.

The conjugation action provides a monoid homomorphism $k^\times \to GW(k)$, and we claim that it is the natural map $u \in k^\times \mapsto \langle u \rangle \in GW(k)$ up to sign, cf. also [Caz12, Section 3.5]. We noted above that for any unit $u \in k^\times$, there is a commutative diagram of pointed maps

$$\begin{tikzcd} SL_2 \arrow[r] \arrow[d] & \mathbb{P}^1 \arrow[d] \\ SL_2 \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

where the horizontal arrows are the natural projections, the left vertical arrow is conjugation with u and the right vertical arrow the map $[x:y] \mapsto [ux:y]$. The projection map $SL_2 \to \mathbb{P}^1$ induces $K_0^{MW}(k) \cong [SL_2, SL_2] \to [SL_2, \mathbb{P}^1] \cong K_{-1}^{MW}(k)$ and $K_0^{MW}(k) \times_{k^\times/(k^\times)^2} k^\times \cong [\mathbb{P}^1,\mathbb{P}^1] \to [SL_2,\mathbb{P}^1] \cong K_{-1}^{MW}(k)$. The first map is multiplication with η, since $SL_2 \to \mathbb{P}^1$ is the Hopf map. Similarly, the second map is multiplication with η. From the discussion in [Mor12, p.195], the $k^\times$-factor of $[\mathbb{P}^1,\mathbb{P}^1]$ is included via multiplication with h, hence it is annihilated by multiplication with η. Commutativity of the diagram implies that the two possible maps $k^\times \to [SL_2, SL_2] \cong GW(k) \to W(k)$ and $k^\times \to [\mathbb{P}^1, \mathbb{P}^1] \cong GW(k) \to W(k)$ agree. In particular, it follows from the results of Cazanave and Morel that under the above identification, the homotopy class of conjugation with $\mathrm{diag}(u, 1)$ is the class of $\eta \langle u \rangle \in W(k)$. Since $\eta : GW(k) \to W(k)$ annihilates exactly $\mathbb{Z} \cdot h$, the morphism $k^\times \to [SL_2, SL_2] \cong GW(k)$ must by $u \mapsto \langle u \rangle + nh$ for some $n \in \mathbb{Z}$. Since conjugation with u is invertible, the augmentation must map the image of u to 1, hence the morphism $k^\times \to [SL_2, SL_2] \cong GW(k)$ must map u exactly to $\langle u \rangle$.

It is clear that the $k^\times$-action on the homotopy of SL_2 is induced from the natural action $[(SL_2, 1), (SL_2, 1)]_\bullet \times SL_2 \to SL_2$ together with the composition $k^\times \to GW(k) \cong [(SL_2, 1), (SL_2, 1)]_\bullet$ associating to u the conjugation by $\mathrm{diag}(u, 1)$. In particular, the $k^\times$-action coming from conjugation extends to a $GW(k)$-module structure on homotopy and homology of $SL_2(k[\Delta^\bullet])$. Using the equivariance of the isomorphisms from Lemma 5.3, we find that the same is true for homotopy of $BSL_2(k[\Delta^\bullet])$. This proves the first assertion.

For the second assertion, recall that since $BSL_2(k[\Delta^\bullet])$ is simply-connected, so the Hurewicz homomorphism $\pi_3(BSL_2(k[\Delta^\bullet])) \to H_3(BSL_2(k[\Delta^\bullet]))$ is surjective, and by Lemma 5.3, it is also equivariant. Therefore, the $GW(k)$-module structure on π_3 descends to a $GW(k)$-module structure on H_3. □

Remark 5.5. *There is probably a very explicit description of the isomorphism* $[SL_2, SL_2] \cong GW(k)$ *along the lines of* [Caz12], *which would allow to prove the above result without the passage through* $\mathbb{P}^1$.

Lemma 5.6. *The conjugation action of* $k^\times$ *on* $SL_2(k)$ *descends to an action of* $\mathbb{Z}[k^\times/(k^\times)^2]$ *on* $H_\bullet(SL_2(k), \mathbb{Z})$. *The natural change-of-topology morphism*

$$H_3(SL_2(k), \mathbb{Z}) \to H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z})$$

is equivariant for the $\mathbb{Z}[k^\times/(k^\times)^2]$*-module structures.*

Proof. For any unit $u \in k^\times$, the conjugation action of u^2 is the same as conjugating with $\mathrm{diag}(u, u^{-1}) \in SL_2(k)$. The squares therefore act via inner automorphisms, hence trivially on the homology. Homology groups are abelian groups, so the action of $k^\times/(k^\times)^2$ can be extended to the group ring linearly.

The change-of-topology morphism is induced from the inclusion of bisimplicial sets $BSL_2(k) \to BSL_2(k[\Delta^\bullet])$, where the first bisimplicial set is constant in the $\Delta^\bullet$-direction. The degree-wise morphisms $BSL_2(k) \to BSL_2(k[\Delta^n])$ are induced from the inclusion of the constants $k \hookrightarrow k[\Delta^n]$. Equivariance for the $k^\times$-module structures is then clear. The $\mathbb{Z}[k^\times/(k^\times)^2]$-module structure on $H_3(SL_2(k))$ has been described above. The $\mathbb{Z}[k^\times/(k^\times)^2]$-module structure on $H_3(BSL_2(k[\Delta^\bullet]))$ comes from the $GW(k)$-module structure in the previous proposition composed with $\mathbb{Z}[k^\times/(k^\times)^2] \to GW(k)$. The corresponding equivariance is then also clear. □

We state an obvious corollary:

Corollary 5.7. *We have the following factorization of the change-of-topology morphism:*

$$H_3(SL_2(k), \mathbb{Z}) \to H_3(SL_2(k), \mathbb{Z}) \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} GW(k) \to H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}).$$

6. Bloch groups and specialization homomorphisms

We review the relationship between $H_3(SL_2(k), \mathbb{Z})$ and the refined Bloch group, $\mathcal{RB}(k)$ and we use this relationship to compute lower bounds for the kernel of the map

$$H_3(SL_2(k), \mathbb{Z}[1/2]) \to H_3(SL_2(k), \mathbb{Z}[1/2]) \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} GW(k).$$

We begin by recalling that for any infinite field k there is a natural surjective homomorphism $H_3(SL_2(k),\mathbb{Z}) \to K_3^{\mathrm{ind}}(k)$ ([HT09, Lemma 5.1]) which induces an isomorphism

$$H_3(SL_2(k),\mathbb{Z}[1/2]) \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} \mathbb{Z} \cong K_3^{\mathrm{ind}}(k) \otimes \mathbb{Z}[1/2]$$

([Mir08]).

Recall from Section 5 that $\mathcal{I}_k \subseteq \mathbb{Z}[k^\times/(k^\times)^2]$ is the augmentation ideal and $\mathcal{J}_k \subseteq \mathbb{Z}[k^\times/(k^\times)^2]$ is the kernel of $\mathbb{Z}[k^\times/(k^\times)^2] \to GW(k)$. Thus, if M is a $\mathbb{Z}[k^\times/(k^\times)^2]$-module we have

$$\mathcal{I}_k M = \ker(M \to M \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} \mathbb{Z})$$

and

$$\mathcal{J}_k M = \ker(M \to M \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} GW(k)).$$

In particular, the natural map

$$M \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} GW(k) \to M \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} \mathbb{Z}$$

(induced by $GW(k) \to \mathbb{Z}$) is an isomorphism if and only if $\mathcal{J}_k M = \mathcal{I}_k M$.

For a field k with at least 4 elements, the *scissors congruence group* or *pre-Bloch group*, $\mathcal{P}(k)$, is the $\mathbb{Z}$-module with generators $[a]$, $a \in k^\times$, subject to the relations

(1) $[1] = 0$, and
(2)
$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] \text{ for } x, y \neq 1.$$

The *refined pre-Bloch group*, $\mathcal{RP}(k)$, is the $\mathbb{Z}[k^\times/(k^\times)^2]$-module with generators $[a]$, $a \in k^\times$, subject to the relations

(1) $[1] = 0$, and
(2)
$$[x] - [y] + \langle x \rangle \left[\frac{y}{x}\right] - \left\langle x^{-1} - 1 \right\rangle \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \langle 1-x \rangle \left[\frac{1-x}{1-y}\right] \text{ for } x, y \neq 1.$$

We let

$$S_2(k) := \frac{k^\times \otimes_{\mathbb{Z}} k^\times}{\langle \{x \otimes y + y \otimes x | x, y \in k^\times\} \rangle},$$

the second (graded) symmetric power. We let $x \circ y$ denote the image of $x \otimes y$ in $S_2(k)$. We endow $S_2(k)$ with the trivial $\mathbb{Z}[k^\times/(k^\times)^2]$-module structure.

The *refined Bloch group*, $\mathcal{RB}(k)$, of the field k (with at least 4 elements) is the kernel of the $\mathbb{Z}[k^\times/(k^\times)^2]$-module homorphism Λ:

$$\begin{array}{rcl} \Lambda = (\lambda_1, \lambda_2) : \mathcal{RP}(k) & \to & \mathcal{I}_k^2 \oplus S_2(k), \\ {[a]} & \mapsto & \left(\langle\langle a \rangle\rangle \langle\langle 1-a \rangle\rangle, a \circ (1-a)\right). \end{array}$$

The following is main result (Theorem 4.3 (1)) of [Hut11b]:

Proposition 6.1. *For an infinite field k there is a natural complex*

$$0 \to \mathrm{Tor}(\mu_k, \mu_k) \to H_3(SL_2(k),\mathbb{Z}) \to \mathcal{RB}(k) \to 0$$

of $\mathbb{Z}[k^\times/(k^\times)^2]$-modules which is exact except possibly at the middle term where the homology is annihilated by 4.

For $1 \neq x \in k^\times$, the element $[x] + [1-x] \in \mathcal{P}(k)$ is independent of x and has order dividing 6 ([Sus90, Lemma 1.3, Lemma 1.5]). We denote this constant by $\mathcal{C}_k$. Furthermore, by [Sus90, Lemma 1.2], for $x \in k^\times$ the elements $\psi(x) := [x] + [x^{-1}] \in \mathcal{P}(k)$ satisfy $2\psi(x) = 0$ and $\psi(xy) = \psi(x) + \psi(y)$. We denote by $\mathcal{S}_k$ the group

$\{\psi(x)|x \in k^\times\} \subset \mathcal{P}(k)$ and by $\tilde{\mathcal{P}}(k)$ the group $\mathcal{P}(k)/\mathcal{S}_k$. Observe that the natural map $\mathcal{P}(k) \to \tilde{\mathcal{P}}(k)$ induces an isomorphism $\mathcal{P}(k) \otimes \mathbb{Z}[1/2] \cong \tilde{\mathcal{P}}(k) \otimes \mathbb{Z}[1/2]$.

Similarly, we let $\psi_1(x)$ denote the element $[x] + \langle -1 \rangle [x^{-1}] \in \mathcal{RP}(k)$. (These elements are not generally of finite order.) Let $\widetilde{\mathcal{RP}}(k)$ denote the $\mathbb{Z}[k^\times/(k^\times)^2]$-module obtained by taking the quotient of $\mathcal{RP}(k)$ modulo the $\mathbb{Z}[k^\times/(k^\times)^2]$-module generated by the set $\{\psi_1(x)|x \in k^\times\}$ and let $\widetilde{\mathcal{RB}}(k)$ denote the image of the composite map $\mathcal{RB}(k) \to \mathcal{RP}(k) \to \widetilde{\mathcal{RP}}(k)$. Then ([Hut11a, Lemma 4.1]) we have:

Lemma 6.2. *The natural map $\mathcal{RB}(k) \to \widetilde{\mathcal{RB}}(k)$ is surjective with kernel annihilated by 4. In particular, it induces an isomorphism $\mathcal{RB}(k) \otimes \mathbb{Z}[1/2] \cong \widetilde{\mathcal{RB}}(k) \otimes \mathbb{Z}[1/2]$.*

Now suppose that k is an infinite field with (surjective) valuation $v : k^\times \to \Gamma$, where Γ is a totally ordered additive abelian group, and corresponding residue field $\bar{k}$. Let $\phi : \Gamma \to \mathbb{Z}/2$ be a group homomorphism. For an abelian group A, we let $A[\phi]$ denote A endowed with the $\mathbb{Z}[k^\times/(k^\times)^2]$-module structure

$$\langle x \rangle \cdot a := (-1)^{\phi(v(x))} a \quad \text{for all } x \in k^\times, a \in A.$$

Then we have ([Hut11b, section 4.3]):

Proposition 6.3. *There is a natural surjective $\mathbb{Z}[k^\times/(k^\times)^2]$-module homomorphism $S_{v,\phi} = S_\phi : \widetilde{\mathcal{RP}}(k) \to \tilde{\mathcal{P}}(\bar{k})[\phi]$ determined by the formula*

$$S_\phi([a]) = \begin{cases} [\bar{a}], & v(a) = 0 \\ \mathcal{C}_{\bar{k}}, & v(a) > 0 \\ -\mathcal{C}_{\bar{k}}, & v(a) < 0 \end{cases}$$

Furthermore, if $\phi \neq 0$ *the image of the induced composite homomorphism*

$$H_3(SL_2(k), \mathbb{Z}) \to \mathcal{RB}(k) \to \tilde{\mathcal{P}}(\bar{k})[\phi]$$

contains $4 \cdot \tilde{\mathcal{P}}(\bar{k})$.

The following corollary, which follows from the case $\phi = 0$ in 6.3, will be needed below:

Corollary 6.4. *Let k be a field with valuation and corresponding residue field $\bar{k}$. There is a natural surjective homomorphism $\mathcal{P}(k) \to \tilde{\mathcal{P}}(\bar{k})$.*

Proof. When $\phi = 0$, $\tilde{\mathcal{P}}(\bar{k})$ has the trivial $\mathbb{Z}[k^\times/(k^\times)^2]$-module structure and hence the homomorphism S_ϕ factors through $\mathcal{RP}(k)_{k^\times} = \mathcal{P}(k)$. □

Corollary 6.5. *Let k be a field with valuation $v : k^\times \to \Gamma$ and residue field $\bar{k}$. Suppose that*

(1) $\Gamma/2\Gamma \neq 0$ *and*
(2) $16 \cdot \tilde{\mathcal{P}}(\bar{k}) \neq 0$.

Then $\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}) \neq 0$.

Proof. Let $\phi : \Gamma \to \mathbb{Z}/2$ be a non-zero homomorphism. Let $y \in \tilde{\mathcal{P}}(\bar{k})$ with $16y \neq 0$. There exists $x \in H_3(SL_2(k), \mathbb{Z})$ with $S_\phi(x) = 4y$.

Choose $\pi \in \mathcal{O}_v$ with $\phi(v(\pi)) = 1$. So $\langle \pi \rangle y = -y$ and hence $\langle\langle \pi \rangle\rangle y = \langle\langle \pi^{-1} \rangle\rangle y = -2y$.

But $v(1-\pi) = 0 = v(\pi-1)$ and hence $v(1-\pi^{-1}) = v(\pi^{-1})$ and $\phi(v(1-\pi^{-1})) = 1$ also. Thus $\langle\langle 1-\pi^{-1} \rangle\rangle y = -2y$ also. It follows that

$$S_\phi(\langle\langle \pi^{-1} \rangle\rangle \langle\langle 1-\pi^{-1} \rangle\rangle x) = \langle\langle \pi^{-1} \rangle\rangle \langle\langle 1-\pi^{-1} \rangle\rangle S_\phi(x) = (-2) \cdot (-2) \cdot 4y = 16y \neq 0$$

and hence $0 \neq \langle\langle \pi^{-1} \rangle\rangle \langle\langle 1-\pi^{-1} \rangle\rangle x \in \mathcal{J}_k H_3(SL_2(k), \mathbb{Z})$. □

The following is Theorem 6.19 in [Hut11a]:

Proposition 6.6. *Let k be a local field with finite residue field $\bar{k}$ of odd order. If $\mathbb{Q}_3 \subset k$, suppose that $[k : \mathbb{Q}_3]$ is odd. Then there is an isomorphism of $\mathbb{Z}[k^\times/(k^\times)^2]$-modules*

$$H_3(SL_2(k), \mathbb{Z}[1/2]) \cong \left(K_3^{\mathrm{ind}}(k) \otimes \mathbb{Z}[1/2]\right) \oplus \left(\mathcal{P}(\bar{k}) \otimes \mathbb{Z}[1/2]\right).$$

In this isomorphism, the map from $H_3(SL_2(k), \mathbb{Z}[1/2])$ to the second factor is induced by S_ϕ where ϕ is the nontrivial homomorphism $\Gamma = \mathbb{Z} \to \mathbb{Z}/2$.

Remark 6.7. *If k is a finite field with q elements, then $\mathcal{P}(k)$ has order $q+1$ and $\mathcal{P}(k) \otimes \mathbb{Z}[1/2]$ is cyclic (*[Hut11b, Lemma 7.4]*) of order $(q+1)'$. Here, n' denotes the odd part of the integer n: $n = 2^a n'$ with $a \geq 0$ and n' odd.*

More generally we have the following ([Hut13]):

Proposition 6.8. *Let $k_0, k_1, \dots, k_n = k$ be a sequence of fields satisfying:*

1. *For each $i \in \{1, \dots, n\}$ there is a complete discrete value v_i on k_i with residue field k_{i-1}.*
2. *k_0 is either finite or real-closed or quadratically closed.*
3. *$\mathrm{char}(k_0) \neq 2$.*
4. *Either $\mathrm{char}(k) = 3$ or $\mathrm{char}(k_0) \neq 3$ or k contains a primitive cube root of unity.*

Then there is a natural split short exact sequence

$$0 \to \bigoplus_{i=0}^{n-1} (\mathcal{P}(k_i) \otimes \mathbb{Z}[1/2])^{\oplus 2^{n-i-1}} \to H_3(SL_2(k), \mathbb{Z}[1/2]) \to K_3^{\mathrm{ind}}(k) \otimes \mathbb{Z}[1/2] \to 0.$$

Remark 6.9. *The direct sum decomposition occurring here is the eigenspace decomposition for the group of characters on $k^\times/(k^\times)^2$ which restrict to the trivial character on $k_0^\times/(k_0^\times)^2$ (see* [Hut13, section 6]*).*

To spell this out, let k be complete with respect to a discrete valuation v with residue field $\bar{k}$ of characteristic not equal to 2. Let $U := \{a \in k^\times | \ v(a) = 0\}$. By Hensel's Lemma $u \in U$ is square if and only if $\bar{u} \in \bar{k}^\times$ is a square. Thus $U/U^2 \cong \bar{k}^\times/(\bar{k}^\times)^2$, and if $\pi \in k^\times$ is a uniformizer there is a natural (split) short exact sequence

$$1 \to \bar{k}^\times/(\bar{k}^\times)^2 \to k^\times/(k^\times)^2 \to \pi^{\mathbb{Z}/2} \to 1$$

where the first injection is obtained by choosing an inverse image x in U of a given element $\bar{x} \in \bar{k}$.

Now let $\mathcal{X}_k := \mathrm{Hom}(k^\times/(k^\times)^2, \mu_2)$. As noted, the conditions on k in the proposition (completeness of v_i and $\mathrm{char}(k_0) \neq 2$) ensure that there are natural injective maps

$$k_{i-1}^\times/(k_{i-1}^\times)^2 \to k_i^\times/(k_i^\times)^2$$

and hence there are surjective restriction homomorphisms

$$\mathcal{X}_{k_i} \to \mathcal{X}_{k_{i-1}}.$$

For each $i \leq n$, let

$$W_i := \ker(\mathcal{X}_k \to \mathcal{X}_{k_i}) = \{\chi \in \mathcal{X}_k | \ \chi|_{k_i^\times} = 1\}.$$

For each $i < n$ and $\chi \in W_i \setminus W_{i+1}$, the χ-eigenspace of $H_3(SL_2(k), \mathbb{Z}[1/2])$ – which we will denote $H_3(SL_2(k), \mathbb{Z}[1/2])_\chi$ – is isomorphic to $\mathcal{P}(k_i) \otimes \mathbb{Z}[1/2]$, and, by definition, the square class $\langle a \rangle$ acts as multiplication by $\chi(a)$ on this factor.

Lemma 6.10. *Let k be as in Proposition 6.8. Let M be a $\mathbb{Z}[k^\times/(k^\times)^2] \otimes \mathbb{Z}[1/2]$-module. Let $i < n$ and $\chi \in W_i \setminus W_{i+1}$. Then $\mathcal{J}_k M_\chi = M_\chi$.*

Proof. Let $\mathcal{O}_n = \{a \in k = k_n | v_n(a) \geq 0\}$ and for $j < n$ define recursively $\mathcal{O}_j = \{a \in \mathcal{O}_{j+1} | v_j(\pi_j(a)) \geq 0\}$, where $\pi_j : \mathcal{O}_{j+1} \to k_j$ is the natural surjection. Let $x \in \mathcal{O}_{i+1}$ with $v_{i+1}(\pi_{i+1}(x)) = 1$. Then the group (of order 2) $k_{i+1}^\times / \left((k_{i+1}^\times)^2 \cdot k_i^\times\right)$ is generated by the class of x and hence $\chi(\langle x\rangle) = \chi(\langle x^{-1}\rangle) = -1$. Since

$$v_{i+1}(\pi_{i+1}(x-1)) = 0,$$

it follows that the class of $x-1$ represents an element - possibly trivial - of $k_i^\times/(k_i^\times)^2$ and hence $\chi(\langle x-1\rangle) = 1$. Hence

$$\chi(\langle 1-x^{-1}\rangle) = \chi(\langle x^{-1}\rangle)\chi(\langle x-1\rangle) = -1 \cdot 1 = -1.$$

Thus $\langle\langle x^{-1}\rangle\rangle$ and $\langle\langle 1-x^{-1}\rangle\rangle$ both act on M_χ as multiplication by -2, and hence $\langle\langle x^{-1}\rangle\rangle \langle\langle 1-x^{-1}\rangle\rangle \in \mathcal{J}_k$ acts on M_χ as multiplication by 4. □

Corollary 6.11. *If k is as in Proposition 6.8 then*

(1)

$$\begin{aligned} H_3(SL_2(k), \mathbb{Z}[1/2]) \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} GW(k) &\cong H_3(SL_2(k), \mathbb{Z}[1/2]) \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} \mathbb{Z} \\ &\cong K_3^{\mathrm{ind}}(k) \otimes \mathbb{Z}[1/2] \end{aligned}$$

and

(2) there is a natural short exact sequence

$$0 \to \bigoplus_{i=0}^{n-1} (\mathcal{P}(k_i) \otimes \mathbb{Z}[1/2])^{\oplus 2^{n-i-1}} \to H_3(SL_2(k), \mathbb{Z}[1/2]) \to$$

$$\to H_3(SL_2(k), \mathbb{Z}[1/2]) \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} GW(k) \to 0.$$

(3) $\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}[1/2]) \cong \bigoplus_{i=0}^{n-1} (\mathcal{P}(k_i) \otimes \mathbb{Z}[1/2])^{\oplus 2^{n-i-1}}$.

Proof. The second statement follows from the first by Proposition 6.8, and the third is an immediate consequence of the second.

To prove the first isomorphism of statement (1), we must show that

$$\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}[1/2]) = \mathcal{I}_k H_3(SL_2(k), \mathbb{Z}[1/2]).$$

Now

$$\begin{aligned} \mathcal{I}_k H_3(SL_2(k), \mathbb{Z}[1/2]) &= \bigoplus_{i=0}^{n-1} (\mathcal{P}(k_i) \otimes \mathbb{Z}[1/2])^{\oplus 2^{n-i-1}} \\ &= \bigoplus_{i=0}^{n-1} \left(\bigoplus_{\chi \in W_i \setminus W_{i+1}} H_3(SL_2(k), \mathbb{Z}[1/2])_\chi \right) \end{aligned}$$

by Proposition 6.8 and the remark which follows it. The result follows by Lemma 6.10. □

Remark 6.12. *Let k satisfy the hypotheses of Proposition 6.8. If k_0 is finite or quadratically closed then the Witt ring $W(k_0)$ of k_0 is 2-torsion. An easy induction using Springer's Theorem on Witt rings of fields complete with respect to a discrete valuation (*[Lam05, Chapter VI, Theorem 1.4]*) implies that $W(k)$ is 2-torsion and hence that $GW(k) \otimes \mathbb{Z}[1/2] = \mathbb{Z}[1/2]$.*

However, in the case that k_0 is real closed, then – by Springer's Theorem again – the fundamental ideal of $W(k)$ contains a free abelian group of rank 2^n and the map $GW(k) \otimes \mathbb{Z}[1/2] \to \mathbb{Z}[1/2]$ has a large kernel.

The following is a special case of [Hut11a, Theorem 5.1]:

Proposition 6.13. *Let $\mathcal{O}$ be a unique factorization domain with field of fractions k. Let P be a set of representatives of the association classes of prime elements of k. For each $p \in P$ there is a discrete valuation $v_p : k^\times \to \mathbb{Z}$ with corresponding residue field $\bar{k}_p$. Let $\phi : \mathbb{Z} \to \mathbb{Z}/2$ be the non-zero homomorphism.*

Then the specialization homomorphisms induce a well-defined surjective *map of $\mathbb{Z}[k^\times/(k^\times)^2]$-modules*

$$S = \sum_{p\in P} S_{p,\phi} : H_3(SL_2(k), \mathbb{Z}[1/2]) \to \bigoplus_{p\in P} \mathcal{P}(\bar{k}_p) \otimes \mathbb{Z}[1/2].$$

Corollary 6.14. *Let $\mathcal{O}$ be a unique factorization domain with field of fractions k. Then the map induced by S*

$$\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}[1/2]) \to \bigoplus_{p\in P} \mathcal{P}(\bar{k}_p) \otimes \mathbb{Z}[1/2]$$

is surjective.

More generally, the map

$$\mathcal{J}_k(H_3(SL_2(k), \mathbb{Z}) \otimes A) \to \bigoplus_{p\in P} (\mathcal{P}(\bar{k}_p) \otimes A)$$

is surjective for any commutative $\mathbb{Z}[1/2]$-algebra A.

Proof. Denote the right-hand side by $\mathcal{P}(\mathcal{O})$. By Proposition 6.13, it is enough to show that $\mathcal{J}_k\mathcal{P}(\mathcal{O}) = \mathcal{P}(\mathcal{O})$.

Let $x \in \mathcal{P}(\mathcal{O})$. There exist primes $p_1, \ldots, p_t \in P$ such that $x = \sum_{i=1}^t x_i$ with $x_i \in \mathcal{P}(\bar{k}_{p_i}) \otimes \mathbb{Z}[1/2]$.

Recall that $\langle a \rangle \in \mathbb{Z}[k^\times/(k^\times)^2]$ acts as multiplication by $(-1)^{v_{\mathfrak{p}_i}(a)}$ on $\mathcal{P}(\bar{k}_{p_i})$. Choose $a \in \mathcal{O}$ with the property that $v_{p_i}(a)$ is odd for $1 \le i \le t$. Let $b = 1/a$. Then $v_{p_i}(b) = v_{p_i}(1-b)$ is odd for all i. It follows that $\langle\langle b \rangle\rangle \langle\langle 1-b \rangle\rangle x_i = (-2)^2 x_i = 4x_i$ for all i and hence that

$$x = \langle\langle b \rangle\rangle \langle\langle 1-b \rangle\rangle \left(\frac{x}{4}\right) \in \mathcal{J}_k\mathcal{P}(\mathcal{O}).$$

□

Remark 6.15. *With a little more care, one can show that the image of the map*

$$\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}) \to \bigoplus_{p\in P} \tilde{\mathcal{P}}(\bar{k}_p)$$

contains $\bigoplus_{p\in P} 16 \cdot \tilde{\mathcal{P}}(\bar{k}_p)$.

7. On the failure of weak homotopy invariance

This section sums up our insights into the failure of weak homotopy invariance for the third homology of SL_2.

7.1. **Module structures.** We show how the failure of weak homotopy invariance derives from the fact that module structure on $\mathbb{A}^1$-invariant group homology descends to a Grothendieck-Witt module structure, by the results of section Section 5, while the one on group homology typically does not, by the results of section Section 6.

We begin by noting that the kernel of the change-of-topology morphism is nontrivial for a quite general class of fields.

Theorem 7.1. *Let k be a field with valuation $v : k^\times \to \Gamma$ and residue field $\bar{k}$. Suppose that*

(1) $\Gamma/2\Gamma \neq 0$ *and*
(2) $16 \cdot \tilde{\mathcal{P}}(\bar{k}) \neq 0$.

Then the kernel of the natural change-of-topology morphism

$$H_3(SL_2(k),\mathbb{Z}) \to H_3(BSL_2(k[\Delta^\bullet]),\mathbb{Z})$$

is not trivial.

Furthermore, if ℓ is an odd prime and if $\mathcal{P}(\bar{k})\otimes\mathbb{Z}/\ell \neq 0$, then the same statement holds with $\mathbb{Z}$ replaced by $\mathbb{Z}/\ell$.

Proof. The first statement is a direct consequence of Corollaries 5.7 and 6.5 above.

For the second statement, the universal coefficient theorem implies that the kernel of $H_3(SL_2(k),\mathbb{Z})\otimes\mathbb{Z}/\ell \to H_3(BSL_2(k[\Delta^\bullet]),\mathbb{Z})\otimes\mathbb{Z}/\ell$ embeds in the kernel of the map $H_3(SL_2(k),\mathbb{Z}/\ell) \to H_3(BSL_2(k[\Delta^\bullet]),\mathbb{Z}/\ell)$. But the former kernel contains $\mathcal{J}_k(H_3(SL_2(k),\mathbb{Z})\otimes\mathbb{Z}/\ell)$, since $GW(k)\otimes\mathbb{Z}/\ell = (\mathbb{Z}[k^\times/(k^\times)^2]\otimes\mathbb{Z}/\ell)/\mathrm{Image}(\mathcal{J}_k)$. This, in turn, maps onto the nonzero group $\mathcal{P}(\bar{k})\otimes\mathbb{Z}/\ell = \tilde{\mathcal{P}}(\bar{k})\otimes\mathbb{Z}/\ell$. □

Next, we observe (Theorem 7.4 below) that when the field k is small, the kernel of the change-of-topology morphism is always large.

Lemma 7.2. *Let k be a global field. Let ℓ be an odd prime such that $[k(\zeta_\ell):k]$ is even, where ζ_ℓ denotes a primitive ℓ-th root of unity. Then there are infinitely many finite places v of k satisying $\ell|q_v+1$, where q_v is the cardinality of the residue field, $\bar{k}_v$, at v.*

Proof. Let $L = k(\zeta_\ell)$. By the Chebotarev density theorem there are infinitely many primes (not dividing ℓ) whose Frobenius in $\mathrm{Gal}(L/k)$ has order 2. It follows that, for such a prime v, $\zeta_\ell \notin \bar{k}_v = \mathbb{F}_{q_v}$, but $\zeta_\ell \in \mathbb{F}_{q_v^2}$. Thus, for such v, $\ell \nmid q_v - 1$, but $\ell | q_v^2 - 1$. □

Remark 7.3. *Of course, for any given number field k, $[k(\zeta_\ell):k] = \ell - 1$ for all but finitely many odd primes ℓ.*

If k is a global field of positive characteristic, there are infinitely many odd primes ℓ such that $[k(\zeta_\ell):k]$ is even.

Theorem 7.4. *Let k be an infinite but finitely-generated field. Then the kernel of the natural change-of-topology morphism*

$$H_3(SL_2(k),\mathbb{Z}[1/2]) \to H_3(BSL_2(k[\Delta^\bullet]),\mathbb{Z}[1/2])$$

is not finitely-generated.

Furthermore, if ℓ is an odd prime for which $[k(\zeta_\ell):k]$ is even, then the same statement holds with $\mathbb{Z}[1/2]$ replaced by $\mathbb{Z}/\ell$.

Proof. Let $A = \mathbb{Z}[1/2]$ or $A = \mathbb{Z}/\ell$ for an odd prime ℓ.

Now the field must contain a subfield, k_0 say, isomorphic either to $\mathbb{Q}$ or to $\mathbb{F}_p(x)$ where $p = \mathrm{char}(k) > 0$. Let d denote the transcendence degree of k over k_0. We will prove the result, together with the statement that $\mathcal{P}(k)\otimes A$ is not finitely generated, by induction on d.

Suppose first that $d = 0$. Then k is a global field. In this case (fixing an infinite prime in the function field case), the ring of integers $\mathcal{O}_k$ is a Dedekind domain with finite class group. Hence there exists $a \in \mathcal{O}_k$ for which $\mathcal{O} := \mathcal{O}_k[a^{-1}]$ is a unique factorization domain. Now for any prime p of $\mathcal{O}$, $\mathcal{P}(\bar{k}_p)\otimes\mathbb{Z}[1/2]$ is cyclic of order $(q_p+1)'$ where n' denotes the prime-to-2 part of the number n (Remark 6.7). This, together with Lemma 7.2 implies that $\mathcal{P}(\bar{k}_p)\otimes A$ is nonzero for infinitely many primes p.

We also observe that for any field k, there is an exact sequence

$$\mathcal{P}(k) \to S_2(k) \to K_2(k) \to 0.$$

But when k is a global field $K_2(k)$ is a torsion group while $S_2(k)$ modulo torsion is a free abelian group of infinite rank. It follows that $\mathcal{P}(k)$ maps onto a free abelian group of infinite rank, and hence $\mathcal{P}(k) \otimes A$ is not finitely generated.

Now suppose $d > 0$ and the result is known for fields of smaller transcendence degree over k_0. Then for any discrete valuation p on k, $\bar{k}_p$ is an infinite finitely-generated field of smaller transcendence degree. By the proof of Corollary 6.5 the kernel of the change-of-topology morphism surjects onto $\mathcal{P}(\bar{k}_p) \otimes A$. By induction, $\mathcal{P}(\bar{k}_p) \otimes A$ is already not finitely generated. Also by Corollary 6.4 we have the surjection $\mathcal{P}(k) \otimes A \to \mathcal{P}(\bar{k}_p) \otimes A$ and the result follows. □

Remark 7.5. *The induction step could also be proved by noting that the conditions on the field k in the theorem imply that there is a subring $\mathcal{O}$ of k with following properties: $\mathcal{O}$ is a unique factorization domain with field of fractions k and for which the set P of association classes of prime elements is infinite.*

Finally, we note that in the case of local fields, we can describe the exact structure of the change-of-topology kernel over $\mathbb{Z}[1/2]$.

Theorem 7.6. *Let k be a higher local field as in Proposition 6.8. Then the change-of-topology morphism factors through $K_3^{\mathrm{ind}}(k) \otimes \mathbb{Z}[1/2]$. Its kernel is isomorphic to*

$$\bigoplus_{i=0}^{n-1} (\mathcal{P}(k_i) \otimes \mathbb{Z}[1/2])^{\oplus 2^{n-i-1}}.$$

In particular, if k is complete with respect to a discrete valuation with residue field k_0 which is either finite of odd order or real-closed or quadratically closed then this kernel is isomorphic to $\mathcal{P}(k_0) \otimes \mathbb{Z}[1/2]$.

Proof. Recall that $\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}[1/2])$ is contained in the change-of-topology morphism by Corollary 5.7. But

$$\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}[1/2]) = \ker \left(H_3(SL_2(k), \mathbb{Z}[1/2]) \to K_3^{\mathrm{ind}}(k) \otimes \mathbb{Z}[1/2] \right)$$

by Corollary 6.11. Since the map to $K_3^{\mathrm{ind}}(k)$ factors through the change-of-topology morphism by Lemma 4.2, it follows that $\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}[1/2])$ is equal to the change-of-topology kernel.

Finally,

$$\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}[1/2]) \cong \bigoplus_{i=0}^{n-1} (\mathcal{P}(k_i) \otimes \mathbb{Z}[1/2])^{\oplus 2^{n-i-1}}$$

by Corollary 6.11 (3). □

7.2. **Number fields: finite generation.** In the case when k is a number field, we can deduce the failure of weak homotopy invariance in another way, which is of independent interest: It follows from simple size considerations - the group $H_3(SL_2(k), \mathbb{Z})$ is not finitely-generated while $H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z})$ is. This last fact is a consequence of finite-generation results in symplectic K-theory:

Proposition 7.7. *(i) Let k be a non-archimedean local field of characteristic $\neq 2$, and let ℓ be an odd prime different from the characteristic. Then the group $H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}/\ell)$ is finite.*

(ii) Let k be a number field. Then the homology group $H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}[1/2])$ is a finitely generated $\mathbb{Z}[1/2]$-module.

Proof. By Corollary 3.11, it suffices to prove the statements for $H_3(Sp_\infty(k), \mathbb{Z}) \cong H_3(BSp_\infty(k[\Delta^\bullet]), \mathbb{Z})$. The simplicial set $BSp_\infty(k[\Delta^\bullet])$ is simply-connected because Sp_∞ is $\mathbb{A}^1$-connected. Therefore, the Hurewicz theorem implies a surjection

$$\pi_3^{\mathbb{A}^1}(BSp_\infty)(\operatorname{Spec} k) \cong \pi_3(BSp_\infty(k[\Delta^\bullet])) \to H_3(BSp_\infty(k[\Delta^\bullet]), \mathbb{Z}).$$

We are thus reduced to show finite generation for $\pi_3^{\mathbb{A}^1}(BSp_\infty)(k) \cong KSp_3(k)$. The symplectic K-theory assertion is proved in Proposition 7.8 resp. Proposition 7.9 below. ☐

Proposition 7.8. *Let k be a number field. Then $KSp_3(k) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ is a finitely generated $\mathbb{Z}[1/2]$-module.*

Proof. To determine $KSp_3(k)$, we use the computations of Hornbostel, cf. [Hor02]. Note that in loc.cit., the group $KSp_3(k)$ is denoted by ${}_{-1}K_3^h(k)$. Let $A = \mathcal{O}_{k,S}$ be a ring of S-integers for a finite set S of places containing all the infinite places and all the places lying above 2. Using [Hor02, Corollary 4.15], we obtain an exact sequence

$$\cdots \to \bigoplus_{\mathfrak{p}} {}_{-1}U_3(A/\mathfrak{p}) \to {}_{-1}K_3^h(A) \to {}_{-1}K_3^h(k) \to \bigoplus_{\mathfrak{p}} {}_{-1}U_2(A/\mathfrak{p}) \to \cdots$$

The U-theory groups of finite fields are determined in [Hor02, Corollary 4.17]:

$${}_{-1}U_2(\mathbb{F}_q) \cong \mathbb{Z}/2, \qquad \text{and} \qquad {}_{-1}U_3(\mathbb{F}_q) \cong Gr_4,$$

where Gr_4 is either $\mathbb{Z}/4$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. In particular, tensoring the above exact sequence with $\mathbb{Z}[1/2]$ yields an isomorphism ${}_{-1}K_3^h(A) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] \cong {}_{-1}K_3^h(k) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$.

Finally, we give an argument to show that the hermitian K-groups of the ring $A = \mathcal{O}_{k,S}$ are finitely generated. For algebraic K-theory, this is proved in [Qui73]. For the symplectic groups, we can use the Borel-Serre compactification [BS73] to see that the group $Sp_{2n}(\mathcal{O}_{k,S})$ is of type FP_∞, hence the homology groups are all finitely generated. The elementary subgroup $Ep_{2n}(\mathcal{O}_{k,S})$ equals the commutator subgroup, hence it is of finite index, hence also of type FP_∞, and it is perfect. The symplectic K-groups can be defined as $KSp_i(\mathcal{O}_{k,S}) = \pi_i(BEp_\infty(\mathcal{O}_{k,S})^+)$. This space is simply-connected, and by FP_∞ (and homology stabilization for the symplectic groups) has finitely generated homology groups. From Serre's theory of classes of abelian groups [Ser53], the homotopy groups of this space are also finitely generated. ☐

Proposition 7.9. *Let k be a non-archimedean local field of characteristic $\neq 2$, and let ℓ be an odd prime different from the characteristic. Then $KSp_3(k) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell$ is finite.*

Proof. We denote by $\mathcal{O}$ the valuation ring, and by $\mathcal{O}/\mathfrak{m}$ the residue field. Using the localization sequence for symplectic K-theory as in Proposition 7.8 before, it suffices to prove the assertion for ${}_{-1}U_2(\mathcal{O}/\mathfrak{m})$ and ${}_{-1}K_3^h(\mathcal{O})$. As in Proposition 7.8, ${}_{-1}U_2(\mathbb{F}_q)$ is a finite 2-group, hence it does not contribute. The valuation ring is a complete discrete valuation ring, therefore we have an isomorphism

$${}_{-1}K_3^h(\mathcal{O}) \otimes \mathbb{Z}/\ell \cong {}_{-1}K_3^h(\mathcal{O}/\mathfrak{m}) \otimes \mathbb{Z}/\ell.$$

The group on the right-hand side is a finite group. ☐

Remark 7.10. *Note that the finite generation argument above more generally proves that the group $\pi_3^{\mathbb{A}^1}(BSL_2)(k) \otimes \mathbb{Z}[1/2]$ is finitely generated if k is a number field. From* [AF12a, Theorem 3]*, there is an exact sequence*

$$0 \to S_4''(k) \to \pi_2^{\mathbb{A}^1}(SL_2)(\operatorname{Spec} k) \to KSp_3(k) \to 0.$$

The group $S_4''(k)$ sits in an exact sequence

$$I^5(k) \to S_4''(k) \to S_4'(k) \to 0,$$

where $I^5(k)$ is the fifth power of the fundamental ideal of the Witt ring $W(k)$ and there is a surjection $K_4^M(k)/12 \twoheadrightarrow S_4'(k)$. Since $I^5(k) \otimes \mathbb{Z}[1/2]$ and $K_4^M(k)/12 \otimes$

$\mathbb{Z}[1/2]$ *are finitely generated, we get the finite generation for the above* $\mathbb{A}^1$*-homotopy group. This seems to be an interesting finite-generation result in* $\mathbb{A}^1$*-homotopy. Together with the surjection*

$$\pi_3^{\mathbb{A}^1}(BSL_2)(k)\otimes\mathbb{Z}[1/2]\to H_3(BSL_2(k[\Delta^\bullet]),\mathbb{Z}[1/2]),$$

this provides another way of proving the below Theorem 7.11 without resorting to the improved stability results Corollary 3.11.

Theorem 7.11. *Let k be a number field. Then the kernel of the natural change-of-topology morphism*

$$H_3(SL_2(k),\mathbb{Z})\to H_3(BSL_2(k[\Delta^\bullet]),\mathbb{Z})$$

is not finitely generated. The same also holds with $\mathbb{Z}/\ell$-coefficients, ℓ an odd prime for which $[k(\zeta_\ell):k]$ is even.

Proof. Let A denote either $\mathbb{Z}[1/2]$ or $\mathbb{Z}/\ell$ where ℓ is an odd prime for which $[k(\zeta_\ell):k]$ is even. By the proof of Theorem 7.4, the group $H_3(SL_2(k),A)$ is not finitely generated while, by Proposition 7.7, $H_3(BSL_2(k[\Delta^\bullet]),A)$ is finitely generated. Thus the kernel of

$$H_3(SL_2(k),A)\to H_3(BSL_2(k[\Delta^\bullet]),A)$$

is not finitely generated. □

8. The cokernel of the change-of-topology morphism

The main results in this article concern estimates of the kernel of the change-of-topology morphism from $H_3(SL_2(k),A)$ to $H_3(BSL_2(k[\Delta^\bullet]),A)$. In this section, we discuss the cokernel of this morphism. In order to do this, we treat some aspects of the spectral sequence associated to the bisimplicial set $BSL_2(k[\Delta^\bullet])$. This spectral sequence has the form

$$E^1_{p,q}=H_q(BSL_2(k[\Delta^p])),\mathbb{Z})\Rightarrow H_{p+q}(dBSL_2(k[\Delta^\bullet]),\mathbb{Z}),$$

with differentials $d^r_{p,q}:E^r_{p,q}\to E^r_{p-r,q+r-1}$. Here, the differentials

$$d^1_{p,q}=\sum_{i=0}^{p}(-1)^iH_q(d_i):H_q(BSL_2(k[\Delta^p])),\mathbb{Z})\to H_q(BSL_2(k[\Delta^{p-1}])),\mathbb{Z})$$

are induced from the simplicial structure of the simplicial algebra $k[\Delta^\bullet]$.

We will assume in this section that the underlying field k is infinite.

Remark 8.1. *For infinite k, homotopy invariance for homology of $SL_2(k[T])$, cf.* [Knu01, Theorem 4.3.1], *implies $E^2_{1,q}\cong 0$ for all $q>1$.*

Since $H_0(SL_2(k[\Delta^n]),\mathbb{Z})\cong\mathbb{Z}$ for all n, we also have

$$E^2_{p,0}\cong\begin{cases}\mathbb{Z} & p=0\\ 0 & otherwise\end{cases}$$

Remark 8.2. *The differential $d^2_{2,1}$ is trivial: we have $H_2(BSL_2(k[\Delta^\bullet]),\mathbb{Z})\cong K_2^{MW}(k)$* [Mor12, p.185]*, and $H_2(SL_2(k),\mathbb{Z})\cong K_2^{MW}(k)$* [Sus87]*. By the stabilization results, the natural change-of-topology morphism*

$$H_2(SL_2(k))\to H_2(BSL_2(k[\Delta^\bullet]))$$

is the identity. By the vanishing in Remark 8.1, we have an exact sequence

$$H_1(SL_2(k[\Delta^2]),\mathbb{Z})/(\operatorname{im} d^1_{3,1})\xrightarrow{d^2_{2,1}}H_2(SL_2(k),\mathbb{Z})\to H_2(BSL_2(k[\Delta^\bullet]),\mathbb{Z})\to 0,$$

which proves the claim.

Proposition 8.3. *There is a short exact sequence*

$$0 \to H_3(SL_2(k))/(\operatorname{im} d^2_{2,2} + \operatorname{im} d^3_{3,1}) \to H_3(BSL_2(k[\Delta^\bullet])) \to$$

$$\to H_1(SL_2(k[\Delta^2]))/(\operatorname{im} d^1_{3,1}) \to 0$$

Moreover, the natural change-of-topology $H_3(SL_2(k)) \to H_3(BSL_2(k[\Delta^\bullet]))$ factors through the injection above.

Proof. By the vanishing in Remark 8.1, we have $E^\infty_{1,2} = E^\infty_{3,0} = 0$. The exact sequence claimed is the one induced from the spectral sequence:

$$0 \to E^\infty_{0,3} \to H_3(BSL_2(k[\Delta^\bullet])) \to E^\infty_{2,1} \to 0.$$

We prove $E^\infty_{2,1} = H_1(SL_2(k[\Delta^2]))/(\operatorname{im} d^1_{3,1})$. We have $H_1(SL_2(k[T])) = 0$, since $SL_2(k[T])$ is perfect. Hence the differential $d^1_{2,1}$ is trivial. No differential except $d^1_{3,1}$ hits the $(2,1)$-entry. By Remark 8.2, the differential $d^2_{2,1}$ is also trivial. This proves the claim.

We identify $E^\infty_{0,3}$. All differentials starting at $(0,3)$ are trivial. Therefore, $E^\infty_{0,3}$ is the quotient of $H_3(SL_2(k))$ by all differentials hitting it. The differentials $d^1_{1,3}$ and $d^1_{4,0}$ are trivial by Remark 8.1. Only the differentials $d^2_{2,2}$ and $d^2_{3,1}$ remain.

The last statement is obvious, since the natural change-of-topology morphism includes $SL_2(k)$ as 0-simplices. Therefore, the natural map factors as

$$H_\bullet(SL_2(k), \mathbb{Z}) \to E^\infty_{0,\bullet} \to H_\bullet(BSL_2(k[\Delta^\bullet]), \mathbb{Z}).$$

□

Remark 8.4. *This is a good place to point out that the above result implies that the spectral sequence*

$$E^1_{p,q} = H_q(BSL_2(k[\Delta^p])), \mathbb{Z}) \Rightarrow H_{p+q}(dBSL_2(k[\Delta^\bullet]), \mathbb{Z}),$$

does not degenerate at the E^2-page. Moreover, the differentials $d^2_{2,2}$ and $d^3_{3,1}$ provide a natural relation between the counterexamples to homotopy invariance for homology of SL_2 and the kernel of the natural change-of-topology morphism

$$H_3(SL_2(k), \mathbb{Z}) \to H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}).$$

In the cases discussed in Theorem 7.4 and Theorem 7.6, the above differentials induce non-trivial morphisms from homotopy-invariance-counterexamples to pre-Bloch groups of residue fields.

The explicit computation of such differentials is very complicated, for various reasons. First of all, our knowledge of the groups $H_2(SL_2(k[\Delta^2]))$ and $H_1(SL_2(k[\Delta^3]))$ is very limited. The constructions of [KM97] *show that these groups tend to be very large, but do not give a precise description of their structure. Second, the essential step in the computation of $d^2_{2,2}$ needs explicit lifts of null-homologous cycles in $\mathcal{Z}_2(SL_2(k[T]))$ to 3-chains. While the amalgam decomposition of $SL_2(k[T])$ can in principle be used to compute such things, the computations easily become too complicated to follow through.*

Proposition 8.5. *Let k be an infinite field of characteristic $\neq 2$. There is a commutative diagram with exact columns*

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H_3(SL_2(k))/(\operatorname{im} d^2_{2,2} + \operatorname{im} d^3_{3,1}) \arrow[r] \arrow[d] & H_3(Sp_\infty(k)) \arrow[d] \\
H_3(BSL_2(k[\Delta^\bullet])) \arrow[r, "s"] \arrow[d] & H_3(BSp_\infty(k[\Delta^\bullet])) \arrow[d] \\
E^\infty_{2,1} \arrow[r] \arrow[d] & 0 \\
0 &
\end{tikzcd}$$

The map s above is surjective. In particular, we have a surjection

$$t : E^\infty_{2,1} \twoheadrightarrow \operatorname{coker}\left(H_3(SL_2(k)) \to H_3(Sp_\infty(k))\right).$$

If k is of characteristic 0, then s and t are isomorphisms.

Proof. We first note that the inclusions of groups $SL_2 \hookrightarrow Sp_\infty$ induces a morphism of the bisimplicial object $BSL_2(k[\Delta^\bullet]) \to BSp_\infty(k[\Delta^\bullet])$. The spectral sequence computing the homology of the diagonal of a bisimplicial object is compatible with morphisms of bisimplicial objects. The exact column on the left is a consequence of Proposition 8.3. By [Kar73], the homology of the infinite symplectic group has $\mathbb{A}^1$-invariance for regular rings in which 2 is invertible. Therefore the spectral sequence associated to the bisimplicial object $BSp_\infty(k[\Delta^\bullet])$ collapses and produces the exact column on the right. The whole diagram is commutative by the abovementioned compatibility of the spectral sequences with the stabilization morphism.

Surjectivity of s is a consequence of stabilization Proposition 3.6, and the characteristic 0 isomorphism is a consequence of our improved stabilization result Corollary 3.11. An application of the snake-lemma then proves the last claim. Note in particular that the top horizontal morphism is induced from the standard inclusion $SL_2 \to Sp_\infty$, hence it is really the stabilization morphism. □

Remark 8.6. *It is interesting to see that the surjectivity of the change-of-topology map can be completely translated into a question on stabilization of the homology of linear groups, namely the question of surjective stabilization for the morphism $H_3(SL_2) \to H_3(Sp_\infty)$. The known stabilization results for the symplectic groups do not seem to decide surjective stabilization in this range.*

References

[AF12a] A. Asok and J. Fasel. A cohomological classification of vector bundles on smooth affine threefolds. Preprint, arXiv:1204.0770.

[AF12b] A. Asok and J. Fasel. Algebraic vector bundles on spheres. Preprint, arXiv:1204.4538.

[BM99] J. Barge and F. Morel. Cohomologie des groupes linéaires, K-théorie de Milnor et groupes de Witt. C. R. Acad. Sci. Paris Sér. I Math. 328 (1999), no. 3, 191–196.

[BS73] A. Borel and J.-P. Serre. Corners and arithmetic groups. Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault. Comment. Math. Helv. 48 (1973), 436–491.

[Cat07] J.-L. Cathelineau. Homology stability for orthogonal groups over algebraically closed fields. Ann. Sci. Éc. Norm. Supér. (4) 40 (2007) 487–517.

[Caz08] C. Cazanave. Classes d'homotopie de fractions rationnelles. C. R. Math. Acad. Sci. Paris 346 (2008), no. 3-4, 129–133.

[Caz12] C. Cazanave. Algebraic homotopy classes of rational functions. Ann. Sci. Éc. Norm. Supér. (4) 45 (2012), no. 4, 511–534.

[DS82] J.L. Dupont and C.-H. Sah. Scissors congruences. II. J. Pure Appl. Algebra 25 (1982), no. 2, 159–195.

[GJ99] P.G. Goerss and J.F. Jardine. Simplicial homotopy theory. Progress in Mathematics, 174. Birkhäuser Verlag, Basel, 1999.

[Hor02] J. Hornbostel. Constructions and dévissage in Hermitian K-theory. K-Theory 26 (2002), no. 2, 139–170.

[HT09] K. Hutchinson and L. Tao. The third homology of the special linear group of a field. J. Pure Appl. Algebra 213 (2009), 1665–1680

[HT10] K. Hutchinson and L. Tao. Homology stability for the special linear group of a field and Milnor-Witt K-theory. Doc. Math. 2010, Extra volume: Andrei A. Suslin sixtieth birthday, 267–315.

[Hut11a] K. Hutchinson. A refined Bloch group and the third homology of SL_2 of a field. J. Pure Appl. Algebra 217 (2013), 2003–2035.

[Hut11b] K. Hutchinson. A Bloch-Wigner complex for SL_2. To appear J. K-theory, 2013. arXiv:1107.0264.

[Hut13] K. Hutchinson. Scissors congruence groups and the third homology of SL_2 of local rings and fields. Preprint, arXiv:1309.5010.

[Jam58] I.M. James. The intrinsic join: a study of the homotopy groups of Stiefel manifolds. Proc. London Math. Soc. (3) 8 1958, 507–535.

[Jar83] J.F. Jardine. On the homotopy groups of algebraic groups. J. Algebra 81 (1983), no. 1, 180–201.

[Kar73] M. Karoubi. Périodicité de la K-théorie hermitienne. In: Algebraic K-theory, III: Hermitian K-theory and geometric applications (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 301–411. Lecture Notes in Math., Vol. 343, Springer, Berlin, 1973.

[KK82] M. Knebusch and M. Kolster. Wittrings. Aspects of Mathematics, 2. Friedr. Vieweg & Sohn, Braunschweig, 1982.

[KM97] S. Krstić and J. McCool. Free quotients of $SL_2(R[x])$. Proc. Amer. Math. Soc. 125 (1997), 1585–1588.

[Knu01] K.P. Knudson. Homology of linear groups. Progress in Mathematics, 193. Birkhäuser Verlag, Basel, 2001.

[Lam05] T.Y. Lam. Introduction to quadratic forms over fields. Graduate Studies in Mathematics, 67. American Mathematical Society, Providence, RI, 2005.

[Mir08] B. Mirzaii. Third homology of general linear groups. J. Algebra 320 (2008), no 5., 1851–1877

[Mor04] F. Morel. Sur les puissances de l'idéal fondamental de l'anneau de Witt. Comment. Math. Helv. 79 (2004), no. 4, 689–703.

[Mor11] F. Morel. On the Friedlander-Milnor conjecture for groups of small rank. Current developments in mathematics, 2010, 45–93, Int. Press, Somerville, MA, 2011.

[Mor12] F. Morel. $\mathbb{A}^1$-algebraic topology over a field. Lecture Notes in Mathematics, 2052. Springer, Heidelberg, 2012.

[Mos] L.-F. Moser. $\mathbb{A}^1$-locality results for linear algebraic groups. Preprint, 2011.

[NS90] Yu. P. Nesterenko and A.A. Suslin. Homology of the general linear group over a local ring, and Milnor's K-theory, Math. USSR-Izv. 34 (1990) 121–145.

[Qui73] D. Quillen. Finite generation of the groups K_i of rings of algebraic integers. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 179–198. Lecture Notes in Math., Vol. 341, Springer, Berlin, 1973.

[Ser53] J.-P. Serre. Groupes d'homotopie et classes de groupes abéliens. Ann. of Math. (2) 58, (1953), 258–294.

[Sta11] A. Stavrova. Isotropic reductive groups over polynomial rings. To appear J. K-theory, 2013. arXiv:1111.4664.

[Sus84] A.A. Suslin. Homology of GL_n, characteristic classes and Milnor K-theory. In: Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), LNMA 1046, pp. 357–375. Springer, 1984.

[Sus87] A.A. Suslin. Torsion in K_2 of fields. K-Theory 1 (1987), no. 1, 5–29.

[Sus90] A.A. Suslin. K_3 of a field and the Bloch group. Trudy Mat. Inst. Steklov., 183:190-199, 229, 1990. Translated in Proc. Steklov Inst. Math. 1991, no. 4, 217-239, Galois Theory, rings, algebraic groups and their applications (Russian)

[VW12] K. Völkel and M. Wendt. On $\mathbb{A}^1$-fundamental groups of isotropic reductive groups. Preprint, arXiv:1207.2364, 2012.

[Wen10] M. Wendt. $\mathbb{A}^1$-homotopy of Chevalley groups. J. K-Theory 5 (2010), no. 2, 245–287.

[Wen11] M. Wendt. Rationally trivial torsors in $\mathbb{A}^1$-homotopy theory. J. K-Theory 7 (2011), no. 3, 541–572.

Kevin Hutchinson, School of Mathematical Sciences, University College Dublin
E-mail address: `kevin.hutchinson@ucd.ie`

Matthias Wendt, Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Eckerstrasse 1, 79104, Freiburg im Breisgau, Germany
E-mail address: `matthias.wendt@math.uni-freiburg.de`