# Rigid body stiffness matrix for Identification of Inertia Properties from Output-only data

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#### Abstract:

Identification of inertia properties (mass, location of the center of mass and inertia tensor) is essential for designing of engineering structures. Using modal testing is a possibility for estimation of the inertia properties in which they can be identified using the orthogonality property of mass-normalized rigid body mode shapes. However, identification of rigid body mode shapes using modal testing is not always possible, because it is not possible to excite the structure at all degrees of freedom. In this paper, output-only modal analysis in which the structure can be excited in different directions is used to identify the rigid body modes of the structure. It is shown that all of the rigid body modes of the structure can be extracted from output-only modal analysis. As the obtained rigid body mode shapes from output-only modal analysis are not scaled, a new method is proposed for scaling them using rigid body stiffness matrix. The inertia properties of the structure are obtained from the scaled mode shapes. The accuracy of the proposed method is studied using a numerical case study of a steel structure as well as an experimental case study of a steel frame.

*Keywords:* Inertia properties, Output-only modal analysis, mode shape scaling, rigid body modes.

# **1** Introduction

The evaluation of the inertia properties of a structure is necessary for static and dynamic designing process (Eicholtz et al., 2012; Loyd et al., 2010; Okuma et al., 2001). Finite Element (FE) Method can be used for estimation of the inertia properties as a theoretical method. But, it is not always possible to establish an accurate model in practice (especially in complicated structures). Modal testing is an experimental alternative to estimate the inertia properties of structures using their real conditions.

The experimental methods for estimation of the inertia properties from vibration testing are categorised in two main groups; time domain methods and frequency domain methods (Almeida et al., 2007). Pendulum method (Holzweissig and Dresig, 1994; Hughes, 1957) is a time domain method that uses the period of oscillation of hanged structure to evaluate the inertia properties. However, this method may not be reliable due to the friction, the air force

and etc. A simple and less expensive extension of the Pendulum method was proposed in (Zhi-Chao et al., 2009). Another time domain method was proposed in (Pandit and Hu, 1994) that uses transformation of the translational motion to the rotational and translational motion in order to identify the inertia properties. Pandit et al. used time domain data in six axes for identification of the inertia properties from impact data (Pandit et al., 1992).

Inertia Restrain Method (IRM), Method of Direct Physical Parameter Identification (MDPPI) and Modal Methods (MM) are three main groups of frequency domain methods. The dynamic response of a structure in free-free condition in low frequency ranges used in IRM or Mass Line method (Fregolent and Sestieri, 1996; Fullekrug and Schedlinski, 2004; Urgueira, 1995; Wei and Reis, 1989).

Measured Frequency Response Functions (FRFs) are directly used in the MDPPI method to estimate the inertia properties (Huang and Lallement, 1997; Mangus et al., 1993). Orthogonality properties of mass-normalized rigid body mode shapes are used in the Modal Methods such as Bertl and Conti method for estimation of the inertia properties (Bretl and Conti, 1987; Conti and Bretl, 1989). All of the six rigid body modes should be identified using a modal identification method In Bertl and Conti method. But it is not always possible to excite all the six rigid body modes during an experimental test (Almeida et al., 2007). Therefore, many exciting points in different directions are required to significantly excite all the six rigid body modes (Almeida et al., 2008). A comparative study for estimation of the inertia properties using two different frequency domain methods from experimental modal testing is carried out in (Ashory et al., 2010) and the accuracy of methods are investigated.

Measuring the excitation forces is difficult or even impossible in many cases. The modal parameters can be identified using only measured responses in the output-only modal methods without the knowledge of inputs. Therefore, it is possible to excite the structure in any directions. As the excitation is not measured in output-only modal analysis, the extracted mode shapes are not scaled. Therefore, the mode shapes obtained should be scaled using a scaling method. Some methods have been proposed for scaling of the operational mode shapes such as; mass change method (Brincker and Andersen, 2003), mass-stiffness change method (Khatibi et al., 2012) and receptance based method (Bernal, 2011).

Estimation of the inertia properties using the output-only modal data first was proposed in (Malekjafarian et al., 2013). It was shown that all the rigid body modes are available from the output-only modal data while the mode shapes obtained are not scaled. Malekjafarian et al. (2013) used the mass change method for scaling of the rigid body mode shapes by adding some extra masses to the structure and repeating the test. Although the method can detect the rigid body modes much easier than the conventional methods, but it includes some errors and difficulties due to the scaling process.

In this paper a new method is proposed for scaling the rigid body mode shapes obtained from the output-only modal testing. The method does not need either adding extra masses to the structure and repeating the test. It is shown that the rigid body mode shapes can be scaled, when the rigid body stiffness matrix of the suspension system of the structure is well determined. A numerical case study of a simple structure is investigated to validate the accuracy of the proposed method. The inertia properties of the structure are obtained using both the conventional modal method and the output-only modal method showing good accuracy for the proposed method. An experimental case study of a frame structure is used to show the efficiency of the method for a real case study. The rigid body stiffness matrix of the suspension system is obtained using testing of all the springs attached to the structure and finally the rigid body properties are obtained using the proposed method. It is demonstrated that the proposed method is as accurate as the traditional method, while has less complexities and difficulties in extracting all the six rigid body mode shapes.

#### 2 Theory

# **2.1 Identification of inertia properties from output-only data using rigid body stiffness matrix**

For a completely free-free structure, the structure can move to any direction without any resistance showing that the rigid body frequencies of this structure are zero. In the other hand, if the structure is suspended from stiff springs in all directions, then for any rigid body movement, there would be a resistance from the springs that makes a real rigid body movement for the whole structure. Depending on the effective mass and spring of the rigid structure in each direction, there is a rigid body frequency for the structure. The structure is usually suspended from some springs to figure the real rigid body vibrations which lead to rigid body modes (including rigid body frequencies and rigid body mode shapes). Depends on the magnitude of the spring stiffness, the rigid body frequencies could be determined, but they are usually in a low frequency range. To excite all of the six rigid body modes (three corresponding to the translational motions and three to the rotational motions of the structure), the structure should be excited in all directions. As the input forces are not measured in output-only modal analysis, the structure can be excited at any arbitrary point and direction. Therefore, all the rigid body modes are detectable in a single test. Although the output-only modal methods have no difficulties in measurement of the excitation forces, but in absent of inputs of the system, the obtained mode shapes are not scaled. Usually some methods such as mass change method (Brincker and Andersen, 2003), mass-stiffness change method (Khatibi et al., 2012) and receptance based method (Bernal, 2011) and etc., are used for scaling of the operational mode shapes that are based on mass or stiffness modification of the structure. Therefore, all the existing scaling methods are based on modification of the structure that always causes some difficulties and errors.

In this work a new method is proposed for scaling of the mode shapes from output-only modal analysis that is just working for the rigid body modes. The proposed method does not use any modification in the structure which can improve the accuracy of the estimation of the inertia properties from output only modal analysis.

#### 2.1.1 Scaling of rigid body mode shapes using rigid body stiffness matrix

For a structure with total mass of m that is suspended from stiff springs (Fig. 1), the equations of motion of the structure under a free-free condition are given in a linear form by (JIN et al., 2013; Okuma and Shi, 1997):

where  $x_{cm}$ ,  $y_{cm}$  and  $z_{cm}$  are the coordinates of the centre of mass, *J* is the moment of inertia of the structure,  $\ddot{q}$  is the linear acceleration of the structure and  $\ddot{q}_{\theta_x}$  is the rotational acceleration of the structure at the location of centre of mass and *F* and *M* represent the external forces applied to the structure. Eq. (1) can be written in a simple form as:

$$\boldsymbol{M_{rb}} \boldsymbol{\ddot{q}}_{rb} = \boldsymbol{F}_{rb} \tag{2}$$

 $M_{rb}$  is the rigid body mass matrix of the rigid structure including its inertia properties (Malekjafarian et al., 2013). Considering the free vibration of such a structure (when damping is ignored), the equation of the rigid body motion for free vibration (which is different from the elastic motion) can be written as:

$$\boldsymbol{M_{rb}\ddot{q}} + \boldsymbol{K_{rb}}q = 0 \tag{3}$$

where  $K_{rb}$  is the rigid body stiffness matrix of the rigid structure that is related to the stiffness of the attached springs. An eigen value solution for Eq. (3) is:

$$\boldsymbol{M_{rb}\phi_n\omega_n}^2 = \boldsymbol{K_{rb}\phi_n} \tag{4}$$

where  $\phi_n$  is the scaled rigid body mode shape vector and  $\omega_n$  is the *n*<sup>th</sup> rigid body natural frequency where  $n=1\sim6$ .



Figure 1: A structure on the elastic supports

Pre-multiplying Eq. (4) by  $\phi_n^T$ :

$$\phi_n^T \boldsymbol{M_{rb}} \phi_n \omega_n^2 = \phi_n^T \boldsymbol{K_{rb}} \phi_n \tag{5}$$

Considering orthogonality of the mode shapes with respect to the mass matrix  $\phi_n^T M_{rb} \phi_n = 1$ , Eq. (5) becomes:

$$\omega_n^2 = \phi_n^{\ T} K_{rb} \phi_n \tag{6}$$

Considering that all the unscaled rigid body modes can be obtained using the outputonly data, we have:

$$\Psi = \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_6 \end{bmatrix}$$
(7)

where  $\psi_1, \psi_2,...$  and  $\psi_6$  are the unscaled rigid body mode shape vectors obtained from the output-only data. The relation between  $\psi$  the unscaled mode shape vector and  $\phi$  the scaled mode shape vector can be given by:

$$\phi = \alpha \psi \tag{8}$$

where  $\alpha$  is the scaling factor. By substituting Eq. (8) into Eq. (6), the following equation is obtained:

$$\omega_n^2 = \alpha^2 \psi_n^T \boldsymbol{K_{rb}} \psi_n \tag{9}$$

Therefore, the scaling factor can be obtained using the following equation:

$$\alpha = \sqrt{\frac{\omega_n^2}{\psi_n^T \boldsymbol{K_{rb}}\psi_n}} \tag{10}$$

The obtained equation is used only for rigid body modes. The rigid body stiffness matrix  $K_{rb}$  (that is related only to the suspension configuration of the structure) can be calculated from the stiffness of all the springs attached to the structure. The scaling factors can be estimated using the rigid body stiffness matrix, the natural frequencies and the unscaled mode shapes.

Finally, the inertia properties can be estimated by applying the scaled mode shapes in the modal method. Consequently, the inertia properties can be estimated from the results of output-only modal analysis without any modification to the structure with acceptable accuracy.

#### 2.1.2 Identification of inertia properties from scaled mode shapes

From the output-only identification method that applied to the measured responses, the rigid body mode shapes are obtained with respect to the physical coordinate system where the measurements are taken. When N three-axial accelerometers are used, dimension of the mode shape matrix  $\mathbf{\Phi}$  (that includes all the modes shapes obtained from the output-only modal methods) is 3N×6. All elements of this matrix are translational degrees of freedom.

$$\boldsymbol{\Phi} = \begin{bmatrix} \alpha_1 \begin{cases} \begin{pmatrix} \psi_x \\ \psi_y \\ \psi_z \end{pmatrix}_1 \\ \vdots \\ \begin{pmatrix} \psi_x \\ \psi_y \\ \psi_z \end{pmatrix}_N \\ \vdots \\ \begin{pmatrix} \psi_x \\ \psi_y \\ \psi_z \end{pmatrix}_N \end{pmatrix}_1 \quad \alpha_2 \begin{cases} \begin{pmatrix} \psi_x \\ \psi_y \\ \psi_z \end{pmatrix}_1 \\ \vdots \\ \begin{pmatrix} \psi_x \\ \psi_y \\ \psi_z \end{pmatrix}_N \end{pmatrix}_2 \quad \cdots \quad \alpha_6 \begin{cases} \begin{pmatrix} \psi_x \\ \psi_y \\ \psi_z \end{pmatrix}_1 \\ \vdots \\ \begin{pmatrix} \psi_x \\ \psi_y \\ \psi_z \end{pmatrix}_N \end{pmatrix}_6 \end{bmatrix}$$
(11)

In order to identify the inertia properties, it is necessary to transform the matrix  $\mathbf{\Phi}$  to a 6×6 mode shape matrix  $\mathbf{\Phi}_0$  which is defined using the following equations (Malekjafarian et al., 2013):

$$\boldsymbol{\Phi}_{\boldsymbol{0}} = \begin{bmatrix} \begin{pmatrix} \phi_{0x} \\ \phi_{0y} \\ \phi_{0y} \\ \phi_{0y} \\ \phi_{0y} \\ \phi_{0x} \\ \phi_{0\theta x} \\ \phi_{0\theta y} \\ \phi_{0\theta z} \\ 1 \end{bmatrix} \begin{pmatrix} \phi_{0x} \\ \phi_{0y} \\ \phi_{0y} \\ \phi_{0x} \\ \phi_{0\theta y} \\ \phi_{0\theta z} \\ 2 \end{bmatrix} \dots \begin{pmatrix} \phi_{0x} \\ \phi_{0y} \\ \phi_{0y} \\ \phi_{0x} \\ \phi_{0\theta y} \\ \phi_{0\theta y} \\ \phi_{0\theta z} \\ \delta_{0\theta y} \\ \phi_{0\theta z} \\ \delta_{0\theta y} \\ \delta_{0\theta$$

where  $\Phi_0$  is the 6×6 mass-normalized mode shape matrix which contains the six rigid body modes of the structure with respect to the selected origin. Transformation can be done using the following equation (Malekjafarian et al., 2013):

$$\Psi \alpha = R_0 \Phi_0 \tag{13}$$

where  $R_0$  is the transformation matrix of the rigid body modes corresponding to the N triaxial accelerometers (Ashory et al., 2010)

$$\boldsymbol{R}_{0} = \begin{bmatrix} 1 & 0 & 0 & 0 & z_{1} & -y_{1} \\ 0 & 1 & 0 & -z_{1} & 0 & x_{1} \\ 0 & 0 & 1 & y_{1} & -x_{1} & 0 \end{bmatrix} \\ \vdots & \vdots & \vdots \\ \begin{bmatrix} 1 & 0 & 0 & 0 & z_{N} & -y_{N} \\ 0 & 1 & 0 & -z_{N} & 0 & x_{N} \\ 0 & 0 & 1 & y_{N} & -x_{N} & 0 \end{bmatrix} \end{bmatrix}$$
(14)

where  $x_i$ ,  $y_i$  and  $z_i$  are the coordinates of the i<sup>th</sup> accelerometer. The first three rows of the mode shape matrix  $\Phi_0$  are related to the translational motions of the structure and the next three rows are related to the rotational motion of the structure.

Considering that the matrix  $R_0$  is not square, matrix  $\Phi_0$  should be obtained using the concept of Pseudo inverse (Malekjafarian et al., 2013):

$$\Phi_0 = \left(R_0^T R_0\right)^{-1} R_0^T \Psi \alpha \tag{15}$$

where  $(R_0^T R_0)^{-1} R_0^T$  is the Pseudo inverse of matrix  $R_0$ .

The classic eigenvalue equation of motion for a structure with the elastic supports (Fig 1) (Malekjafarian et al., 2013):

$$\boldsymbol{M_{rb_0}}\boldsymbol{\phi_0}_n\boldsymbol{\omega_n}^2 = \boldsymbol{K_{rb_0}}\boldsymbol{\phi_0}_n \tag{16}$$

where  $M_{rb_0}$  and  $K_{rb_0}$  are the mass and the stiffness matrices about the selected origin,  $\phi_{0_n}$  is the  $n^{\text{th}}$  rigid body mode shape vector at the selected origin and  $\omega_n$  is the  $n^{\text{th}}$  rigid body natural frequency (Malekjafarian et al., 2013).

Using the orthogonally property of the mass-normalized mode shape, Eq. (17) could be obtained (Almeida et al., 2007):

$$\phi_0^T \boldsymbol{M_{rb_0}} \phi_0 = I \tag{17}$$

As the rigid body mode shapes are linearly independent, the mode shape matrix is invertible and the mass matrix can be derived as (Almeida et al., 2007):

$$M_{rb_0} = \Phi_0^{-T} \Phi_0^{-1} \tag{18}$$

The resulting matrix  $M_{rb_0}$  (Eq. (2)) represents the mass matrix related to the origin based on the six extracted mode shape vectors at the N measured response locations.

The mass of the structure can be calculated by averaging the first three diagonal elements of the mass matrix. From the upper right quadrant and the lower left quadrant of the mass matrix in Eq. (19), the mass center of structure can be calculated (Almeida et al., 2007).

$$x_{cm} = \frac{1}{2m_0} \tag{19}$$

$$x_{cm} = \frac{1}{2m} (M_0^{26} - M_0^{26}), \qquad y_{cm} = \frac{1}{2m} (M_0^{34} - M_0^{16}), \qquad z_{cm} = \frac{1}{2m} (M_0^{15} - M_0^{24})$$
(20)

where  $M_{rb_0}{}^{ij}$  is the *ij*<sup>th</sup> element of the estimated mass matrix  $M_{rb_0}$  and  $x_{cm}$ ,  $y_{cm}$  and  $y_{cm}$  are the coordinates of the centre of mass.

In addition, using the lower right quadrant of mass matrix, the elements of inertia tensor are calculated. In order to determine the inertia tensor with respect to the center of mass, the matrix in Eq. (18) is transferred to the center of mass of the structure.

#### 2.2 Frequency Domain Decomposition (FDD) method

Frequency Domain Decomposition (FDD) is a non-parametric output-only modal method which was first suggested by (Brincker et al., 2000). If the system has r input x and m output y, then:

$$\boldsymbol{G}_{yy}(j\omega) = \boldsymbol{H}(j\omega)^{H}\boldsymbol{G}_{xx}(j\omega)\boldsymbol{H}(j\omega)^{T}$$
(21)

where  $G_{xx}(j\omega)$  is the Power Spectral Density (PSD) matrix of the input,  $G_{yy}(j\omega)$  is the PSD matrix of the responses,  $H(j\omega)$  is the Frequency Response Function (FRF) matrix, the superscript "T" indicates the transpose of the matrix, the superscript "H" indicates the complex conjugate of the matrix and *j* is equal to  $\sqrt{-1}$ . If the input force is assumed to be a white signal, the PSD matrix of the output can be given by (Brincker et al., 2000):

$$\boldsymbol{G}_{yy}(j\omega) = \sum_{k=1}^{n} \left( \frac{\mathrm{d}_{k} \boldsymbol{\psi}_{k} \boldsymbol{\psi}_{k}^{T}}{j\omega - \lambda_{k}} + \frac{\overline{\mathrm{d}}_{k} \overline{\boldsymbol{\psi}}_{k} \overline{\boldsymbol{\psi}}_{k}^{T}}{j\omega - \overline{\lambda}_{k}} \right)$$
(22)

where  $d_k$  is a scalar,  $\boldsymbol{\psi}_k$  is the  $k^{th}$  unscaled mode shape vector,  $\boldsymbol{\lambda}_k$  is the  $k^{th}$  complex resonance frequency, *n* is the number of modes and the over bar "-" indicates the complex conjugate.

The dynamic behaviour of a structure is dominated by one of its mode close to the corresponding natural frequency. Therefore the response of the structure in this frequency is similar to the mode shape of the structure at this mode (Khatibi et al., 2011; Khatibi et al., 2012). The PSD of the response in each frequency can be decomposed to the singular-values and singular-vectors using the following equation:

$$\boldsymbol{G}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}(j\omega_i) = U_i S_i U_i^{\ H} \tag{23}$$

where  $U_i$  is the *i*<sup>th</sup> singular-vectors matrix,  $S_i$  is the *i*<sup>th</sup> singular-values matrix and  $\omega_i$  is the *i*<sup>th</sup> frequency.

As the singular-values are directly related to modal participation factors, the number of non-zero singular-values indicates the number of modes which contribute to the response of the system at that frequency. The peaks of the first singular values of system correspond to the natural frequencies of system. The singular-vectors corresponding to the peaks of the first singular-values estimate the mode shapes.

#### 3. Numerical case study

#### **3.1 Introducing the case study**

A cubic structure that is suspended using 12 springs is considered for the numerical case study. Boundary condition of the structure with 12 springs and stiffnesses of the springs are given in Fig. 2. Dimensions, exact inertia properties of the structure, locations of the measurement and the input forces are given in Fig. 3.



Figure 2 : Suspension condition of the case study.



Figure 3 : Inertia properties and density of the case study.

In order to show the efficiency of the new proposed method, the inertia properties of the structure are obtained using both conventional modal analysis and output-only modal analysis. Finally, the errors of the results obtained from two methods are compared to confirm the accuracy of the proposed approach.

#### 3.2 Identification of inertia properties using conventional modal analysis

In order to estimate the inertia properties of the numerical example using conventional modal analysis, the FRFs of the structure are obtained using different excitation points (which are shown in Fig. 3) in a simulated test. It is known from the conventional modal testing that the peaks in FRFs are related to the frequencies which are dominant in the response. Therefore, the rigid body modes can be detected from the FRFs peaks. The FRFs obtained are shown in Fig. 4. It is shown that all the six rigid body modes cannot be detected in one FRF. As it is mentioned in (Almeida, 2006) when conventional modal analysis is used, it is necessary to estimate the FRFs in different points to detect all the rigid body modes of the structure. Many independent tests should be conducted and some pre-analysing processes are required to know which FRF gives better results.

The first six natural frequencies of the structure that are related to the six rigid body modes are estimated from the FRFs are given in Table 1.

Table 1. Natural frequencies	s of the fight	bouy modes	nom me co	inventional n	ioual allary	515.
Mode #	1	2	3	4	5	6
Natural frequencies (Hz)	1.46	2.74	3.73	4.40	5.91	6.12

Table 1: Natural frequencies of the rigid body modes from the conventional modal analysis.

The rigid body mode shapes obtained from the conventional modal method are used in Eq. 18 to find the rigid body mass matrix. The ten rigid body properties then are estimated

using equations 19 and 20. The error of the estimation of the inertia properties will be compared with those of the output-only modal method in Section 3.3.



Figure 4: Receptance functions obtained for each excitation point; (a) excitation at point 1, (b) excitation at point 2, (c) excitation at point 3.

# 3.3 Identification of inertia properties using output-only data

In order to obtain the inertia properties from the proposed method, the structure is excited in a simulated test by random forces at arbitrary points. The FDD method was applied to the measured responses. The SVD diagram of the FDD method is shown in Fig. 5.



Figure 5 : SVD diagram for the FDD method.

The natural frequencies (Table 2) and unscaled mode shapes are obtained using the FDD method.

Table 2: Natural frequencies of the rigid body modes from output-only modal analysis.

Mode #	1	2	3	4	5	6
Natural frequencies from FDD	1.46	2.74	3.74	4.40	5.91	6.11
method (Hz)						

The rigid body stiffness matrix is calculated using the equation of motions of the rigid body (T. and D, 2004) from the stiffnesses of the springs attached to the structure (a=0.4, b=0.15 and c=0.2).

$$\boldsymbol{K_{rb}} = \begin{bmatrix} k_{11} & \cdots & k_{16} \\ \vdots & \ddots & \vdots \\ k_{61} & \cdots & k_{66} \end{bmatrix}$$
(24)

$$\begin{aligned} k_{11} &= 8k_x, \ k_{12} = k_{13} = k_{14} = 0, \ k_{15} = 4k_x a \ , \\ k_{16} &= -4k_x c \\ k_{21} &= k_{23} = k_{25} = 0, \ k_{22} = 8k_y, \ k_{24} = -4k_y a \ , \ k_{26} = 4k_y b \\ k_{31} &= k_{32} = k_{36} = 0, \ k_{33} = 8k_z, \ k_{34} = 4k_z c \ , \ k_{35} = -4k_z b \\ k_{41} &= 0, \ k_{42} = -4k_y a \ , \ k_{43} = 4k_z c \ , \ k_{44} = 4k_y a^2 + 4k_z c^2 \ , \ k_{45} = -2k_z b c \ , \ k_{46} = -2k_y a b \\ k_{51} &= 4k_x a \ , \ k_{52} = 0 \ , \ k_{53} = -4k_z b \ , \ k_{54} = -2k_z b c \ , \ k_{55} = 4k_x a^2 + 4k_z b^2 \ , \ k_{56} = -2k_x a c \end{aligned}$$

$$k_{61} = -4k_xc$$
,  $k_{62} = 4k_yb$ ,  $k_{63} = 0$ ,  $k_{64} = -2k_yab$ ,  $k_{65} = -2k_xac$ ,  $k_{66} = 4k_yb^2 + 4k_xc^2$  (25)

The scaling factors for all modes are estimated using Eq. (11) and the scaled rigid body mode shapes are estimated. The inertia properties are estimated using the scaled rigid body mode shapes. The errors of the inertia properties calculated from the proposed approach are compared with those of conventional modal analysis in Fig. 6. It is shown that both methods include same level of error which confirms the accuracy of the approach. In addition, the errors of the results obtained from conventional modal analysis are less than those of the output-only modal analysis in some cases. But, it should be considered that differences of the errors are not significant, while user can take advantages of the output-only data.



Figure 6: Errors of the inertia properties obtained from the conventional method and the output-only method.

### 4. Experimental case study

#### 4.1 Introducing the case study

In order to validate the proposed technique experimentally, the method is applied to a 3D structure made of steel. The structure is suspended from another structure which plays a support role for the main structure using 24 springs shown in Fig. 7. The 3D models of the main structure without its support and with its support are separately shown in Fig. 8. The inertia properties of the structure that are obtained from the theory are given in Table 3.

Table 3: The inertia properties of the experimental case study obtained from theory.

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6.750	0.265	0.191	0.161	0.699	1.020	1.160	0.345	0.288	0.209



Figure 7: The experimental case study (green structure) is suspended from its support (gray structure) using 12 springs.





Figure 8: (a) The 3D model of the main case study. (b) The 3D model of the main case study when is suspended from the support structure.

#### 4.2 Estimation of inertia properties using conventional modal analysis

The structure is excited in different excitation points. The excitation signal is measured using force transducer type BK 8200 and the signal is amplified using amplifier type 2647A. The responses of the structure are measured using three accelerometers type DJB/130V. The measuring points are selected using the suggestions in (Lee et al., 1999) and are shown in Fig. 8 (a). The FRFs of the structure are obtained from different excitation points using Pulse8 software (Brüel & Kjær, 2003) and are shown in Fig. 9.



Figure 9: The obtained FRFs from the test.

It is shown that all the six rigid body modes cannot be detected in one FRF. Many independent tests are conducted and some pre-analysing process is done to know which FRF gives better results. The rigid body natural frequencies are estimated using the FRFs (Table 4).

Table 4: Natural frequencie	es of the rigid	l body mode	s from the co	onventional i	modal testi	ng.
Mode #	1	2	3	4	5	6
Natural frequencies from FRF (Hz)	9.54	11.09	13.39	13.56	15.79	17.13

The inertia properties of the structure are estimated using the method explained in Section 2. The errors of the inertia properties obtained from conventional modal testing will be presented in Section 4.3 when they will be compared to the output-only results.

# 4.3 Identification of inertia properties using output-only modal analysis

# 4.3.1 Estimation of rigid body stiffness matrix of the case study

As it is stated in the section 2.1.1, the rigid body stiffness matrix of the suspension of the structure should be identified to scale the operational rigid body mode shapes. For the presented case study, the rigid body stiffness matrix is estimated using the equation of motion of the rigid body using the method presented in section 3.3 (T. and D, 2004). Stiffnesses of all the springs are estimated using a simple 1DOF free vibration test of a mass-spring system as shown in Fig. 10.



Figure 10: Test procedure for estimation of exact stiffness of the used springs.

A specified mass is hanged from the considered spring and the FRF of the system is obtained using hammer modal testing (Fig. 11). Stiffness of each spring can be calculated

from the obtained natural frequency of the system and the well-known formula  $\omega = \sqrt{\frac{k}{m}}$ .



Figure 11: FRF obtained from modal testing of the springs using the procedure shown in Fig. 10.

The magnitudes of stiffnesses of the springs are obtained and given in Table. 5. Location of each point is shown in Fig. 7. Finally the rigid body stiffness matrix of the case study is constructed using Eq. (24) while the preload effect caused by the weight of the structure is neglected.

Point	$k_x (N / m)$	$k_y$ (N/m)	$k_z (N / m)$
1	1304.2	2494.9	6028.5
2	1305.3	2490.3	6145.3
3	1304.6	2502.4	6147.3
4	1312.5	2501.1	6632.4
5	1304.5	2489.6	6273.8
6	1308.9	2494.4	6145.3
7	1303.3	2499.8	6028.5
8	1304.1	2490.0	6145.3

Table 5: Amounts of stiffnesses of 24 springs in 8 attaching points.

# 4.3.1 Identification of inertia properties

The structure is excited in different points by random forces. The responses of the structure are measured using three accelerometers type DJB/130V.The FDD method is applied to the

responses. The SVD diagram of the FDD method is obtained and shown in Fig. 12. The natural frequencies (Table 6) and the mode shapes are obtained using the FDD method.



Figure 12: The SVD diagram of the FDD method.

Table 6: Natural frequencie	es of the rigi	d body mode	es from the o	utput-only n	nodal testir	ıg.
Mode #	1	2	3	4	5	6
Natural frequencies from FDD method (Hz)	9.53	11.09	13.25	13.56	15.81	17.12

The scaling factors for each mode are estimated using Eq. (11) and the rigid body stiffness matrix obtained earlier. The inertia properties have been estimated using the scaled rigid body mode shapes obtained using the scaling factors.



Figure 13: The errors of the obtained inertia properties from output only modal testing.

The error of the estimated inertia properties are shown in Fig. 13. It is shown that the proposed method estimated the mass of the structure more accurate in compared to the conventional method. In addition, the proposed method shows more accuracy in estimation of centre of mass of the structure in most items. On the other hand, using the output-only data caused more error compared to the conventional method, but the obtained results could be still acceptable. Finally, it can be concluded that the proposed method is able to estimate the inertia properties with acceptable accuracy in compared to the conventional methods.

#### 4. Conclusion

In this paper a previously proposed approach for estimation of the inertia properties of a structure using output only modal analysis is improved. The advantage of the approach is that by using output-only modal analysis, it is possible to excite the structure in many different directions. Consequently, the all rigid body mode shapes can be excited and the data are enough to detect and identify all the rigid body modes. A new scaling method is proposed to scale the obtained rigid body mode shapes from output only modal analysis. Then the mass-normalized mode shapes are used in Bertl and Conti for identification of the inertia properties of the structure. It is shown that if the rigid body stiffness matrix that is related to the suspension conditions could be defined, the obtained results from new proposed method is accurate enough for estimation of the inertia properties. The accuracy of the scaled mode shapes from output-only modal analysis is an important parameter in the accuracy of the whole approach that is related to the accuracy of estimated rigid body stiffness matrix of the suspension. Therefore, despite the measurement errors seem not too low, special applications could benefit from the proposed mass properties measurement approach.

# References

- Almeida, R.A.B., 2006. Evaluation of the Dynamic Characteristic of Rigid Bodies Based on Experimental Results, Department of Mechanical Engineering and Industrial, Faculty of Science and technology. New University of Lisbon, Portugal.
- Almeida, R.A.B., Urgueira, A.P.V., Maia, N.M.M., 2007. Identification of rigid body properties from vibration measurements. Journal of Sound and Vibration 299, 884-899.
- Almeida, R.A.B., Urgueira, A.P.V., Maia, N.M.M., 2008. Evaluation of the performance of three different methods used in the identification of rigid body properties. Shock and Vibration 15, 467-479.
- Ashory, M.R., Malekjafarian, A., Harandi, P., 2010. On the accuracy of estimation of rigid body inertia properties from modal testing results. Struct Eng Mech 35, 53-65.
- Bernal, D., 2011. A receptance based formulation for modal scaling using mass perturbations. Mech Syst Signal Pr 25, 621-629.
- Bretl, J., Conti, P., 1987. Rigid body mass properties from test data, Proc. 5th Int. Modal Analysis Conference.
- Brincker, R., Andersen, P., 2003. A Way of Getting Scaled Mode Shapes in Output Only Modal Analysis, of the 21st International Modal Analysis Conference.
- Brincker, R., Zhang, L.M., Andersen, P., 2000. Modal identification from ambient responses using frequency domain decomposition. P Soc Photo-Opt Ins 4062, 625-630.
- Brüel & Kjær, S.V.M.A.S., 2003. PULSE, Version8.0.
- Conti, P., Bretl, J., 1989. Mount stiffness and inertia properties from modal test data. Journal of Vibration, Acoustic, Stress and Reability in Design 11, 134-138.
- Eicholtz, M.R., Caspall, J.J., Dao, P.V., Sprigle, S., Ferri, A., 2012. Test method for empirically determining inertial properties of manual wheelchairs. J Rehabil Res Dev 49, 51-62.
- Fregolent, A., Sestieri, A., 1996. Identification of rigid body inertia properties from experimental data. Mech Syst Signal Pr 10, 697-709.
- Fullekrug, U., Schedlinski, C., 2004. Inertia Parameter Identification from Base Excitation Test data, Proc. of the 5th International Symposium on Environmental Testing for Space Programmes, Noordwijk, Netherlands.
- Holzweissig, F., Dresig, H., 1994. Lehrbuch der Maschinedynamic: Grundlagen und Praxisorientierte Beispiele, mit 40 Aufgaben mit\Losungen und 53 Talellen. Fachbuchverlag neubearb Aufl Leipzig.
- Huang, S.J., Lallement, G., 1997. Direct Estimation of Rigid Body Properties from Harmonic Forced Responses, Proc. 15th Int. Modal Analysis Conference (IMAC XV), pp. 175-180.
- Hughes, G.W., 1957. Trifilar pendulum and its application to experimental determination of moments of inertia. ASME Papers 57-SA-51.
- JIN, Y., ZHANG, J., GUAN, X., 2013. Theoretical calculation and experimental analysis of the rigid body modes of powertrain mounting system. 8.
- Khatibi, M., Ashory, M., Albooyeh, A., 2011. Optimum amount of additive mass in scaling of operational mode shapes. Struct Eng Mech 39, 733-750.
- Khatibi, M.M., Ashory, M.R., Malekjafarian, A., Brincker, R., 2012. Mass-stiffness change method for scaling of operational mode shapes. Mech Syst Signal Pr 26, 34-59.
- Lee, H., Lee, Y.B., Park, Y.S., 1999. Response and excitation points selection for accurate rigid-body inertia properties identification. Mech Syst Signal Pr 13, 571-592.
- Loyd, A.M., Nightingale, R., R'Dale'Bass, C., Mertz, H.J., Frush, D., Daniel, C., Lee, C., Marcus, J.R., Mukundan, S., Myers, B.S., 2010. Pediatric head contours and inertial properties for ATD design. Stapp car crash journal 54, 167.
- Malekjafarian, A., Ashory, M.R., Khatibi, M.M., 2013. Identification of inertia properties from the results of output-only modal analysis. Arch Appl Mech 83, 923-937.

- Mangus, J.A., Passarello, C., Vankarsen, C., 1993. Estimation Rigid Body Properties from Frequency Reaction Measurements, Proc. 11th Int. Modal Analysis Conference (IMAC XI), pp. 469-472.
- Okuma, M., Heylen, W., Sas, P., 2001. Identification of rigid body properties of 3-D frame structure by MCK identification method, PROCEEDINGS OF THE INTERNATIONAL SEMINAR ON MODAL ANALYSIS. KU Leuven; 1998, pp. 1217-1224.
- Okuma, M., Shi, Q., 1997. Identification of principal rigid body modes under free-free boundary condition. Journal of vibration and acoustics 119, 341-345.
- Pandit, S.M., Hu, Z., Yao, Y., 1992. Experimental technique for accurate determination of rigid body characteristics, Proc. 10th Int. Modal Analysis Conference, pp. 307-311.
- Pandit, S.M., Hu, Z.Q., 1994. Determination of Rigid-Body Characteristics from Time-Domain Modal Test Data. Journal of Sound and Vibration 177, 31-41.
- T., T.W., D, D.M., 2004. Theory of Vibration with Applications, 5th Edition ed. Prentice Hall.
- Urgueira, A.P.V., 1995. On the Rigid Body Properties Estimation from Modal Testing, 13th Int. Modal Analysis Conference (IMAC XIII), pp. 1479-1483.
- Wei, Y.S., Reis, J., 1989. Experimental Determination of Rigid Body Inertia Properties, Proc. 7th Int. Modal Analysis Conference (IMAC VII), pp. 603-606.
- Zhi-Chao, H., Yi-ning, L., Yao-xin, L., Dan, L., 2009. A new trifilar pendulum approach to identify all inertia parameters of a rigid body or assembly. Mech Mach Theory 44, 1270-1280.