A Wolff Theorem for finite rank bounded symmetric domains

P. Mellon

Abstract

We present a Wolff Theorem for all infinite dimensional bounded symmetric domains of finite rank. Namely, if B is the open unit ball of any finite rank JB^*-triple and $f : B \to B$ is a compact holomorphic map with no fixed point in B, we prove convex f-invariant subdomains of B (of all sizes and at all points) exist in the form of simple operator balls $c_\lambda + T_\lambda(B)$, for $c_\lambda \in B$ and T_λ an invertible linear map. These are exact infinite dimensional analogues of the invariant discs in Δ, the invariant ellipsoids in the Hilbert ball and invariant domains in finite dimensional triples. Results are new for rank > 2, even for classical spaces such as C^*-algebras and JB^*-algebras.

Introduction

Iteration theory of holomorphic maps on Banach spaces has as its foundation some appropriate analogue of Wolff's theorem on Δ [37, 38]; recall if $f : \Delta \to \Delta$ is a holomorphic fixed-point free map, then there exists $\xi \in \partial\Delta$ such that each disc internally tangent at ξ is f-invariant. In a Hilbert space ellipsoids replace the internally tangent discs [18]. In strictly convex domains, in $\mathbb{C}^n$ [1, 2, 7], or in Banach spaces [9, 10], the internally tangent discs are replaced by horospheres defined in terms of the Kobayashi distance. Although these horospheres are defined for arbitrary Banach spaces [2, 3, 32], if the boundary of the ball is more complicated they are considerably less tractable, even in finite dimensions [17, 4, 16]. While holomorphic iteration on both finite and infinite dimensional Banach spaces has recently continued apace [5, 7, 8, 9, 11, 12, 20, 28, 31, 32, 34], spaces whose balls have, for example, non-strictly convex boundaries, even classical spaces such as the C^*-algebras or $L(H, K)$, still require a Wolff theorem and concrete descriptions of invariant domains to facilitate progress on iteration.

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Banach spaces whose open unit ball is homogeneous, however, (including Hilbert spaces, C^*-, J^*-, and JB^*-algebras, among others) are classified as JB^*-triples and Jordan theory enables significant other techniques. In [30, Theorems 3.8, 3.10], the author gave a Wolff theorem for all finite dimensional JB^*-triples, and more generally, for all triples satisfying a certain additional condition. This paper provides a Wolff theorem for all infinite dimensional JB^*-triples of finite rank, together with an explicit algebraic description of the resulting invariant domains in terms of the Jordan product. The advantage with JB^*-triples is that the Jordan product replaces the Kobayashi distance, enabling us to show that the invariant domains are operator balls

$$E_\lambda = c_\lambda + T_\lambda(B), \text{ for } c_\lambda \in B \text{ and } T_\lambda \in \mathsf{GL}(Z).$$

In infinite dimensions lack of compactness of the closed ball forces us to work with the weak topology, made possible as the norm of a finite rank triple is equivalent to a Hilbert norm [24, 25]. Trickier to overcome, however, is that the Jordan product is not weakly continuous [27]. Jordan theory captures the desired behaviour though, even for classical spaces such as C^*-algebras or $L(H, K)$. For that reason, we summarise part of our main result below in a Jordan-free form. The results are new in infinite dimensions for rank greater than 2. Crucially, despite the underlying force being the very non-linear Kobayashi distance, the resulting invariant domains display straighforward affine structure and exist at every point of B.

0.1 Theorem. *Let Z be any finite rank JB^*-triple with open unit ball B and let $f : B \to B$ be a compact holomorphic fixed-point free map. Then there exists $e \in \partial B$, such that for all $\lambda > 0$, there is $c_\lambda \in B$ and $T_\lambda \in \mathsf{GL}(Z)$ such that the operator ball*

$$E_\lambda := c_\lambda + T_\lambda(B)$$

is a convex f-invariant domain in B containing e in its boundary.

Moreover, for each $y \in B$, there is $\lambda_y > 0$ with $y \in \partial E_{\lambda_y}$.

If Z has rank 1, namely, it is a Hilbert space, then the E_λs are exactly the earlier ellipsoids [18], [32]. The theorem reproduces results for finite dimensional triples [30]. We note that the point e is a tripotent ($\{e, e, e\} = e$), satisfying $e = c_\lambda + T_\lambda(e)$, and the maps T_λ are invertible linear operators defined in terms of the triple product.

1 Notation and background

1.1 JB^*-triples

Every homogeneous open unit ball is biholomorphically equivalent to a bounded symmetric domain, classified as the open unit ball of a JB^*-triple [21]. For X,Y complex

Banach spaces, $\mathcal{L}(X,Y)$ denotes continuous linear maps from X to Y, $\mathcal{L}(X) = \mathcal{L}(X,X)$ and $\mathsf{GL}(X)$ is invertible elements in $\mathcal{L}(X)$.

1.1 Definition. A JB^*-triple is a complex Banach space Z with a real trilinear mapping $\{\cdot,\cdot,\cdot\} : Z \times Z \times Z \to Z$ satisfying

(i) $\{x,y,z\}$ is complex linear and symmetric in the outer variables x and z, and is complex anti-linear in y.

(ii) The map $z \to \{x,x,z\}$, denoted $x \,\Box\, x$, is Hermitian, $\sigma(x \,\Box\, x) \geq 0$ and $\|x \,\Box\, x\| = \|x\|^2$ for all $x \in Z$, where σ denotes the spectrum.

(iii) The product satisfies the following "triple identity"

$$\{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} - \{x,\{b,a,y\},z\} + \{x,y,\{a,b,z\}\}.$$

Classical examples include $\mathcal{L}(H)$, H a complex Hilbert space, with $\{x,y,z\} = 1/2(xy^*z + zy^*x)$, where y^* denotes the usual operator adjoint of y. In fact, the first four of the six types of the so-called Cartan factors are subtriples of $\mathcal{L}(H)$, cf [13, Example 9.2].

The triple product is continuous, giving linear maps: $x \,\Box\, y \in \mathcal{L}(Z) : z \to \{x,y,z\}$, $Q(x) \in \mathcal{L}_{\mathbb{R}}(Z) : z \to \{x,z,x\}$, and the geometrically significant Bergman operators

$$B(x,y) = I - 2x \,\Box\, y + Q(x)Q(y) \in \mathcal{L}(Z).$$

Let $\mathsf{Aut}(B)$ denote all biholomorphic maps from B to B. For z in B, we have $g_z \in \mathsf{Aut}(B)$ defined by

$$g_z(w) = z + B(z,z)^{\frac{1}{2}}(I + w \,\Box\, z)^{-1}w$$

satisfying $g_z(0) = z$, $g_z^{-1} = g_{-z}$ and $g_z'(0) = B(z,z)^{\frac{1}{2}}$ (defined in terms of a functional calculus) [21].

We will refer to the concept of (holomorphic) boundary component, K_x, of $x \in \overline{B}$. For finite rank triples, every such boundary component, K_x, is determined by a unique tripotent $e = \{e,e,e\}$, such that $K_x = K_e$. Boundary components in the finite rank case are classified by the triple product [23].

1.2 Spectral Decomposition

Every tripotent $e = \{e,e,e\}$ induces a splitting of Z, as $Z = Z_0(e) \oplus Z_{\frac{1}{2}}(e) \oplus Z_1(e)$, where $Z_k(e)$ is the k eigenspace of $e \Box e$ and $P_0(e) = B(e,e), P_{\frac{1}{2}}(e) = 2(e \,\Box\, e - Q(e)Q(e))$, and $P_1(e) = Q(e)Q(e)$ are mutually orthogonal projections of Z onto $Z_0(e)$, $Z_{\frac{1}{2}}(e)$, and $Z_1(e)$ respectively. We say e is maximal if $Z_0(e) = 0$ and minimal if $Z_1(e) = \mathbb{C}e$. Elements $x, y \in Z$ are orthogonal, $x \perp y$, if $x \Box y = 0$ ($\Leftrightarrow y \Box x = 0 \Leftrightarrow \{x,x,y\} = 0$). Z is said to have finite rank r if every element $z \in Z$ is contained in a subtriple of (complex) dimension $\leq r$, and r is minimal with this property. If Z has finite rank r, a frame is a set $\{e_1, \ldots, e_r\}$ of non-zero pairwise orthogonal minimal tripotents. Every $z \in Z$ then has a unique spectral decomposition, called its Peirce decomposition, $z = \lambda_1 e_1 +$

$\cdots + \lambda_r e_r$, for some frame $\{e_1, \ldots, e_r\}$ and scalars $0 \leq \lambda_1 \leq \ldots \leq \lambda_r = \|z\|$. A fully fledged spectral theory exists for JB^*-triples [22]. In particular, for $z \in Z$, the closed JB^*-subtriple generated by z, denoted Z_z, is isometrically J^*-isomorphic to $C_0(S)$, for locally compact $S = S_z \subset [0, \|z\|]$ called the triple spectrum of z, where z is identified with the map $z(s) = s$ for $s \in S$, and $\{a, b, c\} = a\bar{b}c$, for $a, b, c \in C_0(S)$. The rank of an element z in Z, rank(z), is defined as the (complex) dimension of Z_z. If $z \perp w$ then rank$(z + w)$ =rank(z)+rank(w). See [29, 22] for details.

1.3 Algebraic Norm on Z

For a JB^*-triple of finite rank r, the norm is equivalent to a Hilbert norm [24, 25], cf [13, Propositions 9.11, 9.13]. Namely, if $z = \lambda_1 e_1 + \cdots + \lambda_r e_r$ is the Peirce decomposition of z, for frame $\{e_1, \cdots, e_r\}$ and $0 \leq \lambda_1 \leq \cdots \leq \lambda_r = \|z\|$ then
$\|z\|_a := \sqrt{\lambda_1^2 + \cdots + \lambda_r^2}$ is a Hilbert norm, referred to as the algebraic norm on Z. Clearly $\|z\| \leq \|z\|_a$, $\|z\| = \|z\|_a$ if, and only if, z is a multiple of a minimal tripotent (rank$(z) \leq 1$) and $\|x + y\|_a^2 = \|x\|_a^2 + \|y\|_a^2$ if $x \perp y$. Since for any Hilbert norm, $\| \cdot \|_H$, if $a = w\text{-}\lim_k a_k$ and $\|a\|_H = \lim_k \|a_k\|_H$ then $a = \lim_k a_k$. In particular, for any sequence (a_k) in Z

$$a = \lim_k a_k \iff a = w\text{-}\lim_k a_k \text{ and } \|a\|_a^2 = \lim_k \|a_k\|_a^2 \ . \tag{1}$$

1.4 Kobayashi balls as operator balls

The Kobayashi distance, κ, is key to f-invariance. On the ball B of a JB^*-triple Z $\kappa(z, w) = \tanh^{-1} \|g_{-z}(w)\|$, for $z, w \in B$. Importantly, Kobayashi balls centered at z are the images of norm balls under automorphism $g_z \in \mathsf{Aut}(B)$.

1.2 Definition. [30, Section 2] The Kobayashi ball about $z \in B$ of radius $\tanh^{-1} r$, for $0 < r < 1$, is denoted $D_{z,r} := B_\kappa(z, \tanh^{-1} r) = g_z (B(0, r))$, where $B(0, r) := \{x \in B : \|x\| < r\}$.

Kobayashi balls have been shown to display affine structure here.

1.3 Theorem. *[30, Results 2.2, 2.3, 2.5] For $z \in B$ and $0 < r < 1$, $D_{z,r} = c + T(B)$, where $c = (1 - r^2) B_{rz}^{-1}(z) \in B$ and $T = r B_z B_{rz}^{-1} \in \mathsf{GL}(Z)$, for $B_z = B(z, z)^{1/2}$. Moreover, $D_{z,r}$ is convex and if $f(z) = z$ then $D_{z,r}$ is f-invariant.*

If Z is finite rank, Kobayashi balls have more concrete algebraic descriptions, facilitating calculations via functional calculus. We use [29, Corollary 3.15] to get the following adaptation of [30, Proposition 2.6].

1.4 Corollary. *Let Z have finite rank r and $z = \gamma_1 e_1 + \cdots + \gamma_r e_r$ be the Peirce decomposition of z. Then, for $\delta > 0$, $D_{z,\delta} = c + T\,(B)$ where*

$$c = \sum_1^r \frac{(1-\delta^2)\gamma_i e_i}{1-\delta^2\gamma_i^2}, \quad v = \sum_{i=1}^r s_i e_i \ \text{ and } \ T = \delta B(v,v)$$

where s_i satisfies $(1-s_i^2)^2 = \frac{1-\gamma_i^2}{1-\delta^2\gamma_i^2}$, $1 \le i \le r$.

Domains of the form $c+T(B)$ are more generally referred to as operator balls. For proofs in this section see [30, Section 2].

2 New Results

Let Z be a JB^*-triple with open unit ball B and $f : B \to B$ be a compact holomorphic map having no fixed point in B. Choose $(\alpha_k)_k$, $0 < \alpha_k < 1$, $\alpha_k \uparrow 1$ and let $f_k := \alpha_k f$ for all k. Then f_k has a fixed point, z_k, in $\alpha_k B$ [14] and, as f is compact, we may assume $z_k \to \xi \in \overline{B}$ and hence $\xi \in \partial B$, as otherwise it would be a fixed point of f. [30, Theorem 3.8] proves that whenever

(2) $$R := \lim_k (1 - \|z_k\|^2) B_{z_k}^{-1} \text{ exists in } \mathcal{L}(Z)$$

then a Wolff theorem exists and, in particular, for $\lambda > 0$

$$E_\lambda := \{w \in B : \|B_w^{-1} B(w,\xi) R\| < \lambda\}$$

is a non-empty convex f-invariant subdomain of B with $\xi \in \partial E_\lambda$. Our first aim is therefore to show that (2) holds, while our second is to describe E_λ as a simple operator ball. We recall $B_z = B(z,z)^{1/2} = g_z'(0)$, $g_z \in Aut(B)$, $g_z(0) = z$, and B_z is invertible $\Leftrightarrow z \in B$ [21]. In fact, since $\|B_z^{-1}\| = \frac{1}{1-\|z\|^2}$ [26], the sequence $\left((1-\|z_k\|^2)B_{z_k}^{-1}\right)_k$ is contained in the closed unit ball of $\mathcal{L}(Z)$. While compactness in finite dimensions guarantees (2), this is not the case in infinite dimensions. For this reason, Z will henceforth be of finite rank r. The norm is then equivalent to a Hilbert norm making $\overline{B}$ weakly compact [24]. We must, of course, overcome the significant fact that Jordan properties do not, in general, pass to weak limits [27].

We begin with the spectral decomposition of ξ, $\xi = \mu_1 e_1 + \cdots + \mu_r e_r$ where $\{e_1, \cdots, e_r\}$ is a frame and $0 \le \mu_1 \le \cdots \le \mu_r = \|\xi\| = 1$. Let $p \in \{0, \ldots, r-1\}$ satisfy $\mu_{p+1} = \cdots = \mu_r = 1$ and if $p \neq 0$ then $\mu_p < 1$. Then

(3) $$\xi = e + v, \text{ for } e = \sum_{i=p+1}^r e_i \text{ where, if } p \neq 0, v = \sum_{i=1}^p \mu_i e_i \text{ (and } v = 0 \text{ if } p = 0).$$

Then $e \perp v$, $v \in B$ and $e \in \partial B$ is the unique tripotent determining the boundary component of ξ with $K_\xi = K_e$ [23]. Moreover p =rank(e).

Similarly, for $k \in N$, each z_k also has Peirce decomposition,

$$z_k = \gamma_{k1} e_{k1} + \cdots + \gamma_{kr} e_{kr} \tag{4}$$

where $0 \leq \gamma_{k1} \leq \cdots \leq \gamma_{kr} = \|z_k\|$ and $\{e_{k1}, \ldots, e_{kr}\}$ is a frame. Define $\gamma_i := \lim_k \gamma_{ki}$ (passing to a subsequence if necessary), for $1 \leq i \leq r$. Clearly $0 \leq \gamma_1 \leq \cdots \leq \gamma_r = 1$. Define $q \in \{0, \ldots, r-1\}$ so that $\gamma_{q+1} = \cdots = \gamma_r = 1$ and if $q \neq 0$ then $\gamma_q < 1$. Write

$$w_k = \sum_{i=q+1}^{r} \gamma_{ki} e_{ki} \text{ and, if } q \neq 0,\ x_k = \sum_{i=1}^{q} \gamma_{ki} e_{ki} \text{ (and } x_k = 0 \text{ if } q = 0). \tag{5}$$

Then

$$z_k = w_k + x_k \quad w_k \perp x_k, \|z_k\| = \|w_k\| = \gamma_{kr},\ \|x_k\| = \gamma_{kq} < \alpha < 1 \text{ (some } 0 < \alpha < 1). \tag{6}$$

As $\overline{B}$ is weakly compact (passing to a subsequence if necessary) $(e_{ki})_k$ has a weak limit, $d_i := w\text{-}\lim_k e_{ki}$ in $\overline{B}$, for $1 \leq i \leq r$. As we must marry the weak topology with the triple product - which is not weakly continuous - effort is both required and justified to establish certain crucial weak limits as norm limits. As the invariant domains turn out (Theorem 2.7) to depend only on the boundary component of ξ, determined by $e = e_{p+1} + \cdots + e_r$, the crucial weak limits will turn out to be $d_{p+1}, \ldots, d_r$. Of course, weak limits are not generally norm limits, but techniques including functional calculus, the algebraic norm and considerations of rank are used to obtain the first central result below. This theorem thereafter enables us to revert, more or less, to the norm topology. For clarity, three lemmata establishing simple facts on the rank of some weak limits, are given after the Theorem.

2.1 Theorem. *With notation as above, then*

$$p = q \text{ and } e_i = d_i = \lim_k e_{ki}, \text{ for } p+1 \leq i \leq r.$$

In particular, $e = \lim_k w_k$ *and* $v = \lim_k x_k$.

Proof. By uniqueness of limits $w\text{-}\lim_k z_k = \lim_k z_k = \xi$ so (3) and (4) give

$$\gamma_1 d_1 + \cdots + \gamma_q d_q + d_{q+1} + \cdots + d_r = e + v. \tag{7}$$

Fix $n \in \mathbb{N}$ and let $h_n(t)$ be the odd real polynomial $h_n(t) = t^{2n+1}$. By continuity of the functional calculus [26, Lemma 2.2] $\lim_k z_k = \xi$ implies $\lim_k h_n(z_k) = h_n(\xi) = e + v^{2n+1}$. On the other hand, from (4),

$$h_n(z_k) = \gamma_{k1}^{2n+1} e_{k1} + \ldots + \gamma_{kq}^{2n+1} e_{kq} + \gamma_{k(q+1)}^{2n+1} e_{k(q+1)} \cdots + \gamma_{kr}^{2n+1} e_{kr}$$

$$\text{so } w\text{-}\lim_k h_n(z_k) = \gamma_1^{2n+1} d_1 + \cdots + \gamma_q^{2n+1} d_q + d_{q+1} \ \cdots + d_r.$$

Therefore $e + v^{2n+1} = \gamma_1^{2n+1} d_1 + \cdots + \gamma_q^{2n+1} d_q + d_{q+1} + \cdots + d_r$ and limiting over n gives

$$e = e_{p+1} + \cdots + e_r = d_{q+1} + \cdots + d_r = w\text{-}\lim_k w_k \tag{8}$$

and hence then from (7) $v = \gamma_1 d_1 + \cdots + \gamma_q d_q = w\text{-}\lim_k x_k$.

Now $\lim_k \|w_k\|_a^2 = \lim_k \sum_{i=q+1}^r \gamma_{ki}^2 = \sum_{i=q+1}^r \gamma_i^2 = r - q$. From (8) $e = w\text{-}\lim_k w_k$, so

$$r - p = \|e\|_a^2 \leq \liminf_k \|w_k\|_a^2 = r - q. \tag{9}$$

Similarly,

$$\|v\|_a^2 \leq \liminf_k \|x_k\|_a^2 = \lim_k \sum_{i=1}^q \gamma_{ki}^2 = \sum_{i=1}^q \gamma_i^2. \tag{10}$$

On the other hand, $\xi = \lim_k z_k$ gives

$$\|\xi\|_a^2 = \lim_k \|z_k\|_a^2 = \left(\sum_{i=1}^q \gamma_i^2 \right) + r - q. \tag{11}$$

From (3), $\xi = e + v$ for $v \perp e$ so

$$\|\xi\|_a^2 = \|e\|_a^2 + \|v\|_a^2. \tag{12}$$

Combining (11), (12), (9) and (10) we have

$$\left(\sum_{i=1}^q \gamma_i^2 \right) + r - q = \|\xi\|_a^2 = \|e\|_a^2 + \|v\|_a^2 \leq \left(\sum_{i=1}^q \gamma_i^2 \right) + r - q.$$

Therefore we must have equality in (9) and (10), namely,

$$p = q, \quad \|e\|_a^2 = \lim_k \|w_k\|_a^2 \quad \text{and} \; \|v\|_a^2 = \lim_k \|x_k\|_a^2.$$

Property (1) and (8) then implies $e = \lim_k w_k$ and $v = \lim_k x_k$.
Lemma 2.3 below gives $d_i = \alpha_i u_i$, for $0 \leq \alpha_i \in \mathbb{R}$ and u_i a non-zero minimal tripotent and (8) becomes

$$e_{p+1} + \cdots + e_r = \alpha_{p+1} u_{p+1} + \cdots + \alpha_r u_r. \tag{13}$$

Moreover, $\text{rank}(d_i) \leq \text{rank}(u_i) = 1$, with equality if, and only if, $\alpha_i \neq 0,\ 1 \leq i \leq r$. From Lemma 2.4 below $\text{rank}(d_{p+1} + \cdots + d_r) \leq \text{rank}(d_{p+1}) + \cdots + \text{rank}(d_r)$, with equality if, and only if, $d_i \perp d_j,\ p+1 \leq i \neq j \leq r$. Thus (13) gives

$$\begin{aligned} r - p = \text{rank}(d_{p+1} + \cdots + d_r) &\leq \text{rank}(d_{p+1}) + \cdots + \text{rank}(d_r) \\ &\leq \text{rank}(u_{p+1}) + \cdots + \text{rank}(u_r) = r - p \end{aligned}$$

and therefore $\alpha_i \neq 0$ and $u_i \perp u_j$, for $p+1 \leq i \neq j \leq r$. In other words, $\{u_{p+1}, \cdots, u_r\}$ is a set of (non-zero) mutually orthogonal minimal tripotents. Uniqueness of the Peirce decomposition and (13) then imply that $\alpha_{p+1} = \cdots = \alpha_r = 1$ and (relabelling if necessary) $e_i = u_i = d_i = w\text{-}\lim_k e_{ki},\ p+1 \leq i \leq r$. Since $1 = \|e_i\|_a = \lim_k \|e_{ki}\|_a$ then (1) implies $e_i = \lim_k e_{ki},\ p+1 \leq i \leq r$ and we are done.

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2.2 Lemma. *Let Z be a finite rank Cartan factor, (f_k) be a sequence of minimal tripotents in Z and $d = w\text{-}\lim_k f_k$. Then rank$(d) \leq 1$.*

Proof. For a description of the Cartan factors see [13, Example 9.2]. Let Z be a finite rank Cartan factor and $d = w\text{-}\lim_k f_k$, where each f_k is a minimal tripotent. If Z is of type II, III, V, or VI and finite rank, then Z is finite dimensional giving $d = \lim_k f_k$ and hence d is a tripotent. Since each f_k is minimal, $\|f_k\|_a = \|f_k\|$ for all k and hence $\|d\|_a = \|d\|$, so rank$(d) \leq 1$. If Z is of type I or IV, it can be embedded as a subtriple $Z \subset \mathcal{L}(H)$, for some Hilbert space H. Assume $d \neq 0$ and let $0 < t =$rank$(d) \leq r$. Spectral decomposition gives $d = \alpha_1 g_1 + \cdots + \alpha_t g_t$, for real numbers $0 < \alpha_1 \leq \ldots \leq \alpha_t = \|d\|$ and $\{g_1, \cdots, g_t\}$ a mutually orthogonal set of non-zero minimal tripotents. Let $g = g_1 + \cdots + g_t$. Then g is a tripotent and let $Z = Z_1 \oplus Z_{1/2} \oplus Z_0$ ($Z_i = Z_i(g)$) be the corresponding Peirce decomposition of Z. As $Z \subset \mathcal{L}(H)$, for $V = \ker(g), U = V^{\perp}, U_1 = g(U), V_1 = U_1^{\perp}$ then $H = U \oplus V = U_1 \oplus V_1$ and $\dim(U) = \dim(U_1) =$rank$(g) = t$. Let $z \in Z$ have Peirce decomposition (with respect to g) $z = z_1 + z_{1/2} + z_0$. Then z can be represented (cf [36, Example 21.10]) as

$$z = \begin{bmatrix} z_1 & z_{12} \\ z_{21} & z_0 \end{bmatrix} \in \begin{bmatrix} \mathcal{L}(U, U_1) & \mathcal{L}(V, U_1) \\ \mathcal{L}(U, V_1) & \mathcal{L}(V, V_1) \end{bmatrix} z_1 \in Z_1, z_{1/2} = z_{12} + z_{21} \in Z_{1/2}, z_0 \in Z_0.$$

In particular, choosing suitable co-ordinates, $g = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, and $d = \begin{bmatrix} \widetilde{d} & 0 \\ 0 & 0 \end{bmatrix}$, for I the $t \times t$ identity matrix and $\widetilde{d}$ the diagonal matrix with $\alpha_1, \ldots, \alpha_t$ on the diagonal. Note that $\dim(d(H)) = \dim(g(H)) = t =$ rank(d), that is, the triple rank and the operator rank of d coincide here. As above, write $f_k = \begin{bmatrix} f_1^k & f_{12}^k \\ f_{21}^k & f_0^k \end{bmatrix}$. Then $d = w\text{-}\lim_k f_k$ in Z implies $\widetilde{d} = w\text{-}\lim_k f_1^k$ in $\mathcal{L}(U, U_1)$. Since $\mathcal{L}(U, U_1)$ is finite dimensional, this gives $\widetilde{d} = \lim_k f_1^k$. $\mathcal{L}(U, U_1)$ may be identified with the set, M_t, of all $t \times t$ matrices, where if $A = \lim_k A_k$ in M_t then $\text{rank}_M(A) \leq \lim_k \text{rank}_M(A_k)$, where $\text{rank}_M(A)$ denotes the usual matrix (or operator) rank of A. Therefore $t = \text{rank}_M(\widetilde{d}) \leq \lim_k \text{rank}_M(f_1^k) \leq \lim_k \text{rank}_M(f_k) \leq 1$, as each f_k is a minimal tripotent and we are done.

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2.3 Lemma. *Let Z be a finite rank JB^*-triple, (f_k) be a sequence of minimal tripotents in Z and $d = w\text{-}\lim_k f_k$. Then rank$(d) \leq 1$.*

Proof. By [24, 25, 15] every finite rank JB^*-triple is isometrically isomorphic to an l_∞ direct sum of finite rank Cartan factors. Thus $Z = \bigoplus_{i \in I}^{\infty} C_i$, where for each $i \in I, C_i$ is a finite rank Cartan factor, and $\|z\| = \sup_i \|z_i\|$, for $z = (z_i)_i, z_i \in C_i$. Let now $d = w\text{-}\lim_k f_k$, where each f_k is a minimal tripotent in Z. Then $d_i = w\text{-}\lim_k (f_k)_i \in C_i, i \in I$. We note that, by minimality, each f_k has at most one non-zero component, namely, if $(f_k)_i \neq 0$ then $(f_k)_i$ is a minimal tripotent in C_i and $(f_k)_j = 0$ for all $j \neq i$.
Assume that $d \neq 0$. Then $d_i \neq 0$ for some $i \in I$. Fix this i. As $d_i = w\text{-}\lim_k (f_k)_i \in C_i$ then $0 < \|d_i\| \leq \liminf_k \|(f_k)_i\|$. Thus there exists $K \in \mathbb{N}$ such that for $k > K, (f_k)_i \neq 0$

and hence $(f_k)_i$ is a minimal tripotent in C_i and $(f_k)_j = 0$ for $j \neq i$. Lemma 2.2 then implies d_i is a multiple of a minimal tripotent, say g_i in C_i. On the other hand, for $j \neq i$, $d_j = w\text{-}\lim_k (f_k)_j = 0$ from above. In other words, d has at most one non-zero component $d_i = \alpha_i g_i$ and we are done.

∎

2.4 Lemma. *With $d_i = w\text{-}\lim_k e_{ki}$ as in (4) earlier, then*

$$\mathit{rank}(d_{p+1} + \cdots + d_r) \leq \mathit{rank}(d_{p+1}) + \cdots + \mathit{rank}(d_r)$$

with equality if, and only if, $d_i \perp d_j$ for all $p+1 \leq i \neq j \leq r$.

Proof. $Z = \bigoplus_{i\in I}^{\infty} C_i$, C_i a finite rank Cartan factor, and $\operatorname{rank}(z) = \sum_{i\in I}\operatorname{rank}(z_i)$ (at most finitely many terms in this sum are non-zero) for $z = (z_i)_i$, $z_i \in C_i$. Fix $i \in I$.
(i) For C_i finite rank of type II, III, V or VI: as C_i is finite dimensional, $(d_n)_i = w\text{-}\lim_k (e_{kn})_i$ implies $(d_n)_i = \lim_k (e_{kn})_i$ and hence $\{(d_n)_i, (d_n)_i, (d_m)_i\} = \lim_k \{e_{kn}, e_{kn}, e_{km}\}_i = 0$ since $e_{kn} \perp e_{km}$, for $1 \leq n \neq m \leq r$ and all k. Thus, $(d_n)_i \perp (d_m)_i$, $1 \leq n \neq m \leq r$ and therefore $\operatorname{rank}((d_{p+1})_i + \cdots + (d_r)_i) = \operatorname{rank}((d_{p+1})_i) + \cdots + \operatorname{rank}((d_r)_i)$.
(ii) For C_i finite rank of type I or IV: as in Lemma 2.2, $C_i \subset L(H)$, for some Hilbert space H. From Lemma 2.2, $\operatorname{rank}((d_n)_i) \leq 1$, $1 \leq n \leq r$. Therefore
$\operatorname{rank}((d_{p+1})_i + \cdots + (d_r)_i) \leq \operatorname{rank}((d_{p+1})_i) + \cdots + \operatorname{rank}((d_r)_i)$ with equality if, and only if, $(d_n)_i \perp (d_m)_i$, for all $p+1 \leq n \neq m \leq r$, cf [19].

$$\begin{aligned}\text{Hence, } \operatorname{rank}(d_{p+1} + \cdots + d_r) &= \sum_{i\in I} \operatorname{rank}((d_{p+1})_i + \cdots + (d_r)_i)\\ &\leq \sum_{i\in I} \operatorname{rank}((d_{p+1})_i) + \cdots + \operatorname{rank}((d_r)_i) \text{ (from (i) and (ii) above)}\\ &= \operatorname{rank}(d_{p+1}) + \cdots + \operatorname{rank}(d_r)\end{aligned}$$

with equality if, and only if, $(d_n)_i \perp (d_m)_i$ for all $i \in I$, namely, $d_n \perp d_m$, $p+1 \leq n \neq m \leq r$.

∎

We return now to (2) and use spectral theory to get an alternative description of B_z^{-1}.

2.5 Theorem. *Let $z \in B$. Then $B_z^{-1} = B(q, y)$, where $q = \frac{z}{\sqrt{1-|z|^2}}$ and $y = \frac{-z}{1+\sqrt{1-|z|^2}}$ are calculated in Z_z. In addition,*

$$\|q\| = \frac{\|z\|}{\sqrt{1-\|z\|^2}} \quad \text{and} \quad \|y\| = \frac{\|z\|}{1+\sqrt{1-\|z\|^2}}. \tag{14}$$

In particular, $\|y\| \leq \|z\|$ so $y \in B$.

Proof. Fix $z \in B$ and let $S := S_z \subseteq [0, \|z\|] \subset [0, 1)$ be the triple spectrum of z so that $Z_z \cong C_0(S)$ with $z(s) = s$, $s \in S$. Consider real odd functions q and y in $C_0(S)$, given by

$$q(s) = \frac{s}{\sqrt{1-s^2}},\ y(s) = \frac{-s}{1+\sqrt{1-s^2}},\ \text{and } (1-q(s)y(s))^2 = \frac{1}{1-s^2},\ s \in S. \tag{15}$$

Then $q, y \in Z$ and restricting maps $B(q,y)$ and B_z^{-1} in $\mathcal{L}(Z)$ to the subspace $Z_z \cong C_0(S)$ gives, from (15)

$$B(q,y)(x)(s) = (1 - q(s)y(s))^2 x(s) = \left(\frac{1}{1-s^2}\right) x(s) = B_z^{-1}(x)(s), \quad \text{for } x \in C_0(S), s \in S.$$

Therefore B_z^{-1} and $B(q,y)$ agree on Z_z and since B_z^{-1} is uniquely determined by its action on Z_z [26, Proposition 2.5, Theorem 3.5], we have $B_z^{-1} = B(q,y)$ in $\mathcal{L}(Z)$. Since q and $-y$ are both strictly increasing on S, it follows that $\|q\| = q(\|z\|)$ and $\|y\| = \|-y\| = -y(\|z\|)$.

■

The next result is a useful linearisation of [31, Lemma 2.5].

2.6 Lemma. *Let $a \perp b$ in Z. Then*
$B_{a+b} = B_a \circ B_b$ in $\mathcal{L}(Z)$. In particular, B_a and B_b commute.

Proof. Let $a \perp b \in B$. [31, Lemma 2.5] implies $g_{a+b} = g_a \circ g_b = g_b \circ g_a$. As $g_a'(z) = B_a B(z, -a)^{-1}$ this gives $B_{a+b} = g_{a+b}'(0) = g_a'(b) \circ g_b'(0) = g_b'(a) \circ g_a'(0)$. In other words, $B_{a+b} = B_a \circ B(b,-a)^{-1} \circ B_b = B_b \circ B(a,-b)^{-1} \circ B_a$. As $a \,\square\, b = 0$, the Jordan triple identity implies $Q(a)Q(b) = 0$ so that $B(a,-b) = B(b,-a) = I$.

■

The following result shows that property (2) holds and therefore, by [30, Theorem 3.8], a Wolff theorem exists for all finite rank triples.

2.7 Theorem. *Let Z be a finite rank JB^*-triple. Let $e = \sum_{p+1}^r e_i$ be the unique tripotent determining the boundary component of ξ. Then*

$$R = \lim_k (1 - \|z_k\|^2) B_{z_k}^{-1} = Q(u)Q(u), \text{ for } u \in \partial B \text{ given by } u = \sum_{i=p+1}^{r} a_i^{1/4} e_i,$$

where $a_i = \lim_k \frac{1-\gamma_{kr}^2}{1-\gamma_{ki}^2}$, $1 \le i \le r$.

Proof. Since (6) $z_k = w_k + x_k$, $w_k \perp x_k$ and $\|z_k\| = \|w_k\|$, Lemma 2.6 gives $B_{z_k} = B_{w_k} \circ B_{x_k} = B_{x_k} \circ B_{w_k}$ and hence

$$\lim_k (1 - \|z_k\|^2) B_{z_k}^{-1} = \lim_k (1 - \|w_k\|^2) B_{x_k}^{-1} B_{w_k}^{-1} = \lim_k (1 - \|w_k\|^2) B_{w_k}^{-1} B_{x_k}^{-1}.$$

The map $x \to B_x^{-1}$ is continuous on B and since, from Theorem 2.1, $v = \lim_k x_k \in B$ then $\lim_k B_{x_k}^{-1} = B_v^{-1}$ and hence

$$\lim_k (1 - \|z_k\|^2) B_{z_k}^{-1} = B_v^{-1} \left(\lim_k (1 - \|w_k\|^2) B_{w_k}^{-1} \right) = \left(\lim_k (1 - \|w_k\|^2) B_{w_k}^{-1} \right) B_v^{-1}. \tag{16}$$

Fix $k \in N$. Theorem 2.5 gives $B_{w_k}^{-1} = B(q_k, y_k)$, for $q_k = q(w_k)$ and $y_k = y(w_k)$ and hence

$$(1 - \|w_k\|^2)B_{w_k}^{-1} = (1 - \|w_k\|^2)I - 2(1 - \|w_k\|^2)q_k \,\Box\, y_k + (1 - \|w_k\|^2)Q(q_k)Q(y_k).$$

From (14)

$$\|q_k \,\Box\, y_k\| \leq \|q_k\|\|y_k\| = \frac{\|w_k\|^2}{(\sqrt{1 - \|w_k\|^2})(1 + \sqrt{1 - \|w_k\|^2})} \leq \frac{1}{\sqrt{1 - \|w_k\|^2}}.$$

In particular, $\|(1 - \|w_k\|^2)q_k \,\Box\, y_k\| \leq \sqrt{1 - \|w_k\|^2} \to_k 0$ and hence

$$\lim_k (1 - \|w_k\|^2)B_{w_k}^{-1} = \lim_k Q(\sqrt{1 - \|w_k\|^2}q_k)Q(y_k). \tag{17}$$

By Theorem 2.1, $\lim_k w_k = e$. Now $y \in C[0, 1]$ so by continuity of the functional calculus [26] $\lim_k y_k = \lim_k y(w_k) = y(e) = -e$ and hence $\lim_k Q(y_k) = Q(-e) = Q(e)$. Since q is not continuous at 1, we cannot use the same method for q_k. From (5) and Theorem 2.1 $w_k = \sum_{i=p+1}^{r} \gamma_{ki}e_{ki}$ and $\|w_k\| < 1$ so [26, Theorem 3.5]

$$q_k = q(w_k) = \sum_{i=p+1}^{r} q(\gamma_{ki})e_{ki} = \sum_{i=p+1}^{r} \frac{\gamma_{ki}}{\sqrt{1 - \gamma_{ki}^2}}e_{ki}.$$

Therefore

$$\sqrt{1 - \|w_k\|^2}q(w_k) = \sqrt{1 - \gamma_{kr}^2}q(w_k) = \sum_{i=p+1}^{r} \sqrt{\frac{1 - \gamma_{kr}^2}{1 - \gamma_{ki}^2}}\gamma_{ki}e_{ki}.$$

Passing to a subsequence if necessary, consider the real limit

$$a_i := \lim_k \frac{1 - \gamma_{kr}^2}{1 - \gamma_{ki}^2} \in [0, 1], \text{ for } 1 \leq i \leq r. \tag{18}$$

Then $0 = a_1 = \cdots = a_p \leq a_{p+1} \leq \cdots \leq a_r = 1$. Theorem 2.1 then gives

$$\lim_k \sqrt{1 - \|w_k\|^2}q(w_k) = \sqrt{a_{p+1}}e_{p+1} + \ldots + \sqrt{a_{r-1}}e_{r-1} + e_r.$$

Let $x := \sqrt{a_{p+1}}e_{p+1} + \ldots + \sqrt{a_{r-1}}e_{r-1} + e_r$. Continuity of the triple product and (17) gives $\lim_k (1 - \|w_k\|^2)B_{w_k}^{-1} = Q(x)Q(e)$ and hence (16) gives $\lim_k (1 - \|z_k\|^2)B_{z_k}^{-1} = B_v^{-1}Q(x)Q(e)$. For $i \neq j$, $e_i \perp e_j$ and applications of the triple identity give $0 = Q(e_i)Q(e_j) = Q(e_i)\{e_i, \cdot, e_j\} = \{e_i, Q(e_j)(\cdot), e_j\}$. Expanding thus gives $Q(x)Q(e) = Q(u)Q(u)$ for $u = \sum_{p+1}^{r} a_i{}^{1/4}e_i$. As $a_r = 1, u \in \partial B$. As $v \perp e_i$ for $p + 1 \leq i \leq r$, so $v \perp u$ and hence $Z_v \perp Z_u$. From Theorem 2.5 $B_v^{-1} = B(q, y)$, for $q = q(v), y = y(v) \in Z_v$ and therefore $q, y \perp u$. The triple identity then yields

$$q \,\Box\, y \circ Q(u)Q(u) = 0 \text{ and } Q(q)Q(y)Q(u)Q(u) = 0.$$

In other words, $B_v^{-1} = B(q, y)$ restricts to the identity map on the subspace $Q(u)Q(u)Z$ and therefore

$$\lim_k (1 - \|z_k\|^2)B_{z_k}^{-1} = B_v^{-1}Q(u)Q(u) = Q(u)Q(u).$$

∎

Our main result below is now a Wolff theorem which provides explicit descriptions of f-invariant subdomains of B in terms of invertible linear maps expressed solely in terms of the triple product. These convex f-invariant domains have the affine structure of operator balls and exist at all points of B.

2.8 Theorem. *Let Z be a finite rank JB^*-triple with open unit ball B. Let $f : B \to B$ be a compact holomorphic fixed-point free map. Then there exists $e \in \partial B$, such that for all $\lambda > 0$, there is $c_\lambda \in B$ and $T_\lambda \in GL(Z)$ such that the operator ball*

$$E_\lambda = c_\lambda + T_\lambda(B)$$

is a convex f-invariant domain in B containing e in its boundary.
Moreover, for each $y \in B$, there is $\lambda_y > 0$ with $y \in \partial E_{\lambda_y}$.

To be precise, let e be the unique tripotent in the boundary component of ξ and let $e = e_1 + \cdots + e_s$ be its Peirce decomposition. Then there are scalars $0 \le a_1 \le \ldots \le a_s = 1$ such that

$$c_\lambda = \sum_{i=1}^{s} \left(\frac{a_i}{a_i+\lambda}\right) e_i, \quad v_\lambda = \sum_{i=1}^{s} s_i e_i, \text{ and } \quad T_\lambda = B(v_\lambda, v_\lambda)$$

where s_i satisfies $(1-s_i^2)^2 = \frac{\lambda}{a_i+\lambda}, 1 \le i \le s$.

In particular, $e = c_\lambda + T_\lambda(e) \in \partial E_\lambda$, for all $\lambda > 0$.

Proof. With notation as earlier, Theorem 2.7 proves that condition (2) is satisfied with $R = \lim_k (1 - \|z_k\|^2) B_{z_k}^{-1} = Q(u)Q(u)$, where $u = \sum_{p+1}^{r} a_i^{1/4} e_i \in \partial B$, for $a_i = \lim_k \frac{1-\gamma_{kr}^2}{1-\gamma_{ki}^2}$, $1 \le i \le r$ ($0 = a_1 = \cdots = a_p \le a_{p+1} \le \cdots \le a_r = 1$). Fix $\lambda > 0$ arbitrary. Theorem 3.8 of [30] then gives that

$$E_\lambda = \{w \in B : \|B_w^{-1} B(w, \xi) R\| < \lambda\}$$

is f-invariant and, in addition, is the limit of the sequence of Kobayashi balls $D_k := D(z_k, r_k)$, for $r_k := \sqrt{1 - \frac{(1-\|z_k\|^2)}{\lambda}}$, namely,

$$\text{(19)} \qquad \begin{aligned} &\text{if } z \in E_\lambda \text{ then } z \in D_k \text{ for all } k \text{ large and, conversely,} \\ &\text{if } z \in D_k \text{ for all } k \text{ large, then } z \in \overline{E_\lambda}. \end{aligned}$$

Corollary 1.4 gives $D_k = c_k + T_k(B)$ for

$$c_k = \sum_{i=1}^{r} \left(\frac{1-r_k^2}{1-r_k^2\gamma_{ki}^2}\right) \gamma_{ki} e_{ki}, \quad v_k = \sum_{i=1}^{r} s_{ki} e_{ki} \text{ and } T_k = r_k B(v_k, v_k),$$

where s_{ki} satisfies $(1-s_{ki}^2)^2 = \frac{1-\gamma_{ki}^2}{1-r_k^2\gamma_{ki}^2}$, $1 \le i \le r$. The following real limits have been computed in [30, Theorem 3.10], namely,

$$\text{(20)} \qquad \lim_k \left(\frac{1-r_k^2}{1-r_k^2\gamma_{ki}^2}\right) = \frac{a_i}{a_i+\lambda} \text{ and } \lim_k \frac{1-\gamma_{ki}^2}{1-r_k^2\gamma_{ki}^2} = \frac{\lambda}{a_i+\lambda}, \ 1 \le i \le r.$$

Then $s_i := \lim_k s_{ki}$ satisfies $(1-s_i^2)^2 = \frac{\lambda}{a_i+\lambda}, 1 \le i \le r$, and $a_i = s_i = 0$, for $1 \le i \le p$. Let now

$$c_\lambda := \sum_{i=p+1}^{r} \left(\frac{a_i}{a_i+\lambda}\right) e_i, \quad v_\lambda := \sum_{i=p+1}^{r} s_i e_i \text{ and } T_\lambda := B(v_\lambda, v_\lambda).$$

Theorem 2.1 gives $\lim_k e_{ki} = e_i$, for $p+1 \le i \le r$ and therefore

$$c_\lambda = \lim_k c_k, \quad v_\lambda = \lim_k v_k \text{ and } \quad T_\lambda = \lim_k T_k.$$

Clearly $c_\lambda,\ v_\lambda \in B$ so $T \in \mathsf{GL}(Z)$ [21]. For convenience, we relabel $e_{p+1}, \dots, e_r$ as $e_1, \dots, e_s$ for $s = r-p$ and relabel a_i, s_i similarly.
For $y \in B$, let $\lambda_y := \|B_y^{-1}B(y,\xi)R\|$. Then $y \in \partial E_{\lambda_y}$. The proof that $E_\lambda = c_\lambda + T_\lambda(B)$ we remove for clarity to Proposition 2.9 below. E_λ is clearly then convex and non-empty. Direct calculation gives $e = c_\lambda + T_\lambda(e)$, so that $e \in \partial E_\lambda$.

■

2.9 Proposition.

$$E_\lambda = c_\lambda + T_\lambda(B) \ \ \textit{for all } \lambda > 0.$$

Proof. All notations are as in Theorem 2.8. Fix $\lambda > 0$ and let $c := c_\lambda$ and $T := T_\lambda$. Let $E := E_\lambda$ and $G := c + T(B) = \{w \in B : \|T^{-1}(w-c)\| < 1\}$. We note that $D_k = c_k + T_k(B) = \{w \in B : \|T_k^{-1}(w-c_k)\| < 1\}$, for all k. We write A^{o} for the interior of the set A. Invertibility of T gives $(\overline{G})^{\mathrm{o}} = (\overline{c+T(B)})^{\mathrm{o}} = c + T((\overline{B})^{\mathrm{o}}) = c + T(B) = G$. Let $z \in E$. From (19) then $z \in D_k$ and hence $\|T_k^{-1}(w-c_k)\| < 1\|$ for all k large, so that $\|T^{-1}(w-c)\| \le 1$ giving $z \in c + T(\overline{B}) = \overline{G}$. In other words, $E \subseteq \overline{G}$ and since E is open

$$E \subseteq (\overline{G})^{\mathrm{o}} = G. \tag{21}$$

In the opposite direction, let $x \in G$ so that $\|T^{-1}(w-c)\| < 1$ and hence, for k large, $\|T_k^{-1}(w-c_k)\| < 1$ and $x \in D_k$. Then (19) gives $x \in \overline{E}$ so that $G \subseteq \overline{E}$. Write $x = c + T(y)$, some $y \in B$ and choose $\epsilon > 0$ such that $B(y,\epsilon) \subset B$. Then

$$c + T(B(y,\epsilon)) \subseteq G \subseteq \overline{E}. \tag{22}$$

Let $\nu = \frac{\epsilon}{\|T^{-1}\|}$ and fix any $z, \|z\| < \nu$. Let $u = \frac{z}{\epsilon}$ and $w = T^{-1}(u)$. Then $z = \epsilon T(w)$, for $\|w\| < 1$. In particular, $x + z = c + T(y + \epsilon w) \in \overline{E}$ from (22). Thus $B(x,\nu) \subset \overline{E}$ and hence $x \in E$. In other words $G \subseteq E$ and from (21) then $E = G$.

■

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School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland. E-mail: pmellon@maths.ucd.ie