ABSTRACT: Boulders are fractured and excavated by large waves along steep coastlines, contributing to coastal erosion. Intense fluid pressures enter cracks in attached bedrock, inducing bending stresses that may result in hydraulic fracture of a block. We consider the dynamic response of an attached rock beam to loading during wave impact. We verify the computational method using a known steady-state solution. We show that fast loading leads to greatest stress in the beam and that the beam oscillates before settling to an equilibrium position. These oscillations can produce large stresses long after the impact time.

INTRODUCTION

Coastal cliffs are slowly but progressively eroded by powerful wave action [1]. During overtopping flows, exposed bedrock along the cliff-top platform is subject to intense forces during fluid impact. Local geology may combine with wave action to enhance erosion. On the Aran Islands (Ireland), as a type locality, the bedrock is composed of stratified layers. Thin shale layers separate thicker limestone layers. These shale layers erode at a faster rate than limestone, increasing the exposed surface area and channelling flows [2].

Powerful amplified waves overtop cliffs [3], erode boulders from the platform bedrock [1], and develop cliff-top storm deposits [2]. These include very large boulders (VLBs) greater than 100 tonnes, up to 40 m above the high-water mark, and 250 m inland from the high-water line [2]. An example on the Aran Islands (Ireland) is shown in Fig. 1.

Wave impact and subsequent pressure propagation is a challenging problem [4]. Flow in a closed-end basal crack loads a beam from beneath. The fluid can take the form of liquid only [5] or air-liquid mixtures [6]. Initial crack filling and subsequent impacts exert large pressures that change in both space and time, as sketched in Fig. 2 (see Fig. 14c,d in [6]). The peak pressure during breaking-wave impact occurs over $10^{-3} - 10^{-2}$ s [1,7]. However, during impact the impulsive force on the roof of a closed-end crack is proportional to the area of the base of the crack and to the mean pressure at the entrance [8].
**Figure 1:** Boulder ridge on Aran Islands (Ireland). The ridge is 150 m inland and 16 m above high water. A person is in the foreground for scale. Photo by R. Cox.

**Figure 2:** Sketch of pressure over time in a crack during impact based on experimental results in [6]. Each curve corresponds to position \( \{0, 1/4, 1/2, 3/4, 1\} \times \) (crack length).
The beam responds to a load by deforming elastically. This deformation induces a bending stress that can develop and propagate defects in the rock, even to complete fracture. Propagation occurs when stress in the beam exceeds material strength [9]. Tensile strength of rock is approximately 10% its compressive strength [1]. This motivates wave action from below a rock beam, generating a bending stress that can fracture the rock more easily. In a previous study, we show the dependence of the fluid pressure on the geometry for fracture to occur in the case of a (quasi-) static load [10]. In this study, we consider the response of a beam to dynamic loading during wave impact.

**THEORY**

**Setup**

Overtopping flow fills basal cracks below attached bedrock beams (Fig. 3). Subsequent wave impacts produce pressure pulses (Fig. 2) that travel within the crack. This loading from beneath induces a bending stress that can cause fracture of the beam [10]. We consider a 2D setup with the wave approaching head-on in the $x$-direction, bending the beam in the $z$-direction.

![Figure 3: Setup of wave overtopping a cliff platform (Fig. 1). The fluid fills an exposed basal crack beneath a beam with microcrack defects. Wave impact sends pressure pulses (Fig. 2) along the basal crack that apply a load to the bottom of the beam.](image)

**Governing Equations**

The dynamic Euler-Bernoulli equation for the deformation $w(x,t)$ of a cantilever beam is given by

$$\mu w_{tt} + EI w_{xxxx} - q = 0,$$

$$w(L,t) = 0, \quad w_x(L,t) = 0, \quad w_{xx}(0,t) = 0, \quad w_{xxx}(0,t) = 0,$$

$$w(x,0) = 0, \quad w_t(x,0) = 0,$$

for $x \in [0,L]$ and $t \in [0,T]$. Eq. (1a) is the beam equation, with material properties of linear mass $\mu$, Young’s modulus $E$, and second moment of area $I$. The load is given by $q(x,t)$, to be specified later. We neglect the weight of the beam as, during wave impact, the weight load can be much smaller than the hydrodynamic load [10]. Eqs. (1b) are the fixed ($x = L$) and free ($x = 0$) end conditions. Eqs. (1c) state that the beam is initially undeformed and static.

We nondimensionalize the system (1) by setting the following scalings:
\[ w = \delta L w', \quad x = L x', \quad t = T t', \quad q = Q q', \]

where \( \delta = H/L \) is the beam aspect ratio, \( T \) is a typical time scale which will be taken as the peak loading time, and \( Q \) is a loading scale. Substituting (2) into (1), and dropping the primes (\('\)'), the governing system reads

\[
\begin{align*}
    w_{tt} + \alpha_1 w_{xxxx} - \alpha_2 q &= 0, \\
    w(1,t) &= 0, \quad w_x(1,t) = 0, \quad w_{xx}(0,t) = 0, \quad w_{xxx}(0,t) = 0, \\
    w(x,0) &= 0, \quad w_t(x,0) = 0,
\end{align*}
\]

for \( x \in [0,1] \) and \( t \in [0,1] \), where \( \alpha_1 = EIT^2/\mu L^4 \) and \( \alpha_2 = QT^2/\mu \delta L \) are dimensionless parameters.

The loading function results from a complicated fluid-air-structure interaction. Models using mass-spring systems match well with experimental results [6]. The pressure in a closed-end crack can have the following characteristics: (i) it can be both positive and negative relative to the reference pressure, (ii) the first pulse after impact travels down the crack as a wave, (iii) after the first pulse the pressure oscillates as a time varying spatial constant, (iv) the magnitude decreases with time. These properties are seen in Fig. 2.

We model two load types. The first is a spatially-constant load that varies from zero to a positive constant. This is a test case for the steady-state deformation, with comparison to that in [10]. The second is an approximation to the wave-impact pressure pulses found experimentally and numerically in [6]. We write the loading functions as

\[
q_{\text{test}} = q_0 \tanh(At), \\
q = q_0 \{ \exp(-\cdot 2x-3)^2) - [\frac{1}{2} \exp(-\cdot 7)^2 - \frac{1}{4} \exp(-\cdot 9)^2] \exp(x-1) \},
\]

where \( q \) (4b) is initially a travelling pulse (first term) followed by time varying oscillations (second and third terms), and \( q_0 \) and \( A \) are constants. The loadings (4) are illustrated in Fig. 4. In Fig. 2, we illustrate (4b) at different positions.

The (dimensionless) bending stress \( \sigma \) in the beam is given by

\[ \sigma = -\zeta w_{xx} z, \]

where \( \zeta = \delta^2 E/\sigma_f \) is a dimensionless number depending on the fracture stress \( \sigma_f \), and \( z \in [-1,1] \) is the (dimensionless) normal position relative to the neutral axis [10]. \( z = -1 \) is the bottom surface of the beam and \( z = 1 \) is the top surface. Positive values of \( \sigma \) indicate that the beam is in tension, while negative values represent compression. The bending stress acts oppositely on either side of the neutral axis.

Fracture may occur when the stress (5) inside the beam exceeds the material toughness [10]. In our dimensionless variables, this condition reads as \( \sigma > 1 \).
Computational Method

We solve the dimensionless governing system (3) using a finite difference method of lines scheme. First, we write (3a) as a system of first order differential equations in the time variable $t$. Second, we discretize the spatial variable $x$. The method of lines scheme solves an ODE for the evolution of the deformation at each grid point through time. Alternative methods include a temporal Fourier transform of (3) for an integral representation [11].

We take a uniform grid spacing $h = 1/(N-1)$ for large integer $N$, and write $x_i = ih$. The discrete variables $w_i(t) = w(x_i, t)$ and $q_i(t) = q(x_i, t)$ are the beam deformation and load, respectively, at time $t$ and position $x_i$. A second-order accurate spatial finite-difference scheme [12] of (3) reads

\[
\begin{align*}
(w)_{i} &= u_{i}, & (2<i<N-1) \\
(\alpha_{2} q_{i} - \alpha_{1} (w_{i-2} - 4 w_{i-1} + 6 w_{i} - 4 w_{i+1} + w_{i+2})/h^4, & (2<i<N-1) \\
(w_{i}) &= (48 w_{3} - 52 w_{4} + 15 w_{5})/11, & (6c) \\
w_{2} &= (28 w_{3} - 23 w_{4} + 6 w_{5})/11, & (6d) \\
w_{N-1} &= w_{N-2}/4, & (6e) \\
w_{N} &= 0, & (6f) \\
w_{i}(t=0) &= 0, & (6g) \\
u_{i}(t=0) &= 0. & (6h)
\end{align*}
\]
where \( u \) is introduced as an auxiliary variable. Eqs. (6a,b) are integrated in time using MATLAB’s\texttt{ode15s} solver, with boundary conditions (6e-f) and static undeformed initial conditions (6g,h).

We calculate \( w_{xx} \) in the stress (5) by a simple matrix-vector multiplication, using (sparse) differentiation matrices [13], of the computed \( w \) at each time step.

**Parameter Estimation**

We consider typical values for wave impact on limestone cliff-tops. The dimensions of exposed beams are of the order \( L = O(1) \) m and mass of order \( O(10^4) \) kg. A typical aspect ratio of width to height is \( \delta = 1/4 \) [2]. Hence \( \mu = O(10^4) \) kg m\(^{-1}\) and \( I = O(10^{-3}) \) m\(^4\). Limestone has a Young's modulus of the order \( E = O(10) \) GPa [14].

Limestone tensile strengths are reported in the range \( O(1-10) \) MPa [15]. As such, we take the fracture stress to be \( \sigma_f = O(1-10) \) MPa. Wave impacts at cliffs generate peak pressures of \( p = O(10^{-1}) \) MPa [16] while modelling shows they can reach \( p = O(10^1) \) MPa [1]. Pressure is further amplified inside flaws such as microcracks in the rock, increasing the likelihood of fracture [10]. We take a loading as \( Q = pL = O(1) \) MPa. In cases such as constant loading, the applied pressure need only be a fraction of the fracture stress to propagate cracks [10]. As such, we take \( q_0 = 0.01 \).

The highest pressures occur in breaking-wave impact but for a short period of time, of order \( O(10^{-2}-10^{-4}) \) s [1,7]. As such, we scale \( T = O(10^2) \) s. Pressure pulses in cracks from wave impacts travel through the crack over \( O(10^3) \) s [6]. The total time of impact may be \( O(1) \) s [1].

Combining the above parameter values determines the dimensionless numbers to be of order \( \alpha_1 = O(10^{-1}), \alpha_2 = O(10^2), \) and \( \zeta = O(10^2) \). A large \( \zeta \) balances a small deformation that gives rise to a stress (5) that is large enough to cause fracture. Variations in \( \alpha_1, \alpha_2, \) and \( \zeta \) may be considered through independent changes of Young's modulus \( E \), loading \( Q \), and fracture stress \( \sigma_f \), respectively.

**RESULTS**

The steady-state solution of (3) for a spatially-constant load \( q_0 \) is

\[
\frac{w}{(\alpha_2/\alpha_1)} q_0 \left( x^4 - 4x + 3 \right) / 24, \tag{7}
\]

as in [10]. We will see that our numerical method, with a constant load, settles to this steady-state deformation, verifying the scheme.

Choosing \( A = 3 \) in (4a) gives a loading \( q \) that goes from 0 to \( q_0 \) in 1 dimensionless time unit (dimensionally, \( 0.01 \) s), as in Fig. 4a. We solve (6) using \( N = 500 \) equispaced \( x \)-grid points. The beam deformation oscillates around the steady-state solution (7), for up to 20 dimensionless time units (dimensionally, \( 0.2 \) s), eventually reaching equilibrium (Fig. 5). During this settling transient period, the stress (5) is greater than the steady-state stress when the beam deformation is greater than in steady state. The maximum (dimensionless) stress is greater than 1, meaning that fracture may occur. This maximum stress changes with loading time, characterised by \( A \). Increasing \( A \) results in a faster loading time (4a). The maximum stress in the beam may be up to 70% greater than the largest steady-state stress (Fig. 6). When
loading occurs over a longer period of time (smaller $A$), the oscillations are reduced as expected.

Figure 5: Beam deformation $w(x,t)$ (left panel) and stress $\sigma(x,z,t)$ (right panel) against position for $t = \{0, 3, 6, 10, 15, 20\}$. We take $A = 3$ (Fig. 4a) and $q_0 = 0.01$ in the load $q_{\text{test}}$ (4a). Solid curves are the beam deformation and the dashed curve is the steady-state deformation (7).

Figure 6: Maximum stress in a dynamic beam against loading time characterised by $A$ (4a), compared to the steady-state stress (straight line). The difference is up to a 70% increase in stress.
The beam response from wave impact (4b) behaves analogously (Fig. 7) to the test case above. During the initial travelling pulse, the location of the maximum load is important in the beam dynamics. The deformation and stress decrease as the maximum load moves away from the free end (x = 0), as expected from the law of the lever. Later, when the pressure equalises in space but varies in time, the beam responds as expected. As the load switches from positive to negative, either side of the neutral axis switches from tensile to compressive stress. In this case, the (dimensionless) stress exceeds 1 so that fracture may occur.

**Figure 7:** Beam deformation \( w(x,t) \) (left panel) and stress \( \sigma(x,z,t) \) (right panel) against position for \( t = \{0, 5, 8, 25, 29, 52\} \). The load is given by \( q \) (4b). The beam is in tension and compression on either side of the neutral axis as it oscillates.

**CONCLUSIONS**

We present a computational method to solve the unsteady Euler-Bernoulli beam equation (1) with a general (spatially and temporally varying) load from beneath. The load represents the pressure inside a crack during wave impact. We use a method of lines finite-difference numerical scheme. The solver tolerances and grid spacing must be tuned for each case to obtain a reasonably accurate solution.

The steady state takes longer to reach than typical timescales for pressure change during impacts, up to 20 times as long (Fig. 5). In dimensional terms, this is an impact over 0.01 s with a steady-state response after 0.2 s.

It was previously reported that for steady-state responses, the applied load need only be a fraction of the material toughness to cause fracture [10]. However, here the dynamic
behaviour results in a significant increase in the maximum stress compared to the steady state, up to 70% (Fig. 6). This gives a further reduction to the applied load required to cause fracture. In our model of a wave-impact load (Fig. 4b), the maximum value is $q_0 = 0.01$ (1/100 the material toughness). However, the (dimensionless) stress inside the beam exceeds 1 (Fig. 7). This means that fracture may occur.

Furthermore, oscillations of the beam mean that if one large wave-impact event induces a defect in the beam, the subsequent motion after the sudden impact can induce fracture. In other words, one impact event may potentially both activate and propagate a microcrack to fracture.

In this study, the weight load has been neglected, which acts to impose a tensile stress above the neutral axis and a compressive stress below the neutral axis. Wave impact causes tensile and compressive stress on each side of the neutral axis, alternating as the fluid pressure changes in the crack. When the pressure is negative, the beam is in tensile stress above the neutral axis. However, in this situation, the weight will add to the tensile stress here, increasing the likelihood of fracture.

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REFERENCES