Caputo and related fractional derivatives in singular systems

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Abstract. By using the Caputo (C) fractional derivative and two recently defined alternative versions of this derivative, the Caputo–Fabrizio (CF) and the Atangana–Baleanu (AB) fractional derivative, firstly we focus on singular linear systems of fractional differential equations with constant coefficients that can be non-square matrices, or square & singular. We study existence of solutions and provide formulas for the case that there do exist solutions. Then, we study the existence of unique solution for given initial conditions. Several numerical examples are given to justify our theory.

Keywords: singular, systems, fractional, derivative, caputo, initial conditions.

1 Introduction

In the last decade many authors have studied problems of fractional differential/difference equations and have derived interesting results on different type of problems for given initial or boundary conditions, see [1], [10], [11], [12], [13], [14], [15], [16], [18], [23], [24], [27], [30], [31], [32], [33], [34].

Focus has also been given in the mathematical modelling of many phenomena by using fractional operators. The theory of fractional differential equations (FDEs) is a promising tool for applications in nanotechnology, see [5], in physics, see [19], and biology, see [20]. For some other recent contributions & applications in fractional calculus, see [25], [26], [36], [37], [38], [39].

Fractional-order operators are not just a generalization of the classical integer-order operators. Because of the way they are defined more elegant techniques are required for qualitative studies. In many practical cases the existing techniques are not enough.

Singular systems of differential/difference equations appear in control theory, see [9], in macroeconomics, see [10], circuit theory, see [22], and in the modeling of power systems, see [28], [29]. In this article we will study singular linear systems of FDEs with constant coefficients that can be non-square matrices, or square & singular.

In our opinion, despite several studies, there are still parts missing for a complete and coherent theory of systems of FDEs in order to use this type of systems as a tool for applications in the applied sciences in a similar way to the classical case. In addition, generalised FDEs and cases such as singularities of certain systems of FDEs have been mostly avoided in the framework of fractional calculus. Hence, explicit and easily testable methods are required in order to solve generalised systems of FDEs, so that applied researchers can redesign their models in cases where the fractional operators provide better results than the classical ones.

Definition 1.1. (see [6], [22]) Let $Y : [0, +\infty) \to \mathbb{R}^{m\times 1}$, $t \to Y$, denote a column of continuous and differentiable functions. Then, the Caputo (C) fractional derivative of order a, $0 < a < 1$, is defined by

$$Y_C^{(a)}(t) := Y^{(a)}(t) = \frac{1}{\Gamma(1-a)} \int_0^t \left[(t-x)^{-a} Y'(x)\right] dx. \tag{1}$$

Recently, a new fractional derivative was defined by Caputo and Fabrizio (see [7]) and it was followed by some related theoretical and applied results (see [2], [3], and the references therein). The aim of this fractional derivative was to introduce a new derivative with exponential kernel. Its anti-derivative was reported in [2] and it was found to be the average of a given function. We believe that the main idea presented in [7] was to find a way to describe even better the dynamics of systems with memory effect than other existing definitions of fractional derivatives in the literature.

Definition 1.2. (see [7], [8]) Let $Y : [0, +\infty) \to \mathbb{R}^{m\times 1}$, $t \to Y$, denote a column of continuous and differentiable functions. Then, the Caputo–Fabrizio (CF) fractional derivative of order a, $0 \leq a \leq 1$, is defined by

$$Y_{CF}^{(a)}(t) := Y^{(a)}(t) = \frac{1}{1-a} \int_0^t \left[e^{-\frac{a}{1-a}(t-x)} Y'(x)\right] dx. \tag{2}$$

Following the question "what is the most accurate kernel which better describes it?", Atangana and Baleanu, see [4], suggested a possible answer to this by introducing a new fractional derivative which has a non-local kernel.

Definition 1.3. (see [4]) Let $Y : [0, +\infty) \to \mathbb{R}^{m\times 1}$, $t \to Y$, denote a column of differentiable functions. Then, the modified Caputo (AB) fractional derivative of order $0 \leq a \leq 1$, is defined by

$$Y_{AB}^{(a)}(t) := Y^{(a)}(t) = \frac{B(a)}{1-a} \int_0^t E_a \left[-a\frac{(t-x)^a}{1-a}\right] Y'(x) dx. \tag{3}$$

Where $E_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+ak)}$, $a, z \in \mathbb{C}$, $Re(a) > 0$ (see [6], [22]). $B(a)$ denotes a normalization function obeying $B(0) = B(1) = 1$.

Throughout the paper, where $Y'(x) = \frac{d}{dx}Y(x)$, with $\mathcal{L}$ we will denote the Laplace transform, see [6], [22] and with 0_{ij} the zero matrix of i rows and j columns. Let $B_{n_1} \in \mathbb{C}^{n_1\times n_1}$,$B_{n_2} \in \mathbb{C}^{n_2\times n_2}$, $\dots$, $B_{n_r} \in \mathbb{C}^{n_r\times n_r}$. Then with the direct sum $B_{n_1} \oplus B_{n_2} \oplus \cdots \oplus B_{n_r}$ we will denote the block diagonal matrix $blockdiag \begin{bmatrix} B_{n_1} & B_{n_1} & \dots & B_{n_r} \end{bmatrix}$.

The article is organised as follows: in Section 2 we use the (C), (CF), (AB) fractional derivatives as defined in (1), (2), (3) respectively, and study a class of singular linear systems of FDEs. We study the existence and uniqueness of solutions and provide two different type of formulas for the case that there exist solutions. Finally, in Section 3 we provide several numerical examples to justify our theory.

2 Main Results

In this section we present our main results. We consider the following system of FDEs

$$FY^{(a)}(t) = GY(t) + V(t). \tag{4}$$

Where $F, G \in \mathbb{R}^{r\times m}$, $Y : [0, +\infty] \to \mathbb{R}^{m\times 1}$, $V : [0, +\infty] \to \mathbb{R}^{r\times 1}$ and $0 < a < 1$. The matrices F, G can be non-square ($r \neq m$) or square ($r = m$) with F singular (det$F$=0). With $Y^{(a)}$ we denote the fractional derivative as defined in (1).

Definition 2.1. Given $F, G \in \mathbb{C}^{r\times m}$, $0 < a < 1$, an arbitrary $s \in \mathbb{C}$ and an inverse function $z = z(s) \in \mathbb{C}$, the matrix pencil $zF - G$ is called:

1. Regular when $r = m$ and $\det(zF - G) \not\equiv 0$;
2. Singular when $r \neq m$ or $r = m$ and $\det(zF - G) \equiv 0$.

Remark 2.1. Given $F, G \in \mathbb{C}^{r\times m}$, $0 < a < 1$, an arbitrary $s \in \mathbb{C}$ and an inverse function $z = z(s) \in \mathbb{C}$, if pencil $zF - G$ is:

(a) Regular, since $\det(zF - G) \not\equiv 0$, there exists a matrix $P : \mathbb{C} \to \mathbb{R}^{m\times m}$ (which can be computed via the Gauss-Jordan Elimination Method, see [35]) such that

$$P(s)(zF - G) = A(s). \tag{5}$$

Where $A : \mathbb{C} \to \mathbb{R}^{m\times m}$ is a diagonal matrix with non-zero elements;

(b) Singular and $r > m$, then there exists a matrix $P : \mathbb{C} \to \mathbb{R}^{r\times r}$ (which can be computed via the Gauss-Jordan Elimination Method) such that

$$P(s)(zF - G) = \begin{bmatrix} A(s) \\ 0_{r_1,m} \end{bmatrix}, \quad \text{with} \quad P(s) = \begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix}. \tag{6}$$

Where $A : \mathbb{C} \to \mathbb{R}^{m_1\times m}$, with $m_1 + r_1 = r$, is a matrix such that if $[a_{ij}]_{1\leq i\leq m_1}^{1\leq j\leq m}$ are its elements, for $i = j$ all elements are non-zero and for $i \neq j$ all elements are zero and $P_1(s) \in \mathbb{R}^{m_1\times r}$, $P_2(s) \in \mathbb{R}^{r_1\times r}$.

We will now study the existence of solutions of system (4). We state the following Theorem:

Theorem 2.1. Consider the system of FDEs (4), and let

(i) $w = s^{a-1}$, if we use the (C) fractional derivative;

(ii) $w = \frac{1}{s+a(1-s)}$, if we use the (CF) fractional derivative;

(iii) $w = \frac{B(a)}{1-a}\frac{s^{a-1}}{s^a+\frac{a}{1-a}}$, if we use the (AB) fractional derivative.

Then there exist solutions for (4) if and only if:

(a) The pencil of the system is regular; In this case the general solution is given by:

$$Y(t) = \Phi_0(t)C + \Phi(t). \tag{7}$$

Where $\Phi_0(t) = \mathcal{L}^{-1}\{wA^{-1}(s)P(s)F\}$, $\Phi(t) = \mathcal{L}^{-1}\{A^{-1}(s)P(s)U(s)\}$, $A(s)$, $P(s)$ are defined in (5) and $C \in \mathbb{R}^{m\times 1}$ is an unknown constant vector. If

$$\mathcal{L}^{-1}\{A^{-1}(s)P(s)\} = \Phi_1(t),$$

by using the convolution theorem an alternative formula is:

$$Y(t) = \Phi_0(t)C + \int_0^\infty \Phi_1(t-\tau)V(\tau)d\tau.$$

(b) The pencil of the system is singular with $r > m$ and

$$P_2(s)F = 0_{m_1,1}, \quad P_2(s)U(s) = 0_{m_1,1} \quad \text{and} \quad m_1 = m. \tag{8}$$

In this case the general solution is given by:

$$Y(t) = \Psi_0(t)C + \Psi(t). \tag{9}$$

Where $\Psi_0(t) = \mathcal{L}^{-1}\{wA^{-1}(s)P_1(s)F\}$, $\Psi(t) = \mathcal{L}^{-1}\{A^{-1}(s)P(s)U(s)\}$, $A(s)$, $P(s)$, $P_1(s)$, $P_2(s)$ are defined in (6), $C \in \mathbb{R}^{m\times 1}$ is an unknown constant vector. If

$$\mathcal{L}^{-1}\{A^{-1}(s)P_1(s)\} = \Psi_1(t),$$

by using the convolution theorem an alternative formula is:

$$Y(t) = \Psi_0(t)C + \int_0^\infty \Psi_1(t-\tau)V(\tau)d\tau.$$

Proof. Let $\mathcal{L}\{Y(t)\} = Z(s)$, $\mathcal{L}\{V(t)\} = U(s)$ be the Laplace transform of $Y(t)$, $V(t)$ respectively. Using the fractional derivative as defined in (1), (2) and (3), by applying the Laplace transform $\mathcal{L}$ into (4), see [4], [6], [7], [8], [22], we get

$$F\mathcal{L}\{Y^{(a)}(t)\} = G\mathcal{L}\{Y(t)\} + \mathcal{L}\{V(t)\},$$

or, equivalently,

$$F(zZ(s) - wY_0) = GZ(s) + U(s).$$

Where $Y_0 = Y(0)$, i.e. the initial condition of (4). Since we assume that Y_0 is unknown we can use an unknown constant vector $C \in \mathbb{R}^{m\times 1}$ and give to the above expression the following form

$$(zF - G)Z(s) = wFC + U(s). \tag{10}$$

Where $z = z(s)$, $w = w(s) \in \mathbb{C}$, are inverse functions defined as, see [4], [6], [7], [8], [22]:

(i) $z = s^a$, $w = s^{a-1}$, if we use the (C) fractional derivative;

(ii) $z = \frac{s}{s+a(1-s)}$, $w = \frac{1}{s+a(1-s)}$, if we use the (CF) fractional derivative;

(iii) $z = \frac{B(a)}{1-a}\frac{s^a}{s^a+\frac{a}{1-a}}$, $w = \frac{B(a)}{1-a}\frac{s^{a-1}}{s^a+\frac{a}{1-a}}$, if we use the (AB) fractional derivative.

We have two cases. The first is (a) $r = m$ and $\det(zF - G)$ to be equal to a fractional polynomial with order less than a (regular pencil). The second case is (b) $r \neq m$ or $r = m$ and $\det(zF - G) \equiv 0$, $\forall$ arbitrary $s \in \mathbb{C}$ (singular pencil).

In the case of (a), since the pencil is assumed regular and $\det(zF - G) \not\equiv 0$, there exists a matrix $P : \mathbb{C} \to \mathbb{R}^{m\times m}$ (which can be computed via the Gauss-Jordan Elimination Method, see [35]) such that

$$P(s)(zF - G) = A(s).$$

Where $A : \mathbb{C} \to \mathbb{R}^{m\times m}$ is a diagonal matrix with non-zero elements in its diagonal. Then by multiplying (10) with $P(s)$ we get

$$P(s)(zF - G)Z(s) = wP(s)FC + P(s)U(s),$$

or, equivalently,

$$A(s)Z(s) = wP(s)FC + P(s)U(s),$$

or, equivalently,

$$Z(s) = wA^{-1}(s)P(s)FC + A^{-1}(s)P(s)U(s).$$

The inverse Laplace Transform of the matrix $wA^{-1}(s)P(s)F = w(zF-G)^{-1}F$ always exists because in all three definitions, (C), (CF), (AB), its elements are fractions of fractional polynomials with the order of the polynomial in the denominator always being higher than the order of the polynomial in the numerator. Let $\mathcal{L}^{-1}\{wA^{-1}(s)P(s)F\} = \Phi_0(t)$ and $\mathcal{L}^{-1}\{A^{-1}(s)P(s)U(s)\} = \Phi(t)$. Then $Y(t)$ is given by (7). Or, alternative, if $\mathcal{L}^{-1}\{A^{-1}(s)P(s)\} = \Phi_1(t)$ then by using the convolution theorem we have

$$Y(t) = \Phi_0(t)C + \int_0^\infty \Phi_1(t-\tau)V(\tau)d\tau.$$

In the case of (b), if $r < m$ there are at least $m - r$ unknown functions and m equations. Hence $Z(s)$ in system (10) can not be defined uniquely. If $r > m$ then there exists a matrix $P : \mathbb{C} \to \mathbb{R}^{r\times r}$ (which can be computed via the Gauss-Jordan Elimination Method) such that

$$P(s)(zF - G) = \begin{bmatrix} A(s) \\ 0_{r_1,m} \end{bmatrix}.$$

Where $A : \mathbb{C} \to \mathbb{R}^{m_1\times m}$, with $m_1 + r_1 = r$, is a matrix such that if $[a_{ij}]_{1\le i\le m_1}^{1\le j\le m}$ are its elements, for $i = j$ all elements are non-zero and for $i \neq j$ all elements are zero.

$$P(s) = \begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix},$$

where $P_1(s) \in \mathbb{R}^{m_1\times r}$, $P_2(s) \in \mathbb{R}^{r_1\times r}$. Then system (10) has a unique solution if and only if (8) holds. In any other case we have more unknown functions than equations or no solutions. If (8) holds then

$$P(s)(zF-G) = \begin{bmatrix} A(s) \\ 0_{r_1,m} \end{bmatrix}$$

and we have

$$P(s)(zF-G)Z(s) = wP(s)FC + P(s)U(s),$$

or, equivalently,

$$A(s)Z(s) = wP_1(s)FC + P_1(s)U(s),$$

or, equivalently,

$$Z(s) = wA^{-1}(s)P_1(s)FC + A^{-1}P_1(s)U(s).$$

The inverse Laplace Transform of $wA^{-1}(s)P_1(s)F$ always exists because in all three definitions, (C), (CF), (AB), it is a matrix with elements fractions of fractional polynomials and with the order of the polynomial in the denominator always being higher than the order of the polynomial in the numerator. Let $\mathcal{L}^{-1}\{wA^{-1}(s)P_1(s)F\} = \Psi_0(t)$ and $\mathcal{L}^{-1}\{A^{-1}(s)P(s)U(s)\} = \Psi(t)$. Then $Y(t)$ is given (9). Or, alternative, if $\mathcal{L}^{-1}\{A^{-1}(s)P_1(s)\} = \Psi_1(t)$. Then by using the convolution theorem we have

$$Y(t) = \Psi_0(t)C + \int_0^\infty \Psi_1(t-\tau)V(\tau)d\tau.$$

If $r = m$ then there exists a matrix $P : \mathbb{C} \to \mathbb{R}^{r\times r}$ (which can be computed via the Gauss-Jordan Elimination Method) such that

$$P(s)(zF-G) = A(s) \oplus 0_{r_2,m_2}.$$

Where $A : \mathbb{C} \to \mathbb{R}^{r_1\times m_1}$ with $r_1 \leq m_1$ (because we apply Gauss-Jordan Elimination Method at the rows). All elements of $A(s)$ are zero except the ones in the diagonal with are all non-zero elements. Also, $r_1 + r_2 = m_1 + m_2 = m$. Then system (10) could have solutions if and only if $r_2 = m_2 = 0$, i.e. $r_1 = m_1 = m$; In any other case we have more unknown functions than equations or no solutions. But since we are in the case where $r = m$ and the pencil is singular, i.e. $\det(zF-G) \equiv 0$, this assumption can never hold. To sum up, there exists solution for the system if the pencil is regular or singular with $r > m$ and $A(s)$ $m\times m$ and $P_2(s)F = P_2(s)U(s) = 0_{m-r,1}$. The proof is completed.

Having identified the conditions under which there exists solutions for singular systems in the form of (4), we can now present the following Remarks:

Remark 2.2. For the (C), (CF), (AB) fractional derivatives, if there exist solutions for system (4), then in the case that the pencil of the system is regular, the general solution is given by (7). In the case that the pencil of the system is singular with $r > m$ and (8) holds, the general solution is given by (9). In both cases, C is an unknown constant vector related to the initial conditions of the system since we used the Laplace transform.

Remark 2.3. Even if there exist solutions, it is not guaranteed that for given initial conditions a singular system of FDEs will have a unique solution. If the given initial conditions are consistent, and there exist solutions for (4), then in the formulas of the general solutions (7) and (9) we replace $C = Y_0$. However, if the given initial conditions are non-consistent but there exist solutions for (4), then the general solutions (7) and (9) hold for $t > 0$ and the system is impulsive.

Remark 2.4. In the end of the next subsection we provide a criterion on how to identify if the given initial conditions are consistent or non-consistent.

Remark 2.5. For the case that the initial conditions are consistent, the matrix functions $\Phi_0(t)$, $\Psi_0(t)$ can have elements defined for $t > 0$. But the columns $\Phi_0(t)Y_0$, $\Psi_0(t)Y_0$ have all its elements always defined for $t \geq 0$.

Other formulas

In this subsection, based on Theorem 2.1 and the assumptions for existence of solutions of system (4) we will provide additional formulas by using matrix pencil theory.

From Theorem 2.1 there exist solutions for system (4) if the pencil is either regular, or singular with $r > m$ and (8) holds. If (4) has a *regular pencil*, then $sF - G$ is also regular and from its regularity there exist non-singular matrices P, $Q \in \mathbb{C}^{m\times m}$ such that

$$\begin{aligned} PFQ &= I_p \oplus H_q, \\ PGQ &= J_p \oplus I_q. \end{aligned} \tag{11}$$

Where $J_p \in \mathbb{C}^{p\times p}$, $H_q \in \mathbb{C}^{q\times q}$ appropriate matrices with H_q a nilpotent matrix with index q_*, constructed by using the algebraic multiplicity of the infinite eigenvalue, J_p is a Jordan matrix, constructed by the finite eigenvalues of the pencil and their algebraic multiplicity and $p + q = m$. Let

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix}. \tag{12}$$

with $P_1 \in \mathbb{C}^{p\times m}$, $P_2 \in \mathbb{C}^{q\times m}$ and $Q_p \in \mathbb{C}^{m\times p}$, $Q_q \in \mathbb{C}^{m\times q}$. If we consider the transformation $Y(t) = QZ(t)$ and substitute it into (4) we obtain

$$FY^{(a)}(t)QZ(t) = GQZ(t) + V(t),$$

whereby, multiplying by P and using (11) and (12), we get

$$\begin{bmatrix} I_p & 0_{p,q} \\ 0_{q,p} & H_q \end{bmatrix} \begin{bmatrix} Z_p^{(a)}(t) \\ Z_q^{(a)}(t) \end{bmatrix} = \begin{bmatrix} J_p & 0_{p,q} \\ 0_{q,p} & I_q \end{bmatrix} \begin{bmatrix} Z_p(t) \\ Z_q(t) \end{bmatrix} + \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} V(t),$$

where

$$Z(t) = \begin{bmatrix} Z_p(t) \\ Z_q(t) \end{bmatrix},$$

with $Z_p(t) \in \mathbb{C}^{p\times 1}$, $Z_q(t) \in \mathbb{C}^{q\times 1}$. From the above expressions we arrive easily at the subsystems

$$Z_p^{(a)}(t) = J_p Z_p(t) + P_1 V(t) \tag{13}$$

and

$$H_q Z_q^{(a)}(t) = Z_q(t) + P_2 V(t). \tag{14}$$

By applying the Laplace transform $\mathcal{L}$ into (13) we get

$$\mathcal{L}\{Z_p^{(a)}(t)\} = J_p \mathcal{L}\{Z_p(t)\} + P_1 \mathcal{L}\{V(t)\},$$

or, equivalently,

$$zW(s) - wZ_{p0} = J_p W(s) + P_1 U(s),$$

Where

(i) $z = s^a$, $w = s^{a-1}$, if we use the (C) fractional derivative;

(ii) $z = \frac{s}{s+a(1-s)}$, $w = \frac{1}{s+a(1-s)}$, if we use the (CF) fractional derivative;

(iii) $z = \frac{B(a)}{1-a}\frac{s^a}{s^a+\frac{a}{1-a}}$, $w = \frac{B(a)}{1-a}\frac{s^{a-1}}{s^a+\frac{a}{1-a}}$, if we use the (AB) fractional derivative.

Furthermore, $\mathcal{L}\{Z_p(t)\} = W(s)$, $\mathcal{L}\{V(t)\} = U(s)$ and $Z_{p0} = Z_p(0)$, i.e. the initial condition of (13). Since we assume that Z_{p0} is unknown, we set $Z_{p0} = C$, where C unknown column, and give to the above expression the following form:

$$(zI_p - J_p)W(s) = wC + P_1 U(s),$$

or, equivalently,

$$W(s) = w(zI_p - J_p)^{-1}C + (zI_p - J_p)^{-1}P_1 U(s),$$

or, equivalently, by taking into account that $(zI_p - J_p)^{-1} = \sum_{k=0}^{\infty} z^{-k-1}J_p^k$,

$$W(s) = \sum_{k=0}^{\infty} wz^{-k-1}J_p^k C + \sum_{k=0}^{\infty} z^{-k-1}J_p^k P_1 U(s). \tag{15}$$

If we use the (C) fractional derivative by replacing in (15) $z = s^a$, $w = s^{a-1}$ we have

$$W(s) = \sum_{k=0}^{\infty} s^{-ak-1}J_p^k C + \sum_{k=0}^{\infty} s^{-ak-a}J_p^k P_1 U(s).$$

Let

$$\begin{aligned} \Phi_0(t) &= \mathcal{L}^{-1}\{\textstyle\sum_{k=0}^{\infty} s^{-ak-1}J_p^k\} = \sum_{k=0}^{\infty} \frac{t^{ak}}{\Gamma(ka+1)}J_p^k, \\ \Phi(t) &= \mathcal{L}^{-1}\{\textstyle\sum_{k=0}^{\infty} s^{-ak-a}J_p^k\} = \sum_{k=0}^{\infty} \frac{t^{ak+a-1}}{\Gamma(ka+a)}J_p^k. \end{aligned} \tag{16}$$

Then by using the convolution theorem we have

$$Z_p(t) = \Phi_0(t)C + \int_0^{\infty} \Phi(t-\tau)P_1 V(\tau)d\tau. \tag{17}$$

If we use the (CF) fractional derivative by replacing in (15) $z = \frac{s}{s+a(1-s)}$, $w = \frac{1}{s+a(1-s)}$ we have

$$W(s) = \sum_{k=0}^{\infty} \frac{1}{s+a(1-s)} \left[\frac{s}{s+a(1-s)}\right]^{-k-1} J_p^k C + \sum_{k=0}^{\infty} \left[\frac{s}{s+a(1-s)}\right]^{-k-1} J_p^k P_1 U(s),$$

or, equivalently,

$$W(s) = \sum_{k=0}^{\infty} \frac{[(1-a)s+a]^k}{s^{k+1}} J_p^k C + \sum_{k=0}^{\infty} \frac{[(1-a)s+a]^{k+1}}{s^{k+1}} J_p^k P_1 U(s),$$

or, equivalently, since $[(1-a)s+a]^k = \sum_{n=0}^{k} \begin{pmatrix} k \\ n \end{pmatrix} (1-a)^n s^n a^{k-n}$

$$\begin{gathered} W(s) = \\ \sum_{k=0}^{\infty} \sum_{n=0}^{k} \begin{pmatrix} k \\ n \end{pmatrix} (1-a)^n s^{n-k-1} a^{k-n} J_p^k C + \\ \sum_{k=0}^{\infty} \sum_{n=0}^{k+1} \begin{pmatrix} k+1 \\ n \end{pmatrix} (1-a)^n s^{n-k-1} a^{k+1-n} J_p^k P_1 U(s). \end{gathered}$$

Let

$$\begin{gathered} \Phi_0(t) = \\ \mathcal{L}^{-1}\{\sum_{k=0}^{\infty} \sum_{n=0}^{k} \begin{pmatrix} k \\ n \end{pmatrix} (1-a)^n s^{n-k-1} a^{k-n} J_p^k\} = \\ \sum_{k=0}^{\infty} \sum_{n=0}^{k} \begin{pmatrix} k \\ n \end{pmatrix} (1-a)^n a^{k-n} \frac{t^{k-n}}{\Gamma(k+1-n)} J_p^k \\ \\ \Phi(t) = \\ \mathcal{L}^{-1}\{\sum_{k=0}^{\infty} \sum_{n=0}^{k+1} \begin{pmatrix} k+1 \\ n \end{pmatrix} (1-a)^n s^{n-k-1} a^{k+1-n} J_p^k P_1 U(s)\} = \\ \sum_{k=0}^{\infty} \sum_{n=0}^{k+1} \begin{pmatrix} k+1 \\ n \end{pmatrix} (1-a)^n a^{k+1-n} \frac{t^{k-n}}{\Gamma(k+1-n)} J_p^k P_1 U(s). \end{gathered} \tag{18}$$

Then by using the convolution theorem the solution is given by (17).

If we use the (AB) fractional derivative by replacing in (15) $z = \frac{B(a)}{1-a} \frac{s^a}{s^a + \frac{a}{1-a}}$, $w = \frac{B(a)}{1-a} \frac{s^{a-1}}{s^a + \frac{a}{1-a}}$ we have

$$W(s) = \sum_{k=0}^{\infty} \frac{B(a)}{1-a} \frac{s^{a-1}}{s^a + \frac{a}{1-a}} \left[\frac{B(a)}{1-a} \frac{s^a}{s^a + \frac{a}{1-a}}\right]^{-k-1} J_p^k C + \sum_{k=0}^{\infty} \left[\frac{B(a)}{1-a} \frac{s^a}{s^a + \frac{a}{1-a}}\right]^{-k-1} J_p^k P_1 U(s),$$

or, equivalently,

$$W(s) = \sum_{k=0}^{\infty} \frac{[(1-a)s^a + a]^k}{B^k(a) s^{ak+1}} J_p^k C + \sum_{k=0}^{\infty} \frac{[(1-a)s^a + a]^{k+1}}{B^{k+1}(a) s^{ak+a}} J_p^k P_1 U(s),$$

or, equivalently, since $[(1-a)s^a+a]^k = \sum_{n=0}^{k}\left(\begin{array}{c} k \\ n \end{array}\right)(1-a)^n s^{an}a^{k-n}$

$$
\begin{gathered}
W(s) = \\
\sum_{k=0}^{\infty}\sum_{n=0}^{k}\left(\begin{array}{c} k \\ n \end{array}\right)\frac{(1-a)^n a^{k-n}}{B^k(a)}s^{an-ak-1}J_p^k C+ \\
\sum_{k=0}^{\infty}\sum_{n=0}^{k+1}\left(\begin{array}{c} k+1 \\ n \end{array}\right)\frac{(1-a)^n a^{k+1-n}}{B^{k+1}(a)}s^{an-ak-a}J_p^k P_1 U(s).
\end{gathered}
$$

Let

$$
\begin{gathered}
\Phi_0(t) = \\
\mathcal{L}^{-1}\{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\left(\begin{array}{c} k \\ n \end{array}\right)\frac{(1-a)^n a^{k-n}}{B^k(a)}s^{an-ak-1}J_p^k\} = \\
\sum_{k=0}^{\infty}\sum_{n=0}^{k}\left(\begin{array}{c} k \\ n \end{array}\right)\frac{(1-a)^n a^{k-n}}{B^k(a)}\frac{t^{ak+2-an}}{\Gamma(ak+1-an)}J_p^k \\
\Phi(t) = \\
\mathcal{L}^{-1}\{\sum_{k=0}^{\infty}\sum_{n=0}^{k+1}\left(\begin{array}{c} k+1 \\ n \end{array}\right)\frac{(1-a)^n a^{k+1-n}}{B^{k+1}(a)}s^{an-ak-a}J_p^k P_1 U(s)\} = \\
\sum_{k=0}^{\infty}\sum_{n=0}^{k+1}\left(\begin{array}{c} k+1 \\ n \end{array}\right)(1-a)^n a^{k+1-n}\frac{t^{ak+a-an+1}}{\Gamma(ak+a-an)}J_p^k P_1 U(s).
\end{gathered}
\tag{19}
$$

Then by using the convolution theorem the solution is given by (17).

Let q_* be the index of the nilpotent matrix H_q, i.e. $H_q^{q_*} = 0_{q,q}$. Then if we obtain the following matrix equations

$$
\begin{array}{c}
H_q Z_q^{(a)}(t) = Z_q(t) + P_2 V(t) \\
H_q^2 Z_q^{(2a)}(t) = H_q Z_q^{(a)}(t) + H_q P_2 V^{(a)}(t) \\
H_q^3 Z_q^{(3a)}(t) = H_q^2 Z_q^{(2a)}(t) + H_q^2 P_2 V^{(2a)}(t) \\
H_q^4 Z_q^{(4a)}(t) = H_q^3 Z_q^{(3a)}(t) + H_q^3 P_2 V^{(3a)}(t) \\
\vdots \\
H_q^{q_*-1} Z_q^{([q_*-1]a)}(t) = H_q^{q_*-2} Z_q^{([q_*-2]a)}(t) + H_q^{q_*-2} P_2 V^{([q_*-2]a)}(t) \\
H_q^{q_*} Z_q^{(q_* a)}(t) = H_q^{q_*-1} Z_q^{([q_*-1]a)}(t) + H_q^{q_*-1} P_2 V^{([q_*-1]a)}(t)
\end{array},
$$

by taking the sum of the above equations and using the fact that $H_q^{q_*} = 0_{q,q}$ we arrive easily at the solution of the subsystem (14) has the unique solution

$$
Z_q(t) = -\sum_{i=0}^{q_*-1} H_q^i P_2 V^{(ia)}(t). \tag{20}
$$

To conclude, by combining (17) and (20), for the case of a regular pencil, system (4) has the solution

$$
Y(t) = QZ(t) = \left[\begin{array}{cc} Q_p & Q_q \end{array}\right]\left[\begin{array}{c} \Phi_0(t)C + \int_0^{\infty}\Phi(t-\tau)P_1 V(\tau)d\tau \\ -\sum_{i=0}^{q_*-1} H_q^i P_2 V^{(ia)}(t) \end{array}\right],
$$

or, equivalently,

$$Y(t) = Q_p\left[\Phi_0(t)C + \int_0^{\infty}\Phi(t-\tau)P_1V(\tau)d\tau\right] - Q_q\sum_{i=0}^{q_*-1}H_q^iP_2V^{(ia)}(t). \tag{21}$$

Next, we consider system (4) with a *singular pencil* and $r > m$. In general, the class of $sF - G$ is then characterized by a uniquely defined element, known as the complex Kronecker canonical form, see [9], [17], [21], [35], specified by the complete set of invariants of the singular pencil $sF - G$. This is the set of the finite–infinite eigenvalues and the minimal column–row indices. In the case of $r > m$ there exist only row minimal indices. Let $\mathcal{N}_l$ be the left null space of a matrix respectively. Then the equations $V^T(s)(sF-G) = 0_{1,m}$, have solutions in $V(s)$, which are vectors in the rational vector spaces $\mathcal{N}_l(sF-G)$. The binary vectors $V^T(s)$ express dependence relationships among the rows of $sF - G$. Note that $V(s) \in \mathbb{C}^{r\times 1}$ are polynomial vectors. Let t=dim$\mathcal{N}_l(sF-G)$. It is known, that $\mathcal{N}_l(sF-G)$ as rational vector spaces, are spanned by minimal polynomial bases of minimal degrees

$$\zeta_1 = \zeta_2 = ... = \zeta_h = 0 < \zeta_{h+1} \leq ... \leq \zeta_{h+k=t},$$

which is the set of *row minimal indices* of $sF - G$. This means there are t row minimal indices, but $t - h = k$ non-zero row minimal indices. We are interested only in the k non zero minimal indices. To sum up the invariants of a singular pencil with $r > m$ are the finite − infinite eigenvalues of the pencil and the minimal row indices as described above. Following the above given analysis, there exist non-singular matrices P, Q with $P \in \mathbb{C}^{r\times r}$, $Q \in \mathbb{C}^{m\times m}$, such that

$$\begin{aligned} PFQ &= F_K = I_p \oplus H_q \oplus F_\zeta, \\ PGQ &= G_K = J_p \oplus I_q \oplus G_\zeta. \end{aligned} \tag{22}$$

Where J_p is the Jordan matrix for the finite eigenvalues, H_q a nilpotent matrix with index q_* which is actually the Jordan matrix of the zero eigenvalue of the pencil $sG - F$. The matrices F_ζ, G_ζ are defined as

$$F_\zeta = \begin{bmatrix} I_{\zeta_{h+1}} \\ 0_{1,\zeta_{h+1}} \end{bmatrix} \oplus \begin{bmatrix} I_{\zeta_{h+2}} \\ 0_{1,\zeta_{h+2}} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} I_{\zeta_{h+k}} \\ 0_{1,\zeta_{h+k}} \end{bmatrix}$$

and (23)

$$G_\zeta = \begin{bmatrix} 0_{1,\zeta_{h+1}} \\ I_{\zeta_{h+1}} \end{bmatrix} \oplus \begin{bmatrix} 0_{1,\zeta_{h+2}} \\ I_{\zeta_{h+2}} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0_{1,\zeta_{h+k}} \\ I_{\zeta_{h+k}} \end{bmatrix},$$

with $p + q + \sum_{i=1}^k[\zeta_{h+i}] + k = r$, $p + q + \sum_{i=1}^k[\zeta_{h+i}] = m$. Finally, the matrices P, Q can be written as

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_p & Q_q & Q_\zeta \end{bmatrix}, \tag{24}$$

with $P_1 \in \mathbb{C}^{p\times r}$, $P_2 \in \mathbb{C}^{q\times r}$, $P_3 \in \mathbb{C}^{\zeta_1\times r}$, $\zeta_1 = k + \sum_{i=1}^k[\zeta_{h+i}]$ and $Q_p \in \mathbb{C}^{m\times p}$, $Q_q \in \mathbb{C}^{m\times q}$, $Q_\zeta \in \mathbb{C}^{m\times\zeta_2}$ and $\zeta_2 = \sum_{i=1}^k[\zeta_{h+i}]$.

By substituting the transformation $Y(t) = QZ(t)$ into (4) we obtain

$$FY^{(a)}(t)QZ(t) = GQZ(t) + V(t),$$

whereby, multiplying by P, using (22), (24) and setting $Z(t) = \begin{bmatrix} Z_p(t) \\ Z_q(t) \\ Z_\zeta(t) \end{bmatrix}$, $Z_p(t) \in \mathbb{C}^{p\times 1}$, $Z_p(t) \in \mathbb{C}^{q\times 1}$ and $Z_\zeta(t) \in \mathbb{C}^{\zeta_2\times 1}$, we arrive at at the subsystems

$$Z_p^{(a)}(t) = J_p Z_p(t) + P_1 V(t), \tag{25}$$

$$H_q Z_q^{(a)}(t) = Z_q(t) + P_2 V(t) \tag{26}$$

and

$$F_\zeta Z_\zeta^{(a)}(t) = G_\zeta Z_\zeta(t) + P_3 V(t). \tag{27}$$

The solution of subsystem (25) is given by (17) and the solution of subsystem (26) is given by (20). For the subsystem (27) let

$$Z_\zeta(t) = \begin{bmatrix} Z_{\zeta_{h+1}}(t) \\ Z_{\zeta_{h+2}}(t) \\ \vdots \\ Z_{\zeta_{h+k}}(t) \end{bmatrix}, \quad Z_{\zeta_{h+i}}(t) \in \mathbb{C}^{(\zeta_{h+i})\times 1}, \quad i = 1, 2, ..., k \tag{28}$$

with

$$Z_{\zeta_{h+i}}(t) = \begin{bmatrix} Z_{\zeta_{h+i},1}(t) \\ Z_{\zeta_{h+i},2}(t) \\ \vdots \\ Z_{\zeta_{h+i},\zeta_{h+i}}(t) \end{bmatrix} \tag{29}$$

and

$$P_3 V(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_k(t) \end{bmatrix}, \quad U_i(t) \in \mathbb{C}^{(\zeta_{h+i}+1)\times 1}, \quad i = 1, 2, ..., k$$

with

$$U_i(t) = \begin{bmatrix} v_{i0} \\ v_{i1} \\ v_{i2} \\ \vdots \\ v_{i\zeta_{h+i}} \end{bmatrix}, \quad i = 1, 2, ..., k.$$

By replacing (23) into (27) we get

$$\begin{bmatrix} I_{\zeta_{h+i}} \\ 0_{1,\zeta_{h+i}} \end{bmatrix} Z_{\zeta_{h+i}}^{(a)}(t) = \begin{bmatrix} 0_{1,\zeta_{h+i}} \\ I_{\zeta_{h+i}} \end{bmatrix} Z_{\zeta_{h+i}}(t) + U_i(t),$$

or, equivalently, by using the above expressions

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} Z^{(a)}_{\zeta_{h+i},1}(t) \\ Z^{(a)}_{\zeta_{h+i},2}(t) \\ \vdots \\ Z^{(a)}_{\zeta_{h+i},\zeta_{h+i}}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} Z_{\zeta_{h+i},1}(t) \\ Z_{\zeta_{h+i},2}(t) \\ \vdots \\ Z_{\zeta_{h+i},\zeta_{h+i}}(t) \end{bmatrix} + \begin{bmatrix} v_{i0} \\ v_{i1} \\ v_{i2} \\ \vdots \\ v_{i\zeta_{h+i}} \end{bmatrix},$$

or, equivalently,

$$\begin{gathered} Z^{(a)}_{\zeta_{h+i},1}(t) = v_{i0} \\ Z^{(a)}_{\zeta_{h+i},2}(t) = Z_{\zeta_{h+i},1}(t) + v_{i1} \\ \vdots \\ Z^{(a)}_{\zeta_{h+i},\zeta_{h+i}}(t) = Z_{\zeta_{h+i},\zeta_{h+i}-1}(t) + v_{i(\zeta_{h+i}-1)} \\ 0 = Z_{\zeta_{h+i},\zeta_{h+i}}(t) + v_{i\zeta_{h+i}} \end{gathered}.$$

We have a system of $\zeta_{h+i}+1$ FDEs and ζ_{h+i} unknowns. Starting from the last equation we get the solutions

$$\begin{gathered} Z_{\zeta_{h+i},\zeta_{h+i}}(t) = -v_{i\zeta_{h+i}} \\ Z_{\zeta_{h+i},\zeta_{h+i}-1}(t) = -v_{i(\zeta_{h+i}-1)} - v^{(a)}_{i\zeta_{h+i}} \\ Z_{\zeta_{h+i},\zeta_{h+i}-2}(t) = -v_{i(\zeta_{h+i}-2)} - v^{(a)}_{i(\zeta_{h+i}-1)} - v^{(2a)}_{i\zeta_{h+i}} \\ \vdots \\ Z_{\zeta_{h+i},1}(t) = -v_{i1} - v^{(a)}_{i2} - \dots - v^{([\zeta_{h+i}-1]a)}_{i\zeta_{h+i}} \end{gathered}. \tag{30}$$

In order to solve the system we used the last ζ_j equations. By applying these results in the first equation we get

$$v_{i0} = -v^{(a)}_{i1} - v^{(2a)}_{i2} - \dots - v^{(\zeta_{h+i}a)}_{i\zeta_{h+i}},$$

or, equivalently,

$$\sum_{\rho=0}^{\zeta_{h+i}} v^{(\rho a)}_{i\rho} = 0, \tag{31}$$

which is a necessary and sufficient condition for the system (27) to have a solution. If (31) does not hold, then the system has no solution. If (31) holds, then system (27) has a unique solution given by (28), (29) and (30).

To conclude, in the case of a singular pencil with $r > m$, system (4) has the solution

$$Y(t) = QZ(t) = \begin{bmatrix} Q_p & Q_q & Q_\zeta \end{bmatrix} \begin{bmatrix} Z_p(t)\Phi_0(t)C + \int_0^\infty \Phi(t-\tau)V(\tau)d\tau \\ -\sum_{i=0}^{q_*-1} H^i_q P_2 V^{(ia)}(t) \\ Z_\zeta \end{bmatrix},$$

or, equivalently,

$$Y(t) = Q_p\left[\Phi_0(t)C + \int_0^\infty \Phi(t-\tau)V(\tau)d\tau\right] - Q_q\sum_{i=0}^{q_*-1} H^i_q P_2 V^{(ia)}(t) + Q_\zeta Z_\zeta. \tag{32}$$

To sum up, we proved the following Theorem:

Theorem 2.2. There exist solutions for the system of FDEs (4) if and only if

(a) The pencil of the system is regular;

(b) The pencil of the system is singular with $r > m$ and (31) holds.

Then the general solution is given by (21) in the case of (a) and by (32) in the case of (b). In both (a), (b) where $\Phi_0(t)$, $\Phi(t)$ are given by

(i) (16) if we use the (C) fractional derivative;

(ii) (18) if we use the (CF) fractional derivative;

(iii) (19) if we use the (AB) fractional derivative.

Having identified the conditions under which there exist solutions for singular systems in the form of (4), we can now present the following Corollary:

Corollary 2.1. If there exist solutions for system (4), then in the case that:

(a) The pencil of the system is regular, for given initial conditions $Y(t_0) = Y_0$ the solution is unique if and only if:

$$Y_0 \in colspan Q_p - Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V^{(ia)}(0). \tag{33}$$

The solution is then given by (21) and C is the unique solution of the linear system

$$Q_p C = [Y_0 + Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V^{(ia)}(0)]. \tag{34}$$

(b) The pencil of the system is singular with $r > m$ and (31) holds, for given initial conditions $Y(t_0) = Y_0$ the solution is unique if and only if:

$$Y_0 \in colspan Q_p - Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V^{(ia)}(0) + Q_\zeta Z_\zeta. \tag{35}$$

The solution is then given by (32) and C is the unique solution of the linear system

$$Q_p C = [Y_0 + Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V^{(ia)}(0) + Q_\zeta Z_\zeta]. \tag{36}$$

Proof. This is a direct result from Theorem 2.2. For (a) if we use the formula (21) for $t = 0$ we get:

$$Y(0) = Q_p C - Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V^{(ia)}(0),$$

and we arrive at condition (33) because C is assumed an unknown vector. The above linear system has always a unique solution for C since the matrix Q_p has linear independent columns. Similar for (b) if we use the formula (32) for $t = 0$

$$Y(0) = Q_p C - Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V^{(ia)}(0) + Q_\zeta Z_\zeta,$$

we arrive at condition (35). As previously, the above linear system has always a unique solution for C since the matrix Q_p has linear independent columns. The proof is completed.

3 Numerical examples

In this Section we provide numerical examples to justify our theory. We will use the (C) fractional derivative.

Example 1

We consider system (4) with $V(t) = 0_{7,1}$ and

$$F = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 3 & 2 & 2 & 0 & 1 & 1 \\ 1 & 2 & 3 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then $\det(s^a F - G) \equiv 0$ and from Theorem 2.1 there do not exist solutions for the system since the pencil is singular and the coefficients are square matrices.

Example 2

We consider system (4) with $V(t) = 0_{4,1}$ and

$$F = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix}.$$

The matrices F, G are non-square. Hence, we are in the case where $r > m$, i.e. $4 > 3$ and a singular pencil with

$$s^a F - G = \begin{bmatrix} s^a - 1 & s^a - 2 & s^a - 2 \\ 0 & s^a - 2 & s^a - 2 \\ s^a - 1 & s^a - 2 & s^a - 2 \\ 0 & s^a - 2 & s^a - 3 \end{bmatrix}.$$

From Theorem 2.1 there exists the matrix

$$P(s) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 - s^a & 0 & -s^a + 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

computed via the Gauss-Jordan Elimination Method, such that

$$P(s)(s^a F - G) = \begin{bmatrix} A(s) \\ 0_{1,1} \end{bmatrix}.$$

Where

$$A(s) = \begin{bmatrix} s^a - 1 & 0 & 0 \\ 0 & s^a - 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In addition, for

$$P(s) = \begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix},$$

with

$$P_1(s) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 - s^a & 0 & -s^a + 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad P_2(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix},$$

we have

$$P_2(s)F = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \neq 0_{1,3}.$$

Hence, (8) does not hold and from Theorem 2.1 the system does not have solutions.

Example 3

We consider system (4) with $V(t) = 0_{3,1}$ and

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 1 & 2 \end{bmatrix}.$$

The matrices F, G are non-square. Hence, we are in the case where $r > m$, i.e. $3 > 2$ and a singular pencil with

$$s^a F - G = \begin{bmatrix} s^a - 1 & s^a - 2 \\ 0 & s^a - 2 \\ s^a - 1 & s^a - 2 \end{bmatrix}.$$

From Theorem 2.1 there exists the matrix

$$P(s) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

computed via the Gauss-Jordan Elimination Method, such that

$$P(s)(s^aF-G)=\begin{bmatrix} A(s) \\ 0_{1,2} \end{bmatrix}.$$

Where

$$A(s)=\begin{bmatrix} s^a-1 & 0 \\ 0 & s^a-2 \end{bmatrix}.$$

In addition, for

$$P(s)=\begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix},$$

with

$$P_1(s)=\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_2(s)=\begin{bmatrix} -1 & 0 & 1 \end{bmatrix},$$

we have

$$P_2(s)F=0_{1,2}.$$

Hence, (8) holds and from Theorem 2.1 there exist solutions for the system given by (9):

$$\Psi_0(t)=\mathcal{L}^{-1}\{s^{a-1}A^{-1}P_1(s)F\}=\mathcal{L}^{-1}\{\begin{bmatrix} \frac{1}{s^{1-a}(s^a-1)} & 0 \\ 0 & \frac{1}{s^{1-a}(s^a-2)} \end{bmatrix}\},$$

or, equivalently,

$$\Psi_0(t)=\mathcal{L}^{-1}\{\begin{bmatrix} \sum_{n=0}^{\infty} s^{-an-1} & 0 \\ 0 & \sum_{n=0}^{\infty} 2^n s^{-an-1} \end{bmatrix}\},$$

or, equivalently,

$$\Psi_0(t)=\begin{bmatrix} \sum_{n=0}^{\infty} \frac{t^{an}}{\Gamma(an+1)} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{2^n t^{an}}{\Gamma(an+1)} \end{bmatrix}.$$

Then for $C=\begin{bmatrix} c_1 & c_2 \end{bmatrix}^T$ we have:

$$Y(t)=\begin{bmatrix} \sum_{n=0}^{\infty} \frac{t^{an}c_1}{\Gamma(an+1)} \\ \sum_{n=0}^{\infty} \frac{2^n t^{an}c_2}{\Gamma(an+1)} \end{bmatrix}.$$

Example 4

We consider system (4) with $V(t)=0_{2,1}$ and

$$F=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, G=\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Then

$$s^aF-G=\begin{bmatrix} s^a-1 & s^a-1 \\ 0 & 1 \end{bmatrix}$$

with $\det(s^a F - G) \not\equiv 0$, and thus the pencil is regular. From Theorem 2.1 there exists the matrix

$$P(s) = \begin{bmatrix} 1 & -(s^a - 1) \\ 0 & 1 \end{bmatrix}$$

computed via the Gauss-Jordan Elimination Method, such that

$$P(s)(s^a F - G) = A(s).$$

Where

$$A(s) = \begin{bmatrix} s^a - 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence from Theorem 2.1 there exist solutions for the system given by (7):

$$\Phi_0(t) = \mathcal{L}^{-1}\{s^{a-1}A^{-1}(s)P(s)F\} = \mathcal{L}^{-1}\{\begin{bmatrix} \frac{1}{s^{1-a}(s^a-1)} & -s^{a-1} \\ 0 & 0 \end{bmatrix}\},$$

or, equivalently,

$$\Phi_0(t) = \mathcal{L}^{-1}\{\begin{bmatrix} \sum_{n=0}^{\infty} s^{-an-1} & -s^{a-1} \\ 0 & 0 \end{bmatrix}\},$$

or, equivalently,

$$\Phi_0(t) = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{t^{an}}{\Gamma(an+1)} & -\frac{t^{-a}}{\Gamma(-a)} \\ 0 & 0 \end{bmatrix}.$$

Then we have the general solution:

$$Y(t) = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{t^{an}}{\Gamma(an+1)} & -\frac{t^{-a}}{\Gamma(-a)} \\ 0 & 0 \end{bmatrix} C.$$

Let the initial conditions of the system be

$$Y(0) = Y_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

The pencil $sF - G$ has one finite eigenvalue, $s = 1$ and one infinite. The column vector space of the eigenvectors of the finite eigenvalue is:

$$colspanQ_p = < \begin{pmatrix} 1 \\ 0 \end{pmatrix} > .$$

Then

$$Y_0 \in colspanQ_p,$$

and the system has a unique solution given by

$$Y(t) = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{t^{an}}{\Gamma(an+1)} & -\frac{t^{-a}}{\Gamma(-a)} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

or, equivalently,

$$Y(t) = \left[\begin{array}{c} -\sum_{n=0}^{\infty} \frac{t^{an}}{\Gamma(an+1)} \\ 0 \end{array} \right].$$

Next assume the initial conditions

$$Y_0 = \left[\begin{array}{c} 0 \\ 1 \end{array} \right].$$

Since

$$Y_0 \notin colspanQ_p,$$

from Corollary 2.1 the system has infinite solutions. From the results of this example we see that although there exists solution for system (4), for given initial conditions, the existence of a unique solution is not automatically satisfied. This is very important because in real applications in cases where the solution is not unique a redesign of the system maybe needed.

Example 5

We consider now the non-linear system of FDEs:

$$f(t, Y, Y^{(a)}) = 0_{m,1}, \tag{37}$$

with known initial conditions

$$Y(0) = Y_0, \quad Y^{(a)}(0) = Y_1. \tag{38}$$

Where $f : [0, +\infty] \times \mathbb{R}^{m\times 1} \times \mathbb{R}^{m\times 1} \to \mathbb{R}^{m\times 1}$ has continuous partial derivatives of second order. Let $f_0 = f(0, Y(0), Y^{(a)}(0)) = f(0, Y_0, Y_0^{(a)})$ and

$$F = -\frac{\partial f_0}{\partial Y^{(a)}}, \quad G = \frac{\partial f_0}{\partial Y}, \quad V(t) = f_0 + t\frac{\partial f_0}{\partial t} - \frac{\partial f_0}{\partial Y} Y_0 - \frac{\partial f_0}{\partial Y^{(a)}} Y_1. \tag{39}$$

Where $F, G \in \mathbb{R}^{m\times m}$ and $V : [0, +\infty] \to \mathbb{R}^{m\times 1}$. We may use Theorem 2.1, or Theorem 2.2, to solve approximately, for small values of t, the non-linear system of FDEs (37). If we linearize locally the function f in (37) at $(t, Y(t), Y^{(a)}(t)) = (0, Y_0, Y_0^{(a)})$ we get:

$$f(0, Y_0, Y_0^{(a)}) + \frac{\partial f_0}{\partial t} t + \frac{\partial f_0}{\partial Y}(Y - Y_0) + \frac{\partial f_0}{\partial Y^{(a)}}(Y^{(a)} - Y_1) = 0_{m,1},$$

or, equivalently,

$$[-\frac{\partial f_0}{\partial Y^{(a)}}]Y^{(a)} = [\frac{\partial f_0}{\partial Y}]Y + [f_0 + t\frac{\partial f_0}{\partial t} - \frac{\partial f_0}{\partial Y} Y_0 - \frac{\partial f_0}{\partial Y^{(a)}} Y_1].$$

Note that $-\frac{\partial f_0}{\partial Y^{(a)}}, \frac{\partial f_0}{\partial Y} \in \mathbb{R}^{m\times m}$. Then by adopting the notation (39), the above expression is the singular system of FDEs (4) with the coefficient matrices being square. Hence, if the pencil is regular, i.e. $\det(sF - G) \not\equiv 0$, and (33) holds, the solution of (4) with initial conditions (38) is an approximate solution of system (37) with initial conditions (38) for small values of t.

Let $0 < a < 1$, $Y : [0, +\infty] \to \mathbb{R}^{3\times 1}$ with $Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$, and assume the following system of non-linear FDEs

$$\begin{aligned} y_1^{(a)} + e^{y_1} - y_2 &= 3t \\ y_2^{(a)} y_3^{(a)} + e^{y_2} &= 2e^t \\ y_2^{(a)} + y_3^{(a)} + y_1 y_3 - 2e^{y_2} &= 5t \end{aligned} \tag{40}$$

with initial conditions

$$Y(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad Y^{(a)}(0) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}. \tag{41}$$

We compute the following values:

$$f(0, Y(0), Y^{(a)}(0)) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{\partial f(0, Y(0), Y^{(a)}(0))}{\partial t} = \begin{bmatrix} -3 \\ -2 \\ -5 \end{bmatrix}$$

and

$$\frac{\partial f(0, Y(0), Y^{(a)}(0))}{\partial Y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}, \quad \frac{\partial f(0, Y(0), Y^{(a)}(0))}{\partial Y^{(a)}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

We consider the matrices F, G and $V(t)$ as defined in (38), i.e.:

$$F = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}, V(t) = \begin{bmatrix} 3t - 1 \\ 2t + 2 \\ 5t + 2 \end{bmatrix}.$$

Since

$$s^a F - G = -\begin{bmatrix} s^a + 1 & 0 & 0 \\ 0 & s^a + 1 & s^a \\ 0 & s^a - 2 & s^a \end{bmatrix}$$

is a regular pencil, the solution of system (4) with initial conditions (41) is an approximate solution of system (40) with initial conditions (41). Hence, by using Theorem 2.1 there exists the matrix

$$P(s) = -\frac{1}{3} \begin{bmatrix} 3 & 0 & 0 \\ 0 & s^a + 1 & -s^a - 1 \\ 0 & -s^a + 2 & s^a + 1 \end{bmatrix},$$

computed via the Gauss-Jordan Elimination Method, such that

$$P(s)(s^a F - G) = A(s).$$

Where

$$A(s) = \begin{bmatrix} s^a+1 & 0 & 0 \\ 0 & s^a+1 & 0 \\ 0 & 0 & s^a \end{bmatrix}.$$

Furthermore

$$\Phi_0(t) = \mathcal{L}^{-1}\{s^{a-1}A^{-1}(s)P(s)F\} = \mathcal{L}^{-1}\{\begin{bmatrix} \frac{s^a}{(s^a+1)s} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{s} & \frac{1}{s} \end{bmatrix}\},$$

or, equivalently,

$$\Phi_0(t) = \mathcal{L}^{-1}\{\begin{bmatrix} \sum_{n=0}^{\infty}(-1)^n s^{-an-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{s} & \frac{1}{s} \end{bmatrix}\},$$

or, equivalently,

$$\Phi_0(t) = \begin{bmatrix} \sum_{n=0}^{\infty}(-1)^n \frac{t^{an}}{\Gamma(an+1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and

$$\Phi(t) = \mathcal{L}^{-1}\{A^{-1}(s)P(s)U(s)\}.$$

Where

$$U(s) = \mathcal{L}\{V(t)\} = \mathcal{L}\{\begin{bmatrix} 3t-1 \\ 2t+2 \\ 5t+2 \end{bmatrix}\} = \begin{bmatrix} \frac{3}{s^2} - \frac{1}{s} \\ \frac{2}{s^2} + \frac{2}{s} \\ \frac{5}{s^2} + \frac{2}{s} \end{bmatrix}.$$

Hence

$$\Phi(t) = \mathcal{L}^{-1}\{\begin{bmatrix} \frac{1}{s^2(s^a+1)} - \frac{1}{s(s^a+1)} \\ -\frac{s^a+1}{s^2(s^a+1)} \\ \frac{s^a+3}{s^{a+2}} + \frac{2}{s^{a+1}} \end{bmatrix}\}.$$

Equivalently,

$$\Phi(t) = \begin{bmatrix} \sum_{n=0}^{\infty}(-1)^n[s^{-(an+a+2)} - s^{-(an+a+1)}] \\ \sum_{n=0}^{\infty}(-1)^n[-s^{-(an+2)} - s^{-(an+a+2)}] \\ \frac{1}{s^2} + \frac{3}{s^{a+2}} + \frac{2}{s^{a+1}} \end{bmatrix},$$

or, equivalently,

$$\Phi(t) = \begin{bmatrix} \sum_{n=0}^{\infty}(-1)^n[\frac{t^{an+a+1}}{\Gamma(an+a+2)} - \frac{t^{an+a}}{\Gamma(an+a+1)}] \\ \sum_{n=0}^{\infty}(-1)^n[-\frac{t^{an+1}}{\Gamma(an+2)} - \frac{t^{an+a+1}}{\Gamma(an+a+2)}] \\ t + \frac{3t^{a+1}}{\Gamma(a+2)} + \frac{2t^a}{\Gamma(a+1)} \end{bmatrix}.$$

Then by using (7) and the initial conditions (41), an approximate solution of (40) for small values of t is given by

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \cong \begin{bmatrix} \sum_{n=0}^{\infty}(-1)^n[\frac{t^{an+a+1}}{\Gamma(an+a+2)} - \frac{t^{an+a}}{\Gamma(an+a+1)}] \\ \sum_{n=0}^{\infty}(-1)^n[-\frac{t^{an+1}}{\Gamma(an+2)} - \frac{t^{an+a+1}}{\Gamma(an+a+2)}] \\ t + \frac{3t^{a+1}}{\Gamma(a+2)} + \frac{2t^a}{\Gamma(a+1)} \end{bmatrix}.$$

Conclusions

By using three different definitions of fractional derivatives, the (C) fractional derivative and two recent updated versions of this derivative, the (CF) and (AB) fractional derivative, we studied a class of singular linear systems of FDEs, i.e. systems with constant coefficients that can be non-square matrices, or square and singular. We studied the existence and uniqueness of solutions and provided two different type of formulas for the case that there exist solutions. Numerical examples where given in the final section of the article to justify our theory.

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