Research Article

Mathematical Methods in the Applied Sciences

Received XXXX

(www.interscience.wiley.com) DOI: 10.1002/sim.0000
MOS subject classification: 70K20; 70K70

Geometric relation between two different types of initial conditions of singular systems of fractional nabla difference equations.

I. K. Dassios*

In this article we study the geometric relation between two different types of initial conditions (IC) of a class of singular linear systems of fractional nabla difference equations whose coefficients are constant matrices. For this kind of systems, we analyze how inconsistent and consistent IC are related to the column vector space of the finite and the infinite eigenvalues of the pencil of the system and analyze the geometric connection between these two different types of IC. Numerical examples are given to justify the results. Copyright © 2015 John Wiley & Sons, Ltd.

Keywords: singular, fractional, discrete time system, nabla, initial conditions.

1. Introduction

Difference equations of fractional order have recently proven to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism and so forth, see [3], [12], [15], [21], [27], [28]. There has been a significant development in the study of fractional differential/difference equations and inclusions in recent years; For some recent contributions on fractional differential/difference equations, see [5], [6], [7], [8], [13], [14], [16], [18], [19], [20], [22], [23], [24], [25], [26], [27] and the references therein.

If we define $\mathbb{N}_\alpha$ by $\mathbb{N}_\alpha = \{\alpha, \alpha+1, \alpha+2, ...\}$, α integer, and n such that $0 < n < 1$ or $1 < n < 2$, then the nabla fractional operator in the case of Riemann-Liouville fractional difference of n-th order for any $Y_k : \mathbb{N}_a \to \mathbb{R}^m$ is defined by, see [2],

$$\nabla_\alpha^{-n} Y_k = \frac{1}{\Gamma(n)} \sum_{j=\alpha}^{k} (k-j+1)^{\overline{n-1}} Y_j.$$

We denote $\mathbb{R}^{m\times 1}$ with $\mathbb{R}^m$. Where the raising power function is defined by

$$k^{\bar{\alpha}} = \frac{\Gamma(k+\alpha)}{\Gamma(k)}.$$

We consider the singular fractional discrete time system of the form

$$F\nabla_0^n Y_k = GY_k, \quad k = 1, 2, ..., \tag{1}$$

with known IC. Where $F, G \in \mathbb{R}^{r\times m}$ and $Y_k \in \mathbb{R}^m$. The matrices F, G can be non-square ($r \neq m$) or square ($r = m$) with F singular (det$F$=0).

In this article we will study the geometric relation between two different types of IC of system (1), the consistent and the inconsistent. The paper is organized as follows: section 2 provides the necessary preliminaries used throughout the paper. section 3 contains the main results. We analyze how inconsistent and consistent IC are related to the column vector space of the finite and the infinite eigenvalues of the pencil of the system and provide a geometric connection of these two different types of IC. section 4 contains examples to justify the results of the previous section and we close the paper with section 5 and the conclusions.

*Correspondence to: *Mathematics Applications Consortium for Science and Industry (MACSI), Department of Mathematics & Statistics, University of Limerick, Ireland.*

†Please ensure that you use the most up to date class file, available from the MMA Home Page at
www3.interscience.wiley.com/journal/2197/home

2. Preliminaries

Throughout the paper we will use in several parts matrix pencil theory to establish our results. A matrix pencil is a family of matrices $sF - G$, parametrized by a complex number s, see [10], [11].

Definition 2.1. Given $F, G \in \mathbb{R}^{r\times m}$ and an arbitrary $s \in \mathbb{C}$, the matrix pencil $sF - G$ is called:

1. Regular when $r = m$ and $\det(sF - G) \neq 0$;
2. Singular when $r \neq m$ or $r = m$ and $\det(sF - G) \equiv 0$.

In this article we consider the system (1) with a *regular pencil*, where the class of $sF - G$ is characterized by a uniquely defined element, known as the Weierstrass canonical form, see [10], [11], specified by the complete set of invariants of $sF - G$. This is the set of elementary divisors of type $(s - a_j)^{p_j}$, called *finite elementary divisors*, where a_j is a finite eigenvalue of algebraic multiplicity p_j $(1 \leq j \leq \nu)$, and the set of elementary divisors of type $\hat{s}^q = \frac{1}{s^q}$, called *infinite elementary divisors*, where q is the algebraic multiplicity of the infinite eigenvalue. $\sum_{j=1}^{\nu} p_j = p$ and $p + q = m$.

From the regularity of $sF - G$, there exist non-singular matrices P, $Q \in \mathbb{R}^{m\times m}$ such that

$$PFQ = \begin{bmatrix} I_p & 0_{p,q} \\ 0_{q,p} & H_q \end{bmatrix}, \qquad PGQ = \begin{bmatrix} J_p & 0_{p,q} \\ 0_{q,p} & I_q \end{bmatrix}. \tag{2}$$

J_p, H_q are appropriate matrices with H_q a nilpotent matrix with index q_*, J_p a Jordan matrix and $p + q = m$. With $0_{q,p}$ we denote the zero matrix of $q \times p$. The matrix Q can be written as

$$Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix}. \tag{3}$$

$Q_p \in \mathbb{R}^{m\times p}$ is a matrix with columns the p linear independent (generalized) eigenvectors of the p finite eigenvalues of $sF - G$; $Q_q \in \mathbb{R}^{m\times q}$ is a matrix with columns the q linear independent (generalized) eigenvectors of the q infinite eigenvalues of $sF - G$. Moreover note that while Q is a matrix with columns the m linear independent (generalized) eigenvectors of the m (finite and infinite) eigenvalues of $sF - G$, it is easy to observe that

$$colspanQ = \mathbb{R}^m. \tag{4}$$

Furthermore from (3), (4)

$$colspanQ_p \oplus colspanQ_q = \mathbb{R}^m, \tag{5}$$

where

$$dim(colspanQ_p) = p, \quad dim(colspanQ_q) = q$$

and $\oplus$ is the direct sum of $colspanQ_p$ and $colspanQ_q$.

Definition 2.2. (See [1], [4]) Let J_p be a Jordan matrix as defined in (2). Then with $F_{n,n}(J_p(k+n)^{\overline{n}})$ we will denote the discrete Mittag-Leffler function with two parameters defined by

$$F_{n,n}(J_p(k+n)^{\overline{n}}) = \sum_{i=0}^{\infty} J_p^i \frac{(k+n)^{\overline{in}}}{\Gamma((i+1)n)}. \tag{6}$$

The following results have been proved.

Theorem 2.1. (See [5], [6], [7], [8]) We consider the system (1) with a regular pencil. Then, its solution exists if and only if all finite eigenvalues of the pencil are distinct and lie within the open disk $S = \{s \in \mathbb{R} : |s| < 1\}$; Then, the solution of system (1) for $k \geq 0$, is given by the formula

$$Y_k = Q_p(k+1)^{\overline{n-1}} F_{n,n}(J_p(k+n)^{\overline{n}})(I_p - J_p)C.$$

Where $C \in \mathbb{R}^p$ is a constant vector. The matrices Q_p, J_p are given by (2), (3). The discrete Mittag-Leffler function with two parameters is defined by (6).

Definition 2.3. Consider the system (1) with known IC. Then the IC are called consistent if there exists a solution for the system (1) which satisfies the given conditions.

Proposition 2.1. (See [5], [6], [7], [8]) The IC of system (1) are consistent if and only if

$$Y_0 \in colspanQ_p.$$

Proposition 2.2. (See [5], [6], [7], [8]) Consider the system (1) with given IC. Then if there exists a solution for the initial value problem, it is unique if and only if the IC are consistent. Then, the unique solution is given by the formula

$$Y_k = Q_p(k+1)^{\overline{n-1}} F_{n,n}(J_p(k+n)^{\overline{n}})(I_p - J_p)Z_0^p. \tag{7}$$

Where Z_0^p is the unique solution of the linear system $Y_0 = Q_p Z_0^p$. For inconsistent IC ($Y_0 \notin colspanQ_p$) it is has been proved that the system (1) has infinite solutions.

From the above already established results, we can conclude that if there exists solutions for system (1), then this solution is unique and given by (7) if and only if the IC are consistent. Inconsistent IC lead to infinite solutions. This makes the relation of this two different type of IC important. This relation has also been studied for singular discrete time systems, see [9]. Another known result that we will use in the next section is the orthogonal projection Theorem.

Theorem 2.2. (see [17]) Let W be an inner product space and let V be a finite dimensional subspace of W. Then $\forall w \in W$ there exists unique vectors $v_1 \in V$ and $v_2 \in V^{\perp}$, where $V^{\perp}$ is the orthogonal complement of V, such that $w = v_1 + v_2$ and v_1 is the orthogonal projection of w on V, i.e.

$$v_1 = proj_V w.$$

3. Geometric relation between a consistent and an inconsistent initial condition

In this section we will study the relation between a consistent and an inconsistent IC of the singular fractional system (1). It has been proved (see the previous section for references) that if for the singular system (1) with known IC there exists a solution, then it is unique and given by (7) if and only if the IC lie inside the domain $colspanQ_p$ (consistent IC). However it is possible for a system to have IC that pro exist and are not in the above mentioned domain; i.e. to be inconsistent. Then the system at $k=0$, almost instantaneously is being transferred into another new situation at time $k=1$, described by system (1). This phenomenon is called impulsive behavior of the system at $k=0$. In order to study the relation of these two different type of IC we have to study further the case of the inconsistent IC of the system.

Lemma 3.1. Let J_p be a Jordan matrix as defined in (2) with $\|J_p\| < 1$. Then for $k=0$, the discrete Mittag-Leffler function with two parameters $F_{n,n}(J_p(k+n)^{\overline{n}})$, defined in (6), takes the form

$$F_{n,n}(J_p(n)^{\overline{n}}) = \frac{1}{\Gamma(n)}(I_p - J_p)^{-1}.$$

Proof. By replacing $k=0$ into (6) we get

$$F_{n,n}(J_p(n)^{\overline{n}}) = \sum_{i=0}^{\infty} J_p^i \frac{(n)^{\overline{in}}}{\Gamma((i+1)n)},$$

or, equivalently,

$$F_{n,n}(J_p(n)^{\overline{n}}) = \sum_{i=0}^{\infty} J_p^i \frac{\Gamma(n+in)}{\Gamma((i+1)n)\Gamma(n)},$$

or, equivalently,

$$F_{n,n}(J_p(n)^{\overline{n}}) = \frac{1}{\Gamma(n)} \sum_{i=0}^{\infty} J_p^i$$

and since it is assumed $\|J_p\| < 1$,

$$F_{n,n}(J_p(n)^{\overline{n}}) = \frac{1}{\Gamma(n)}(I_p - J_p)^{-1}.$$

The proof is completed.

Proposition 3.1. Assume system (1) and let Y_0 be inconsistent IC. Then if there exist solutions for (1)

$$Y_0 \in N_r colspanQ_p^{-1}. \tag{8}$$

Where Q_p is defined by (3), Q_p^{-1} is the left inverse of the matrix Q_p, i.e. $Q_p^{-1}Q_p{=}I_p$ and N_r is the right kernel of the set $colspanQ_p^{-1}$.

Proof. The IC are assumed inconsistent and thus they don't satisfy (1). Hence

$$Y_0 \notin colspanQ_p.$$

From (4), (5)

$$Y_0 \in \mathbb{R}^m - colspanQ_p,$$

or, equivalently,

$$Y_0 \in colspanQ_q. \tag{9}$$

By using the transform

$$Y_k = QZ_k, \tag{10}$$

if

$$Z_k = \begin{bmatrix} Z_k^p \\ Z_k^q \end{bmatrix},$$

where $Z_k^p \in \mathbb{R}^p$, $Z_k^q \in \mathbb{R}^q$, by using (3) we get

$$Y_0 = Q_pZ_0^p + Q_qZ_0^q.$$

But from (9) we have $Z_0^p{=}0$ and

$$Y_0 = Q_qZ_0^q. \tag{11}$$

By replacing (10) into (1) we get

$$F\nabla_0^nQZ_k = GQZ_k,$$

or, equivalently,

$$FQ\nabla_0^nZ_k = GQZ_k.$$

Whereby multiplying by P and using (2) we obtain

$$\begin{bmatrix} I_p & 0_{p,q} \\ 0_{q,p} & H_q \end{bmatrix}\begin{bmatrix} \nabla_0^nZ_k^p \\ \nabla_0^nZ_k^q \end{bmatrix} = \begin{bmatrix} J_p & 0_{p,q} \\ 0_{q,p} & I_q \end{bmatrix}\begin{bmatrix} Z_k^p \\ Z_k^q \end{bmatrix}.$$

From above expressions, we arrive easily at the subsystems

$$\nabla_0^nZ_k^p = J_pZ_k^p \tag{12}$$

and

$$H_q\nabla_0^nZ_k^q = Z_k^q. \tag{13}$$

The subsystem (12) takes values for $k \geq 1$ and has the solution

$$Z_k^p = (k+1)^{\overline{n-1}}F_{n,n}(J_p(k+n)^{\bar{n}})(I_p - J_p)Z_1^p, \quad \forall k \geq 1.$$

For a proof of this solution see [2]. Since $Z_0^p{=}0$, by using the Heaviside function H_k,

$$H_k = \left\{ \begin{array}{ccc} 1 & , & k \geq 0 \\ 0 & , & k < 0 \end{array} \right\},$$

we give the solution the following form

$$Z_k^p = H_{k-1}(k)^{\overline{n-1}}F_{n,n}(J_p(k-1+n)^{\bar{n}})(I_p - J_p)Z_1^p, \quad \forall k \geq 0, \tag{14}$$

i.e. a solution $\forall k \geq 0$. The subsystem (13) takes values for $k \geq 1$ and its solution is given by

$$Z_k^q = 0_{q,1}, \quad \forall k \geq 1$$

For the proof see [5], [6], [7], [8]. But from (11)

$$Z_0^q \neq 0_{q,1}$$

and thus by using the Dirac function δ_k,

$$\delta_k = \left\{ \begin{array}{ccc} 1 & , & k = 0 \\ 0 & , & k \neq 0 \end{array} \right\},$$

we can have the solution of (13) in the following form

$$Z_k^q = \delta_k Z_0^q, \quad \forall k \geq 0. \tag{15}$$

Therefore the solution of system (1) $\forall k \geq 0$ can be written as

$$Y_k = QZ_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} Z_k^p \\ Z_k^q \end{bmatrix},$$

or, equivalently,

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} H_{k-1}(k)^{\overline{n-1}} F_{n,n}(J_p(k-1+n)^{\bar{n}})(I_p - J_p)Z_1^p \\ \delta_k Z_0^q \end{bmatrix},$$

or, equivalently,

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} H_{k-1}(k)^{\overline{n-1}} F_{n,n}(J_p(k-1+n)^{\bar{n}})(I_p - J_p) & 0_{p,q} \\ 0_{q,p} & \delta_k I_q \end{bmatrix} \begin{bmatrix} Z_1^p \\ Z_0^q \end{bmatrix},$$

or, equivalently,

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} H_{k-1}(k)^{\overline{n-1}} F_{n,n}(J_p(k-1+n)^{\bar{n}})(I_p - J_p) & 0_{p,q} \\ 0_{q,p} & \delta_k I_q \end{bmatrix} \left(\begin{bmatrix} Z_1^p \\ 0_{q,1} \end{bmatrix} + \begin{bmatrix} 0_{p,1} \\ Z_0^q \end{bmatrix} \right).$$

Let Y_1 be a consistent value for the system (1). Then from Proposition 2.1

$$Y_1 \in colspanQ_p.$$

By using (11) and the above expression combined with (3), (10), i.e. $Y_1 = Q_p Z_1^p$, we have

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} H_{k-1}(k)^{\overline{n-1}} F_{n,n}(J_p(k-1+n)^{\bar{n}})(I_p - J_p) & 0_{p,q} \\ 0_{q,p} & \delta_k I_q \end{bmatrix} Q^{-1}(Y_1 + Y_0).$$

Since it is assumed that there exists solutions for system (1), from Theorem 2.1, $\|J_p\| < 1$. Then for $k = 1$ and Lemma 3.1 we obtain

$$Y_1 = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} (1)^{\overline{n-1}} \frac{1}{\Gamma(n)}(I_p - J_p)^{-1}(I_p - J_p) & 0_{p,q} \\ 0_{q,p} & \delta_1 I_q \end{bmatrix} Q^{-1}(Y_1 + Y_0),$$

or, equivalently,

$$Y_1 = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} \frac{\Gamma(1+n-1)}{\Gamma(1)} \frac{1}{\Gamma(n)}(I_p - J_p)^{-1}(I_p - J_p) & 0_{p,q} \\ 0_{q,p} & 0_{q,q} \end{bmatrix} Q^{-1}(Y_1 + Y_0),$$

or, equivalently,

$$Y_1 = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} I_p & 0_{p,q} \\ 0_{q,p} & 0_{q,q} \end{bmatrix} Q^{-1}(Y_1 + Y_0).$$

Let

$$Q^{-1} = \begin{bmatrix} Q_p^{-1} \\ Q_q^{-1} \end{bmatrix},$$

where $Q_p^{-1} \in \mathbb{R}^{p \times m}$, $Q_q^{-1} \in \mathbb{R}^{q \times m}$. From $Q^{-1}Q = I_m$ we have that the matrix Q_p^{-1} is the left inverse of the matrix Q_p, i.e. $Q_p^{-1}Q_p = I_p$. Then

$$Y_1 = \begin{bmatrix} Q_p & 0_{m,q} \end{bmatrix} \begin{bmatrix} Q_p^{-1} \\ Q_q^{-1} \end{bmatrix} (Y_1 + Y_0),$$

or, equivalently,

$$Y_1 = Q_p Q_p^{-1}(Y_1 + Y_0)$$

and by multiplying from the left by Q_p^{-1} we arrive at

$$Q_p^{-1} Y_0 = 0_{p,1},$$

or, equivalently,

$$Y_0 \in N_r colspanQ_p^{-1}.$$

The proof is completed.

Proposition 3.2. Assume system (1) and let Y_0 be consistent IC. Then if there exist solutions for (1)

$$Y_0 \in N_r colspanQ_q^{-1}. \tag{16}$$

Q_q is defined by (3), Q_q^{-1} is the left inverse of the matrix Q_q, i.e. $Q_q^{-1}Q_q=I_q$ and N_r is the right kernel of the set $colspanQ_q^{-1}$.

Proof. If $Z_k = \begin{bmatrix} Z_k^p \\ Z_k^q \end{bmatrix}$, where $Z_k^p \in \mathbb{R}^p$, $Z_k^q \in \mathbb{R}^q$, by using the transform (10) and we get

$$Y_0 = Q_p Z_0^p + Q_q Z_0^q.$$

But from Proposition 2.1 we have $Z_0^q = 0_{q,1}$ and

$$Y_0 = Q_p Z_0^p. \tag{17}$$

Let Y_{-1} be an inconsistent condition for (1). Then from Proposition 3.1 and (8)

$$Y_{-1} \in colspanQ_q$$

and $Z_{-1}^p = 0_{p,1}$. Thus

$$Y_{-1} = Q_q Z_{-1}^q. \tag{18}$$

By replacing (10) into (1) we get

$$F\nabla_0^n Q Z_k = GQZ_k,$$

or, equivalently,

$$FQ\nabla_0^n Z_k = GQZ_k.$$

Whereby multiplying by P and using (2) we obtain

$$\begin{bmatrix} I_p & 0_{p,q} \\ 0_{q,p} & H_q \end{bmatrix} \begin{bmatrix} \nabla_0^n Z_k^p \\ \nabla_0^n Z_k^q \end{bmatrix} = \begin{bmatrix} J_p & 0_{p,q} \\ 0_{q,p} & I_q \end{bmatrix} \begin{bmatrix} Z_k^p \\ Z_k^q \end{bmatrix}.$$

From the above expressions, we arrive easily at the subsystems (12) and (13). The subsystem (12) takes values for $k \geq 0$ and has the solution

$$Z_k^p = (k+1)^{\overline{n-1}} F_{n,n}(J_p(k+n)^{\bar{n}})(I_p - J_p)Z_0^p, \quad k \geq 0.$$

Since Z_{-1}^p=0, by using the Heaviside function H_k we can give to the solution the following form

$$Z_k^p = H_k Z_k^p = (k+1)^{\overline{n-1}} F_{n,n}(J_p(k+n)^{\bar{n}})(I_p - J_p)Z_0^p, \quad \forall k \geq -1 \tag{19}$$

and thus have a solution for every $k \geq -1$. The subsystem (13) takes values for $k \geq 0$ and has the solution

$$Z_k^q = 0_{q,1}, \quad k \geq 0.$$

But as we stated earlier, $Z_{-1}^q \neq 0_{q,1}$ and thus by using the Dirac function δ_k we can give to the solution the following form

$$Z_k^q = \delta_{k-1} Z_{-1}^q, \quad \forall k \geq -1 \tag{20}$$

and thus have a solution for every $k \geq -1$. Then by using (10), (19) and (20), the solution of system (1) can be written as

$$Y_k = QZ_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} Z_k^p \\ Z_k^q \end{bmatrix},$$

or, equivalently,

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} (k+1)^{\overline{n-1}} F_{n,n}(J_p(k+n)^{\bar{n}})(I_p - J_p)Z_0^p \\ \delta_{k-1} Z_{-1}^q \end{bmatrix},$$

or, equivalently,

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} (k+1)^{\overline{n-1}} F_{n,n}(J_p(k+n)^{\bar{n}})(I_p - J_p)Z_0^p & 0_{p,q} \\ 0_{q,p} & \delta_{k-1} I_q \end{bmatrix} \begin{bmatrix} Z_0^p \\ Z_{-1}^q \end{bmatrix},$$

or, equivalently,

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} (k+1)^{\overline{n-1}} F_{n,n}(J_p(k+n)^{\bar{n}})(I_p - J_p)Z_0^p & 0_{p,q} \\ 0_{q,p} & \delta_{k-1} I_q \end{bmatrix} \left(\begin{bmatrix} Z_0^p \\ 0_{q,1} \end{bmatrix} + \begin{bmatrix} 0_{p,1} \\ Z_{-1}^q \end{bmatrix} \right)$$

and by using (17), (18)

$$Y_k = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} (k+1)^{\overline{n-1}} F_{n,n}(J_p(k+n)^{\bar{n}})(I_p - J_p)Z_0^p & 0_{p,q} \\ 0_{q,p} & \delta_{k-1} I_q \end{bmatrix} Q^{-1}(Y_0 + Y_{-1}).$$

Then for $k = -1$ we obtain

$$Y_{-1} = \left[\begin{array}{cc} Q_p & Q_q \end{array}\right] \left[\begin{array}{cc} 0_{p,p} & 0_{p,q} \\ 0_{q,p} & I_q \end{array}\right] Q^{-1}(Y_0 + Y_{-1})$$

and if we assume

$$Q^{-1} = \left[\begin{array}{c} Q_p^{-1} \\ Q_q^{-1} \end{array}\right],$$

where $Q_p^{-1} \in \mathbb{R}^{p\times m}$, $Q_q^{-1} \in \mathbb{R}^{q\times m}$, then from $Q^{-1}Q = I_m$ we have that the matrix Q_q^{-1} is the left inverse of the matrix Q_q, i.e. $Q_q^{-1}Q_q = I_q$. Hence

$$Y_{-1} = \left[\begin{array}{cc} 0_{m,p} & Q_q \end{array}\right] \left[\begin{array}{c} Q_p^{-1} \\ Q_q^{-1} \end{array}\right](Y_0 + Y_{-1}),$$

or, equivalently,

$$Y_{-1} = Q_q Q_q^{-1}(Y_0 + Y_{-1}).$$

By multiplying from the left with Q_q^{-1} we get

$$Q_q^{-1} Y_0 = 0_{q,1},$$

or, equivalently,

$$Y_0 \in N_r colspan Q_q^{-1}.$$

The proof is completed.

Theorem 3.1. Let Y_0 be a consistent condition of system (1) and Y_0^* an inconsistent. If there exist solutions for (1), Q is the a matrix as defined in (2), (3) and orthogonal, then

$$Y_0 = proj_{colspanQ_p}(Y_0 + Y_0^*), \tag{21}$$

i.e. Y_0 is the orthogonal projection of $Y_0 + Y_0^*$ on the set $colspanQ_p$ and

$$Y_0^* = proj_{colspanQ_q}(Y_0 + Y_0^*), \tag{22}$$

i.e. Y_0^* is the orthogonal projection of $Y_0 + Y_0^*$ on the set $colspanQ_q$.

Proof. As we already stated in (5)

$$colspanQ_p \oplus colspanQ_q = \mathbb{R}^m.$$

While Y_0 is a consistent condition, from Proposition 2.1 we have that

$$Y_0 \in colspanQ_p$$

and while Y_0^* is an inconsistent condition, from (9) we have that

$$Y_0^* \in colspanQ_q.$$

Furthermore

$$Y_0 + Y_0^* \in \mathbb{R}^m.$$

Let Q be an orthogonal matrix, then $Q^T Q = I_m$, where Q^T is the transposed matrix of Q. If we assume

$$Q^{-1} = \left[\begin{array}{c} Q_p^{-1} \\ Q_q^{-1} \end{array}\right],$$

where $Q_p^{-1} \in \mathbb{R}^{p\times m}$, $Q_q^{-1} \in \mathbb{R}^{q\times m}$, then from $Q^{-1}Q = I_m$ we have that $Q_p^{-1} = Q_p^T$, $Q_q^{-1} = Q_q^T$ and from Proposition 3.2

$$Y_0 \in N_r colspan Q_q^{-1},$$

i.e.

$$Y_0 \in N_r colspan Q_q^T,$$

or, equivalently,

$$colspanQ_q^T = (colspanQ_p)^\perp.$$

But $colspanQ_q^T$=$rowspanQ_q$ and thus

$$rowspanQ_q = (colspanQ_p)^\perp. \tag{23}$$

From (9)

$$Y_0^* \in colspanQ_q,$$

or, equivalently,

$$(Y_0^*)^T \in rowspanQ_q.$$

Then from (5), (9), (23), Proposition 2.1 and Theorem 2.2

$$Y_0 = proj_{colspanQ_p}(Y_0 + Y_0^*)$$

and thus we proved (21). From Proposition 3.1 we have

$$Y_0^* \in N_r colspanQ_p^{-1},$$

i.e.

$$Y_0^* \in N_r colspanQ_p^T,$$

or, equivalently,

$$colspanQ_p^T = (colspanQ_q)^\perp.$$

But $colspanQ_p^T = rowspanQ_p$ and thus

$$rowspanQ_p = (colspanQ_q)^\perp. \tag{24}$$

From Proposition 2.1

$$Y_0 \in colspanQ_p,$$

or, equivalently,

$$Y_0^T \in rowspanQ_p.$$

Then from (5), (9), (24), Proposition 2.1 and Theorem 2.2

$$Y_0^* = proj_{colspanQ_q}(Y_0 + Y_0^*)$$

and thus we proved (22). The proof is completed.

4. Numerical Example

We assume the system (1) with

$$F = \frac{2}{3}\begin{bmatrix} \frac{1}{2} & 1 & 1 \\ -2 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, G = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{4}{3} & -\frac{2}{3} & \frac{4}{3} \\ 2 & -2 & 1 \end{bmatrix}.$$

Then $det(sF - G) = s(s - \frac{1}{2})$ and the pencil is regular. Hence, from Theorem 2.1 there exists a solution for system (1). By calculating the eigenvectors of the finite and infinite eigenvalues we get the matrices

$$Q_p = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}, Q_q = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix},$$

respectively.

Example 4.1.

We will begin with a simple example to justify the results of Theorem 3.1. We assume the IC

$$Y_0 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}, Y_0^* = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

It is easy to observe that $Y_0 \in colspanQ_p$ (consistent IC), $Y_0^* \in colspanQ_q$ (inconsistent IC) and $(Y_0 + Y_0^*)^T = \begin{bmatrix} 1 & -1 & 5 \end{bmatrix}$. Then $\forall \alpha \in \mathbb{R}$ such that $u_1 = \alpha Y_0 \in colspanQ_p$, we have

$$proj_{colspanQ_p}(Y_0 + Y_0^*) = \frac{(Y_0 + Y_0^*)^T u_1}{\|u_1\|_2^2} u_1 = \frac{\begin{bmatrix} 1 & -1 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}}{18}\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = Y_0,$$

which justifies (21). In addition, $\forall u_2 \in colspanQ_q$ we have

$$proj_{colspanQ_q}(Y_0+Y_0^*) = \frac{(Y_0+Y_0^*)^T u_2}{\|u_2\|_2^2} u_2 = \frac{\begin{bmatrix} 1 & -1 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}{9}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = Y_0^*,$$

which justifies (22).

Example 4.2.

We assume now the IC

$$Y_0^* = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}.$$

It is easy to observe that $Y_0^* \in colspanQ_q$, i.e. the IC are inconsistent. We will use Theorem 3.1 to seek a consistent IC Y_0 such that system (1) will have a unique solution. Let

$$Y_0 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad x, y, z \in \mathbb{R}.$$

From (22) and $\forall u \in colspanQ_q$ we have

$$Y_0^* = proj_{colspanQ_q}(Y_0+Y_0^*) = \frac{(Y_0+Y_0^*)^T u}{\|u\|_2^2} u,$$

or, equivalently,

$$\begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} = \frac{1}{9}(2x-2y+z+18)\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

or, equivalently,

$$2x-2y+z=0.$$

Hence

$$Y_0 \in < \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} >= colspanQ_p.$$

5. Conclusions

In this article we studied the relation between two different types of IC of a class of singular nabla fractional discrete time systems. We proved that these vectors are related to the column vector spaces of the finite and the infinite eigenvalues respectively and also that a consistent initial value (and an inconsistent initial value) can be viewed as the orthogonal projection of the sum of a consistent with an inconsistent initial value over a certain subspace.

Acknowledgments

I. Dassios is supported by Science Foundation Ireland (award 09/SRC/E1780). The author would like to thank very much the anonymous referees for their comments that clearly improved the article.

References

1. T. Abdeljawad, *On Riemann and Caputo fractional differences*, Computers & Mathematics with Applications, 62 (2011) 1602-1611.
2. F. M. Atici, P.W. Eloe, *Linear systems of fractional nabla difference equations*, The Rocky Mountain Journal of Mathematics, Volume 41, Number 2, pp. 353–370, (2011).
3. D. Baleanu, K. Diethelm, E. Scalas, *Fractional Calculus: Models and Numerical Methods,* World Scientific (2012).
4. F. Chen, X. Luo, Y. Zhou, Existence results for nonlinear fractional difference equation, Adv Diff Equ, 713201 (2011).

5. I. K. Dassios, D. Baleanu, *Duality of singular linear systems of fractional nabla difference equations.* Applied Mathematical Modeling, Elsevier, Volume 39, Issue 14, pp. 4180-4195 (2015).
6. I. K. Dassios, D. Baleanu, *On a singular system of fractional nabla difference equations with boundary conditions,* Boundary Value Problems, 2013:148, Springer (2013).
7. I. K. Dassios, *Optimal solutions for non-consistent singular linear systems of fractional nabla difference equations*, Circuits systems and signal processing, Springer, Volume 34, Number 6, 1769–1797 (2015).
8. I. K. Dassios, D. Baleanu, G. Kalogeropoulos, *On non-homogeneous singular systems of fractional nabla difference equations,* Applied Mathematics and Computation, Volume 227, 112–131 (2014).
9. I. K. Dassios, G. Kalogeropoulos, *On the relation between consistent and non consistent initial conditions of singular discrete time systems,* Dynamics of continuous, discrete and impulsive systems Series A: Mathematical Analysis, Volume 20, Number 4a, pp. 447–458 (2013).
10. R. F. Gantmacher, *The theory of matrices I, II*, Chelsea, New York, (1959).
11. G. I. Kalogeropoulos, *Matrix pencils and linear systems*, Ph.D Thesis, City University, London, (1985).
12. J. Klamka, A., Czornik, M. Niezabitowski, A. Babiarz, *Controllability and mini-mum energy control of linear fractional discrete-time infinite-dimensional systems.* 11th IEEE International Conference on Control & Automation, Taichung, Taiwan, pp. 1210–1214 (2014).
13. C. Lizama, *lp-maximal regularity for fractional difference equations on UMD spaces. Math. Nachr.* (2015) doi: 10.1002/mana.201400326
14. C. Lizama, *The Poisson distribution, abstract fractional difference equations, and stability.* Proc. Amer. Math. Soc. Forthcoming (2015).
15. J. A. Machado, M. E. Mata, and A. M. Lopes. *Fractional State Space Analysis of Economic Systems.* Entropy 17, Number 8 (2015): 5402–5421.
16. J. A. Tenreiro Machado, A. M. S. F. Galhano, J. J. Trujillo. *On development of fractional calculus during the last fifty years.* Scientometrics 98, Number 1 (2014): 577-582.
17. C. D. Meyer, Jr. *Matrix Analysis and Applied Linear Algebra*, SIAM publications, Package edition (2001).
18. W. Lv, *Existence and Uniqueness of Solutions for a Discrete Fractional Mixed Type Sum-Difference Equation Boundary Value Problem.* Discrete Dynamics in Nature and Society 501 (2015): 376261.
19. W. Lv, *Existence of solutions for discrete fractional boundary value problems with a p-laplacian operator,* Advances in Difference Equations, Volume 2012, article 163, 2012.
20. W. Lv and J. Feng, *Nonlinear discrete fractional mixed type sum-difference equation boundary value problems in Banach spaces,* Advances in Difference Equations, Volume 2014, article 184, 12 pages, 2014.
21. I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*,p. xxiv+340. Academic Press, San Diego, Calif, USA (1999).
22. Rahmat, Mohamad Rafi Segi, and Mohd Salmi Md Noorani. *Caputo type fractional difference operator and its application on discrete time scales.* Advances in Difference Equations 2015.1 (2015): 1-15.
23. Rahmat, Mohamad Rafi Segi. *The (q, h)-Laplace transform on discrete time scales.* Computers & Mathematics with Applications 62.1 (2011): 272–281.
24. G.C. Wu, D. Baleanu, Z.G. Deng, S.D. Zeng, *Lattice fractional diffusion equation in terms of a Riesz–Caputo difference*, Physica A, 438 (2015): 335–339.
25. G.C. Wu, D. Baleanu, S.D. Zeng, Z.G. Deng, *Discrete fractional diffusion equation*, Nonlinear Dynamics, 80 (2015) 281–286.
26. Yin, Chun, et al. *Robust stability analysis of fractional-order uncertain singular nonlinear system with external disturbance.* Applied Mathematics and Computation Volume 269 pp. 351–362 (2015)
27. C. Yin, S. Zhong, W. Chen *Design of sliding mode controller for a class of fractional-order chaotic systems* Commun. Nonlinear Sci. Numer. Simul., 17 (2012), pp. 356–366
28. C. Yin, Y.Q. Chen, S.M. Zhong *Fractional-order sliding mode based extremum seeking control of a class of nonlinear systems* Automatica, 50 (2014), pp. 3173–3181

