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A stability result for a network of two triple junctions on the plane

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In this article, we study the problem of a bounded network of two triple junctions in a planar domain with fixed angle conditions at the junctions and at the points at which the curves intersect with the boundary. We introduce the evolution problem of this type of networks, identify the steady states and study their stability in terms of the geometry of the boundary. Copyright © 2015 John Wiley & Sons, Ltd.

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1. Introduction

The study of the geometric evolution problem of networks of curves in planar domains has always been very important in modeling of many phenomena in various fields of science, physics and engineering, see [1], [2], [3], [4], [5].

About 20 years ago, motivated by dynamical models in materials science describing phase separation and the motion of interfaces separating phases, Bronsard et. al. [6], [7], [8] introduced the problem of networks of curves in a planar domain with normal velocity proportional to the curvature and fixed angle conditions at the point at which the curves intersect. From the underlying model they derived the equations of motion, as well as the boundary conditions. The angles formed by the curves at a node are constant throughout the evolution and intersect the boundary of the domain orthogonally at all times. Our interest is in studying a network of curves which is in motion, with the normal velocity equal to the curvature. For some recent contributions, see [9], [10], [11], [12], [13], [14], [15] and the references therein. A network of two triple junctions can be seen in Figure 1. Mathematically, the parametrization of the curves can be formulated as follows. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with

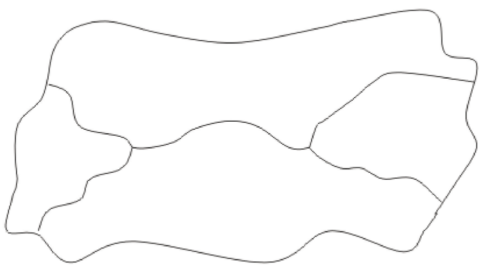

Figure 1. A network of two triple junctions.

sufficiently smooth boundary $\partial\Omega$. Consider a function $p : [0, +\infty) \to \Omega$ with regularity C^1 and five increasingly smooth functions

$$L_i : [0, +\infty) \to [0, +\infty), \quad i = 1, 2, 3, 4, 5,$$

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satisfying $L_i(0) = 0$. For each $t \geq 0$, let

$$G_i(\cdot, t) : [0, L_i(t)] \to \Omega, \quad i = 1, 2, 3, 4, 5,$$

be smooth functions such that $G_i(\cdot, t) \mid_{(0, L_i(t))}$ is an embedding and $\|\partial_s G_i(s, t)\| = 1$; $L_i(t)$ is the length of the curve G_i; $t \geq 0$ is time, s, $0 \leq s \leq L_i(t)$, arc length parameter and $G_i(s, t)$, contained in Ω, meeting at one point and intersecting with $\partial\Omega$ at the other ends; the evolution of $G_i(s, t)$ is described by

$$G_{it} = G_{iss}, \quad i = 1, 2, 3, 4, 5, \tag{1}$$

subject to four conditions, namely that $G_i(0, t) = p(t)$, the angle between $G_{is}(0, t)$ and $G_{js}(0, t)$ is $\frac{2\pi}{3}$ $(i \neq j)$, $G_i(L_i(t), t) \in \partial\Omega$, and that the curves $s \longmapsto G_i(s, t)$ meet $\partial\Omega$ orthogonally. Or, equivalently:

1. Incidence at the point at which the curves intersect:

$$\begin{aligned} G_1(0, t) = G_2(0, t) = G_3(0, t) \\ G_3(s_3, t) = G_4(0, t) = G_5(0, t); \end{aligned} \tag{2}$$

2. Angle conditions at the point at which the curves intersect:

$$\begin{aligned} G_{is}(0, t) \cdot G_{(i+1)s}(0, t) = cos\tfrac{2\pi}{3}, \quad i = 1, 2, \\ G_{3s}(s_3, t) \cdot G_{4s}(0, t) = cos\tfrac{2\pi}{3}, \\ G_{4s}(0, t) \cdot G_{5s}(0, t) = cos\tfrac{2\pi}{3}; \end{aligned} \tag{3}$$

3. Incidence at $\partial\Omega$, $\forall i = 1, 2, 4, 5$:

$$b(G_i(L_i(t), t) = 0; \tag{4}$$

4. Angle conditions at $\partial\Omega$, $\forall i = 1, 2, 4, 5$:

$$< G_{is}(L_i(t), t), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla b(G_i) >= 0. \tag{5}$$

Where $G_{is} = \frac{\partial G_i}{\partial s}$, $G_{iss} = \frac{\partial^2 G_i}{\partial s^2}$, $G_{it} = \frac{\partial G_i}{\partial t}$, s_3 is length of curve described by G_3, $\partial\Omega$ is the boundary of Ω, $b(\cdot, \cdot)$ is a C^1 real function of two variables that describes locally the boundary $\partial\Omega$ and $< \cdot, \cdot >$ is Euclidean inner product.

1. $G_i(s, t)$ are embeddings in the plane and the network is in motion with the normal velocity equal to the curvature law,

$$V_i^N = k_i, \quad i = 1, 2, 3, 4, 5.$$

   $V_i^N = G_{it} \cdot N_i$ is the normal velocity of the curve G_i, N_i is the unit normal vector to G_i, T_i=G_{is} is the unit tangent vector and the vector (N_i, T_i) has the orientation of the coordinate system which is valid locally.

2. V_i^T is the tangential velocity of curve G_i and k_i curvature of curve G_i. Moreover, the velocity V_i of the curve G_i is given by

$$V_i = (V_i^N, V_i^T),$$

   or, equivalently,

$$V_i = (G_{it} \cdot T_i, G_{it} \cdot N_i).$$

   Note that $G_{it} \cdot T_i = 0$, because G_{iss} is perpendicular to T_i and thus

$$V_i = (0, G_{it} \cdot N_i),$$

$$G_{it} \cdot N_i = G_{iss} \cdot N_i,$$

   or, equivalently,

$$V_i^N = k_i.$$

3. The curves G_i, $i = 1, 2, 3, 4, 5$, meet at one point and (2) is describing this property. The relation (3) describes the Plateau angle conditions. The three angles formed at the node $\frac{2\pi}{3}$. Note that these angles can be replaced by arbitrary prearranged values ϑ_1, ϑ_2, ϑ_3 as long as $\vartheta_1 + \vartheta_2 + \vartheta_3 = 360^o$. The relation (4) describes the contact of each curve to the boundary $\partial\Omega$ of the domain Ω. Finally the orthogonal intersection of each curve G_i to the boundary $\partial\Omega$ is described by (5).

4. The network reduces its perimeter (the total length) along the evolution

$$\frac{d}{dt}L(t) = -\int_G kV^N = -\sum_{i=1}^{3}\int_{G_i} k^2 \leq 0.$$

5. It would be more convenient to formulate the problem in a way that the arc length parameter s takes its values in a domain independent from time t. For this purpose let $\Gamma_i = (g_i^1(x,t), g_i^2(x,t))$, $t \geq 0$ and $x \in [0, l_i]$, $\forall i = 1,2,3,4,5$. Then:

$$G_{is} = G_{ix}\frac{dx}{ds}.$$

Furthermore

$$G_{iss} = \frac{\partial G_{ix}}{\partial s}\frac{dx}{ds} + G_{ix}\frac{\partial}{\partial s}\frac{1}{ds/dx},$$

or, equivalently,

$$G_{iss} = G_{ixx}(\frac{dx}{ds})^2 - G_{ix}\frac{1}{(ds/dx)^3}\frac{d^2s}{dx^2},$$

or, equivalently,

$$\Gamma_{iss} = \frac{\Gamma_{ixx}}{|\Gamma_{ix}|^2} - \Gamma_{ix}\frac{1}{(ds/dx)^3}\frac{d^2s}{dx^2}.$$

$s(x) = \int_0^x |\Gamma_x(p,t)|dp$ and $\frac{ds}{dx} = |\Gamma_x(x,t)|$. System of equations (1) will then take the form

$$\Gamma_{it} = \frac{\Gamma_{ixx}}{|\Gamma_{ix}|^2} - \Gamma_{ix}\frac{1}{(ds/dx)^3}\frac{d^2s}{dx^2}, \quad i = 1,2,3,4,5$$

and will be defined in the set $\mathcal{D}$=$[0, l_i] \times [0, +\infty)$. Note that by multiplying by N_1 and taking into account that $-\frac{1}{(ds/dx)^3}\Gamma_{ix}\frac{d^2s}{dx^2} \cdot N_1 = 0$, we get

$$\Gamma_{it} \cdot N_1 = \frac{\Gamma_{ixx}}{|\Gamma_{ix}|^2} \cdot N_1 + \Gamma_{ix}\frac{d^2x}{ds^2} \cdot N_1,$$

or, equivalently,

$$\Gamma_{it} \cdot N_1 = \frac{\Gamma_{ixx}}{|\Gamma_{ix}|^2} \cdot N_1.$$

The tangential term in the equation can be assigned at will without affecting the equations $V_i^N = k_i$, $i = 1,2,3,4,5$. Thus three equation that are compatible with motion by curvature are the following

$$\Gamma_{it} = \frac{\Gamma_{ixx}}{|\Gamma_{ix}|^2}, \quad i = 1,2,3,4,5, \quad x \in [0, l_i], \quad t \geq 0. \tag{6}$$

Equations (6) have to be supplement with conditions (2), (3), (4) which will take the following form:

1. Incidence at the node:

$$\Gamma_1(0,t) = \Gamma_2(0,t) = \Gamma_3(0,t),$$
$$\Gamma_3(l_3,t) = \Gamma_4(0,t) = \Gamma_5(0,t); \tag{7}$$

2. Angle conditions at the node:

$$\frac{\Gamma_{ix}(0)}{|\Gamma_{ix}(0)|} \cdot \frac{\Gamma_{(i+1)x}(0)}{|\Gamma_{(i+1)x}(0)|} = cos\frac{2\pi}{3}, \quad i = 1,2,$$
$$\frac{\Gamma_{3x}(l_3)}{|\Gamma_{3x}(l_3)|} \cdot \frac{\Gamma_{4x}(0)}{|\Gamma_{4x}(0)|} = cos\frac{2\pi}{3}, \tag{8}$$
$$\frac{\Gamma_{4x}(0)}{|\Gamma_{4x}(0)|} \cdot \frac{\Gamma_{5x}(0)}{|\Gamma_{5x}(0)|} = cos\frac{2\pi}{3};$$

3. Incidence at $\partial\Omega$ for $i = 1,2,4,5$ at $x = l_i$:

$$b(\Gamma_i) = 0; \tag{9}$$

4. Angle conditions at $\partial\Omega$ for $i = 1,2,4,5$ at $x = l_i$:

$$< \Gamma_{ix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla b(\Gamma_i) >= 0. \tag{10}$$

The condition $V_i^N = k_i$, $i = 1,2,3,4,5$ is not sufficient by itself to determine the evolution. Different equations for the embedding are expected to lead to different evolutions for the curves in which the nodes do not affect the evolution. Hence we conclude to equations (6) along with conditions (7), (8), (9), (10).

2. Linearization

In this section we present the linearized equations of (6) in order to study the stability of the steady states. We define the family of perturbations $\tilde{\Gamma}_i^\epsilon = \tilde{\Gamma}_i$:

$$\tilde{\Gamma}_i = \Gamma_i + \epsilon(h_i^N N_i + h_i^T T_i), \quad 0 < \epsilon << 1.$$

h_i^N, $h_i^T : [0, l_i] \to \mathbb{R}$ are real-valued functions of class C^2 and l_i is length of curve $\tilde{\Gamma}_i$. Then from [9, 10, 14], $\forall i = 1, 2, 3, 4, 5$, we conclude to the following eigenvalue problem

$$\begin{aligned} h_i''^N - h_i^T k_{ix} + h_i^N k_i^2 &= -\lambda h_i^N, \\ -2h_i'^N k_i - h^N k_{ix} + h_i''^T - h_i^T k_i^2 &= -\lambda h_i^T. \end{aligned} \tag{11}$$

Conditions:

1. Incidence at the node:

$$\begin{aligned} h_1^N(0) + h_2^N(0) + h_3^N(0) = 0, \\ h_1^T(0) + h_2^T(0) + h_3^T(0) = 0; \\ \\ h_3^N(l_3) + h_4^N(0) + h_5^N(0) = 0, \\ h_3^T(l_3) + h_4^T(0) + h_5^T(0) = 0; \end{aligned} \tag{12}$$

2. Angle conditions at the node at $x = 0$:

$$\begin{aligned} h_1'^N(0) + h_1^T(0)k_1 = h_2'^N(0) + h_2^T(0)k_2 = h_3'^N(0) + h_3^T(0)k_3; \\ h_3'^N(l_3) + h_3^T(l_3)k_3 = h_4'^N(0) + h_4^T(0)k_4 = h_5'^N(0) + h_5^T(0)k_5; \end{aligned} \tag{13}$$

3. Incidence at $\partial\Omega$ for $i = 1, 2, 4, 5$:

$$h_i^T(l_i) = 0; \tag{14}$$

4. Angle conditions at $\partial\Omega$ for $i = 1, 2, 4, 5$:

$$K_i^{\partial\Omega} h_i^N(l_i) = h_i'^N(l_i). \tag{15}$$

The following Lemma has been proved in [9], [11], [15]:

Lemma 2.1. If $h_i^N(x) \equiv 0$, $i = 1, 2, 3, 4, 5$ and $x \in [0, l_i]$, then $h_i^T(x) \equiv 0$.

3. Stability of the steady state of a network of three curves that meet at a node

A network of two triple junction that lies inside of a domain on the plane contains five curves and does not have all of its curves meeting at one end at the boundary of the domain. More specifically, if Ω is a bounded and smooth domain on the plane that contains a network of two junctions and for $i = 1, 2, 3, 4, 5$ and $x \in [0, l_i]$, Γ_i are the curves that built the network. Then, Γ_3 has its one end meeting the first junction and its other end meeting at the second junction. In addition, Γ_i, $i = 1, 2$ have their one end meeting at the first junction and their other meeting orthogonally at the boundary $\partial\Omega$ of the domain Ω. Finally, Γ_i, $i = 4, 5$ have their one end meeting at the second junction and their other meeting orthogonally at the boundary $\partial\Omega$ of the domain Ω. The angles formed by the curves at the triple junction are constant $\frac{2\pi}{3}$ throughout the evolution. The situation can be formulated mathematically by (11), (12), (13), (14), (15). The eigenvalues of the linearized operator (11) will give information about the stability of the network. Actually we will prove that stability depends on the geometry of the boundary Ω.

Definition 3.1. We define the sign of a curvature as follows. At the points that a curve is convex we define the sign of the curvature positive and at the points that the a curve is non-degenerate concave we define the sign of the curvature negative.

Theorem 3.1. Let Ω be a bounded and smooth domain on the plane that contains a network (see Figure 1) of two junctions and five curves that intersect at a point as described in (1) with conditions (2), (3), (4), (5). Then

1. If the domain Ω is convex (an ellipse for example) at the points where the steady state of the network meets the boundary, then the steady state is unstable.
2. If the domain Ω is non-degenerate concave at the points where the steady state of the network meets the boundary, then the steady state is stable.
3. If the domain Ω is flat at the points where the steady state of the network meets the boundary, then the steady state is neutrally stable.

Proof. Since we study the stability of the steady states we shall rewrite the linearized operator and its conditions substituting $k_i = 0$, $\forall i = 1, 2, 3, 4, 5$. Also note that by using Lemma 2.1, we can limit our studies by using the functions h_i^N. Hence

$$h_i''^N = -\lambda h_i^N. \tag{16}$$

Conditions:

1. Incidence at the node:
$$h_1^N(0) + h_2^N(0) + h_3^N(0) = 0,$$
$$h_3^N(l_3) + h_4^N(0) + h_5^N(0) = 0;$$
2. Angle conditions at the node at $x = 0$:
$$h_1'^N(0) = h_2'^N(0) = h_3'^N(0),$$
$$h_3'^N(l_3) = h_4'^N(0) = h_5'^N(0);$$
3. Angle conditions at $\partial\Omega$ for $i = 1, 2, 4, 5$:
$$K_i^{\partial\Omega} h_i^N(l_i) = h_i'^N(l_i).$$

For the proof of (a), since Ω is a strictly convex domain, $K_i^{\partial\Omega} > 0$, $\forall i = 1, 2, 4, 5$ (see Figure 2). We will establish the existence of $\lambda < 0$ such that $h_i^N \neq 0$ $\forall$ $i = 1, 2, 3, 4, 5$. If $\lambda < 0$ then the solution of (16) is equal to:

$$h_i^N = C_i cosh(x\sqrt{-\lambda}) + D_i sinh(x\sqrt{-\lambda}), \quad i = 1, 2, 3, 4, 5. \tag{17}$$

C_i,D_i are unknown real variables. From the conditions of (16) we have

$$\sum_{i=1}^{3} C_i = 0,$$

$$C_3 cosh(l_3\sqrt{-\lambda}) + D_3 sinh(l_3\sqrt{-\lambda}) + C_4 + C_5 = 0,$$

$$D_1 = D_2 = D_3,$$

$$C_3 sinh(l_3\sqrt{-\lambda}) + D_3 cosh(l_3\sqrt{-\lambda}) = D4 = D5,$$

$$C_i(K_i^{\partial\Omega} - \sqrt{-\lambda} tanh\sqrt{-\lambda}) - D_i(\sqrt{-\lambda} - K_i^{\partial\Omega} tanh\sqrt{-\lambda}) = 0, \quad i = 1, 2, 4, 5.$$

We have an homogeneous linear system of ten unknowns (the real variables C_i, D_i, $i = 1, 2, 3, 4, 5$) and ten equations. By

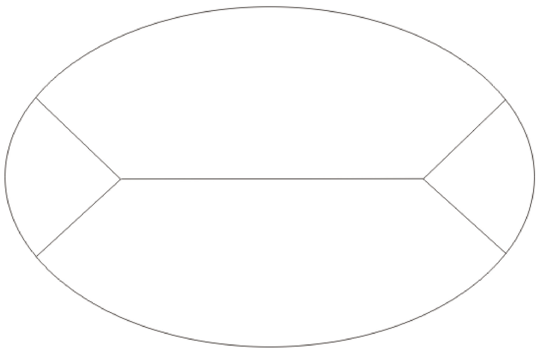

Figure 2. A strictly convex domain on the plane that contains a steady state of two triple junctions.

solving this system we find that for $\lambda < 0$ the system has infinite many solutions, i.e. we have non-zero solutions. In fact by rewriting the above homogeneous linear system of equations in matrix form and by using Matlab, we get that the determinant of the matrix is zero $\forall l_i, i = 1, 2, 3, 4, 5$ if and only if

$$\sqrt{-\lambda} tanh(l_i\sqrt{-\lambda}) - K_i^{\partial\Omega} = 0, \quad K_i^{\partial\Omega} > 0, \quad i = 1, 2, 4, 5.$$

Thus the eigenvalue problem has negative eigenvalues which are given from the solutions of the above equations. Thus the linearized operator has negative eigenvalues and the steady state is unstable.

For the proof of (b), we need to show that the eigenvalue value problem does not have negative and zero eigenvalues, i.e. that for $\lambda \leq 0$ we get $h_i^N = h_i^T = 0$, $i = 1, 2, 3, 4, 5$. As seen in Figure 3 the curvature of the boundary at the points meets the

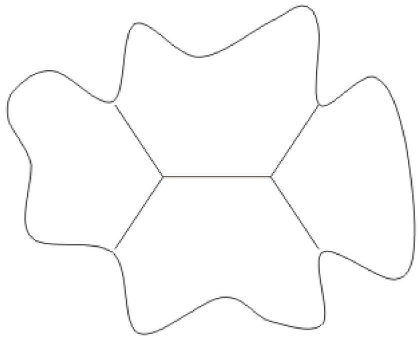

Figure 3. Concave boundary at the points where it meets the steady state of the network.

network is negative ($K_i^{\partial\Omega} < 0$, $\forall i = 1, 2, 4, 5$). For $\lambda < 0$ the solution of the differential equation (16) is given by (17). From the conditions of (16) we get equations that have the same structure as in the case of (a), i.e. the same linear system (10 equations-10 unknowns). The only difference is of course the sign of $K_i^{\partial\Omega}$ which is negative now. By taking into account this change, this time the homogeneous linear system has only the zero solution and thus

$$C_i^N = D_i^N = 0, \quad i = 1, 2, 3, 4, 5,$$

or, equivalently,

$$h_i^N = 0, \quad i = 1, 2, 3, 4, 5.$$

Thus the problem does not have negative eigenvalues. In the case of $\lambda = 0$, (16) takes the form

$$h_{ixx}^N = 0, \quad i = 1, 2, 4, 5,$$

or, equivalently,

$$h_i^N = A_i^N x + B_i^N, \quad i = 1, 2, 4, 5.$$

Furthermore, the conditions of (16) take the form

$$\sum_{i=1}^{3} B_i^N = 0,$$

$$A_3^N l_3 + \sum_{i=3}^{5} B_i^N = 0,$$

$$A_1^N = A_2^N = A_3^N,$$

$$A_3^N = A_4^N = A_5^N,$$

$$A_i^N = K_i^{\partial\Omega}(A_i^N l_i + B_i^N), \quad i = 1, 2, 4, 5.$$

The above homogeneous linear system has only the zero solution, i.e.

$$A_i^N = B_i^N = 0, \quad i = 1, 2, 3, 4, 5.$$

Thus, for $\lambda = 0$ we have $h_i^N = 0$, $\forall i = 1, 2, 3, 4, 5$, i.e. the linearized operator does not have zero eigenvalues.

For the proof of (c), we have that the boundary is flat at the points where it meets the network. This means that $K_i^{\partial\Omega} = 0$, $\forall i = 1, 2, 4, 5$ (see Figure 4). The conditions of (16) will take the following form

1. Incidence at the node:
$$h_1^N(0) + h_2^N(0) + h_3^N(0) = 0,$$
$$h_3^N(l_3) + h_4^N(0) + h_5^N(0) = 0;$$
2. Angle conditions at the node at $x = 0$:
$$h_1'^N(0) = h_2'^N(0) = h_3'^N(0),$$
$$h_3'^N(l_3) = h_4'^N(0) = h_5'^N(0);$$
3. Angle conditions at $\partial\Omega$ for $i = 1, 2, 4, 5$:
$$h_i'^N(l_i) = 0.$$

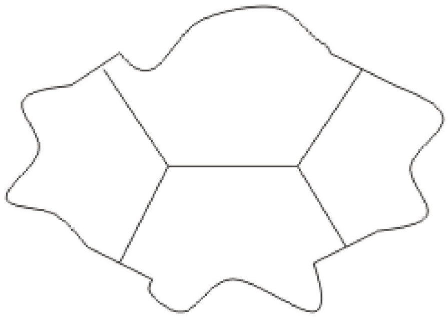

Figure 4. Flat boundary at the points where it meets the network.

For $\lambda < 0$, (16) has the solution (17) with C_i, D_i unknown real variables. By replacing the solution (17) into the above conditions of (16) we get

$$\sum_{i=1}^{3} C_i = 0,$$

$$C_3 cosh(l_3\sqrt{-\lambda}) + D_3 sinh(l_3\sqrt{-\lambda}) + C_4 + C_5 = 0,$$

$$D_1 = D_2 = D_3,$$

$$C_3 sinh(l_3\sqrt{-\lambda}) + D_3 cosh(l_3\sqrt{-\lambda}) = D4 = D5,$$

$$C_i tanh\sqrt{-\lambda} + D_i = 0, \quad i = 1, 2, 4, 5.$$

The above equations form an homogeneous linear system with a unique solution

$$C_i^N = D_i^N = 0, \quad i = 1, 2, 3, 4, 5$$

and therefore

$$h_i^n = 0, \quad i = 1, 2, 3, 4, 5.$$

Thus the problem does not have negative eigenvalues. In the case of $\lambda = 0$ the differential equation (16) takes the form

$$h_{ixx}^N = 0, \quad i = 1, 2, 4, 5,$$

or, equivalently,

$$h_i^N = A_i^N x + B_i^N, \quad i = 1, 2, 4, 5.$$

Furthermore, the conditions of (16) take the form

$$\sum_{i=1}^{3} B_i^N = 0,$$

$$A_3^N l_3 + \sum_{i=3}^{5} B_i^N = 0,$$

$$A_1^N = A_2^N = A_3^N,$$

$$A_3^N = A_4^N = A_5^N,$$

$$A_i^N = 0, \quad i = 1, 2, 4, 5.$$

From the above equations we get the solutions

$$A_i^N = 0, \quad i = 1, 2, 3, 4, 5.$$

$$B_1^N = \alpha,$$

$$B_2^N = -\alpha - \beta,$$

$$B_3^N = \beta,$$

$$B_4^N = -\beta - \gamma,$$

$$B_5^N = \gamma.$$

Where $\alpha, \beta, \gamma \in \mathbb{R}$ are parameters. Hence the problem has the zero eigenvalue with the following eigenvectors

$$\begin{aligned} h_1^N &= \alpha, \\ h_2^N &= -\alpha - \beta, \\ h_3^N &= \beta, \\ h_4^N &= -\beta - \gamma, \\ h_5^N &= \gamma. \end{aligned}$$

The zero eigenvalue has geometric multiplicity 3. This means that for $\lambda = 0$ we have

$$\begin{pmatrix} h_1^N \\ h_2^N \\ h_3^N \\ h_4^N \\ h_5^N \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \alpha + \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \beta + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \gamma.$$

The eigenspace of the zero eigenvalue is

$$< \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} > .$$

Note that the eigenfunctions and the multiplicity of the zero eigenvalue show the three different ways that the network rotates. The proof is completed.

Conclusions

In this article, we introduced the evolution problem of a network of two triple junctions in a bounded and smooth domain, with fixed angle conditions at the points at which they intersect and the normal velocity proportional to the curvature. We studied the stability in terms of the geometry of the boundary and provided conditions under which the steady state of this type of networks is stable, unstable and neutral stable.

A further extension of this article is to identify these kind of networks for 3D domains. In this case instead of network of curves we have a network of surfaces and the steady states are a network of planes (flat, two-dimensional surfaces). For all these there is already some research in progress.

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