A characterization of annular domains by quadrature identities

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Abstract

This note verifies a conjecture of Armitage and Goldstein, that annular domains may be characterized as quadrature domains for harmonic functions with respect to a uniformly distributed measure on a sphere.

1 Introduction

Let $B$ be an open ball of centre $0$ in $\mathbb{R}^N$ ($N \geq 2$) and $m$ denote volume measure on $\mathbb{R}^N$. Then

$$\frac{1}{m(B)} \int_B u \, dm = u(0)$$

for any integrable harmonic function $u$ on $B$. Further, the following theorem of Kuran [8] shows that this property actually characterizes balls. (Its original formulation required $\Omega$ to be connected, but this hypothesis is redundant.) It belongs to a long tradition of results that are surveyed in Netuka and Veselý [10].

**Theorem A** Let $\Omega$ be an open set such that $m(\Omega) < \infty$ and $0 \in \Omega$. If

$$\frac{1}{m(\Omega)} \int_{\Omega} u \, dm = u(0) \quad \text{for any integrable harmonic function } u \text{ on } \Omega,$$

then $\Omega$ is a ball of centre $0$.

If $r > 0$, then let $S(r) = \partial B(r)$, where $B(r) = \{ x \in \mathbb{R}^N : \|x\| < r \}$, and $M(u, r)$ denote the mean value of an integrable function $u$ over $S(r)$ with respect to surface area measure. For annular domains of the form

$$A(r_1, r_2) = \{ x \in \mathbb{R}^N : r_1 < \|x\| < r_2 \} \quad (0 \leq r_1 < r_2),$$

it is known (see, for example, Corollary 2.1 in [3]) that

$$\frac{1}{m(A(r_1, r_2))} \int_{A(r_1, r_2)} u \, dm = M(u, r)$$

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for any integrable harmonic function $u$ on $A(r_1, r_2)$, where

$$
 r = \begin{cases} 
  \left( \frac{2 r_2^N - r_1^N}{N r_2^2 - r_1^2} \right)^{1/(N-2)} & (N \geq 3) \\
  \exp \left( \frac{r_2^2 \log r_2 - r_1^2 \log r_1}{r_2^2 - r_1^2} - \frac{1}{2} \right) & (N = 2)
\end{cases}
$$

(2)

(The necessity of (2) is clear from consideration of the function $x \mapsto \|x\|^{2-N}$ when $N \geq 3$, or $x \mapsto \log \|x\|$ when $N = 2$. Further, the strict convexity of $t \mapsto t^{N/2}$ if $N \geq 3$, or of $t \mapsto t \log t$ if $N = 2$, ensures that $r_1 < r < r_2$.)

Sakai [11] used an argument based on holomorphic functions to show that the above quadrature identity characterizes annuli among multiply connected planar domains that contain $S(r)$. In higher dimensions Armitage and Goldstein [3] subsequently showed that a similar quadrature identity for an open set implies that $\Omega$ is of the form $A(r_1, r_2)$, where (2) holds. They asked, in Problem 3.35 of [5] (see also [9]), whether annular domains themselves could be characterized in this way, having pointed out errors in an earlier paper of Avci [4] on this problem (see pp.142,145 of [3]). We answer their question affirmatively below.

**Theorem 1** Let $\Omega$ be an open set such that $m(\Omega) < \infty$ and $S(r) \subset \Omega$. If

$$
 \frac{1}{m(\Omega)} \int_{\Omega} u \, dm = M(u, r) \text{ for any integrable harmonic function } u \text{ on } \Omega,
$$

(3)

then either

(i) $\Omega$ is of the form $A(r_1, r_2)$, where $0 \leq r_1 < r_2$ and (2) holds, or

(ii) $\Omega$ is an open ball centred at 0.

We define $h_y(x) = \psi_N(\|x - y\|)$, where $\psi_N(t) = t^{2-N}$ when $N \geq 3$ and $\psi_2(t) = -\log t$.

**Theorem 2** Let $\Omega$ be an open set such that $m(\Omega) < \infty$ (or $\Omega$ is bounded, if $N = 2$) and $S(r) \subset \Omega$. If

$$
 \frac{1}{m(\Omega)} \int_{\Omega} h_y \, dm = M(h_y, r) \quad (y \in \mathbb{R}^N \setminus \Omega),
$$

(4)

then either

(i) $\Omega$ is of the form $A(r_1, r_2)$, where $0 \leq r_1 < r_2$ and (2) holds, or

(ii) $\Omega$ is of the form $B \setminus T$, where $B$ is a ball centred at 0 and $T \subset S(r_0)$ for some $r_0 \in (0, r)$. (The set $T$ may be empty.)
In connection with part (ii) of the above result we note that the identity (4) holds for \( \Omega = B(r_2) \setminus S(r_0) \), where

\[
  r_0 = \begin{cases} 
    r_2 \sqrt{\frac{N/2 - (r_2/r)^{N-2}}{N/2 - 1}} & (N \geq 3) \\
    r_2 \sqrt{2 \log(r/r_2) + 1} & (N = 2)
  \end{cases}
\]

provided that \( r_0 \) exists and \( 0 < r_0 < r \).

\section{An intermediate result}

Let \( E \subset \mathbb{R}^N \) be Lebesgue measurable, where \( 0 < m(E) < \infty \), and \( U_E(y) = \int_E h_y \, dm \). Since this potential may not be finite when \( N = 2 \), we define \( \tilde{U}_E(y) = \int_E (h_y - h_z) \, dm \), where \( z \) is some fixed point of \( \mathbb{R}^N \setminus E \). We also define

\[ \tilde{E} = E \cup \{ x \in \mathbb{R}^N : m(B_x \setminus E) = 0 \text{ for some } B_x \text{ centred at } x \}, \]

whence \( E \subset \tilde{E} \) and \( m(\tilde{E} \setminus E) = 0 \).

The main result of Hansen and Netuka [7] is the following analogue of Theorem A. (Its converse is immediate from (1).) We give below a short alternative proof of it and then establish an analogue for annular regions.

\textbf{Theorem 3} Let \( B \) be the open ball of centre 0 such that \( m(B) = m(E) \). If

\[
  \frac{1}{m(E)} \int_E (U_C - U_D) \, dm = (U_C - U_D) (0)
\]

whenever \( C \) and \( D \) are compact subsets of \( \mathbb{R}^N \setminus E \) and \( U_C - U_D \) is bounded, then \( m(B \setminus E) = 0 \).

\textbf{Proof.} We first consider the case where \( N \geq 3 \). Let \( y \) be a Lebesgue point of \( \mathbb{R}^N \setminus (E \cup \{ 0 \}) \). We choose a sequence \( (C_n) \) of (non-negligible) compact sets satisfying

\[
  C_n \subset \{ x \in \mathbb{R}^N \setminus E : \| x - y \| < n^{-1} \} \quad \text{and} \quad \frac{m(B(n^{-1}))}{m(C_n)} \to 1 \quad (n \to \infty).
\]

We note that \( U_{C_n} \leq m(B(n^{-1}))h_y \), and

\[
  \int_E \frac{U_{C_n}}{m(C_n)} \, dm = m(E) \frac{U_{C_n}(0)}{m(C_n)},
\]

by (5). Since \( m(E) < \infty \) we can use dominated convergence to conclude that \( U_E(y) = m(E)h_0(y) \). Hence \( U_E = m(E)h_0 \) almost everywhere outside \( E \).
Since \( U_B \leq m(B)h_0 = m(E)h_0 \) on \( \mathbb{R}^N \), it follows by continuity that 
\( U_B \leq U_E \) outside \((E)^o\). This inequality extends to \( \mathbb{R}^N \), by the minimum principle applied to \( U_E - U_B \) on \((E)^o\). (Although we have not assumed that \( E \) is bounded, we know that \( U_E - U_B \geq -U_B \to 0 \) at infinity.) Since the non-negative function \( U_E - U_B \) is superharmonic on \( \mathbb{R}^N \backslash \overline{B} \) and attains the value 0 there, it follows from the minimum principle that \( U_E = U_B \) on \( \mathbb{R}^N \backslash \overline{B} \). Hence \( m(E \backslash \overline{B}) = 0 \), and so \( m(B \backslash E) = 0 \), as required. (We note, in passing, that the argument in this paragraph provides a short proof of the main result of [1].)

When \( N = 2 \) we choose a further Lebesgue point \( z \) of \( \mathbb{R}^2 \backslash ((E \cup \{0\}) \) and then two sequences \((C_n), (D_n)\) of compact sets satisfying (6) and

\[
D_n \subset \{ x \in \mathbb{R}^N \setminus E : \|x - z\| < n^{-1} \} \quad \text{and} \quad m(D_n) = m(C_n).
\]

Since \( \|U_{C_n}/m(C_n) - h_0\| \leq \log 2 \) outside \( B(2 + 2 \|y\|) \), we see that

\[
|U_{C_n} - U_{D_n}|/m(C_n) \leq 2 \log 2 \text{ outside } B(R), \quad \text{where } R = 2+2 \max\{\|y\|, \|z\|\}.
\]

On \( B(R) \) we have

\[
|U_{C_n}(x)| \leq \int \frac{2R}{\|x - t\|} dm(t) + m(B(n^{-1})) \log(2R)
\]

\[
\leq m(B(n^{-1}))(\|h_y(x)\| + 2 \log(2R)),
\]

and so

\[
|U_{C_n} - U_{D_n}| \leq m(B(n^{-1}))(\|h_y\| + \|h_z\| + 4 \log(2R)).
\]

We can now use (5) and dominated convergence as before to see that \( U_E^z = m(E)(h_0 - h_0(z)) \) almost everywhere outside \( E \).

Let \( u = U_E^z - U_B + m(E)h_0(z) \) and

\[
u_n = \frac{m(E)}{m(E \cap B(n))} U_{E \cap B(n)} - U_B + m(E)h_0(z) \quad (n \in \mathbb{N}).\]

Then \( u_n \to u \) locally uniformly on \( \mathbb{R}^2 \). In particular, there exists \( c > 0 \) such that \( |u_n| \leq c \) on \( S(1) \) for all \( n \). Each function \( u_n \) is superharmonic outside \( \overline{B} \) and tends to 0 at infinity, so \( u_n \geq -c \) on \( \mathbb{R}^2 \backslash \overline{B} \) by the minimum principle. Hence \( u \geq -c \) on \( \mathbb{R}^2 \backslash \overline{B} \). Since \( u \geq 0 \) outside \((E)^\circ\) and \( \{\infty\} \) is polar, we can argue as before to see that \( u \geq 0 \) on \( \mathbb{R}^2 \), and then that \( m(B \backslash E) = 0 \). \qed

It is clear from the above proof that, in Theorem 3, we might as well replace (5) by the requirement that \( U_E = m(E)h_0 \) almost everywhere outside \( E \) when \( N \geq 3 \), or that \( U_E^z = m(E)(h_0 - h_0(z)) \) almost everywhere outside \( E \) when \( N = 2 \), where \( z \) is a Lebesgue point of \( \mathbb{R}^2 \backslash ((E \cup \{0\}) \). The analogous result for annular domains is given below. Its proof combines an argument
We define

\[ M(h_y, \rho) = \min\{\psi_N(\rho), \psi_N(\|y\|)\} \quad (\rho > 0) \quad (7) \]

(see Example 4.2.9 in [2]).

**Theorem 4** Let \( r > 0 \). If

\[
\begin{align*}
U_E(x) &= m(E)M(h_x, r) \quad \text{a.e. outside } E \quad (N \geq 3) \\
U_E^+(x) &= m(E)M(h_x - h_z, r) \quad \text{a.e. outside } E \quad (N = 2) 
\end{align*}
\]

where \( z \) is a Lebesgue point of \( \mathbb{R}^2 \setminus E \), then either

(i) there exist \( r_1, r_2 \) satisfying \( 0 < r_1 < r_2 \) and (2), such that \( m(A(r_1, r_2)) = m(E) \) and \( m(A(r_1, r_2) \setminus E) = 0 \), or

(ii) there exists \( r_2 \geq r \) such that \( m(B(r_2)) = m(E) \) and \( m(B(r_2) \setminus E) = 0 \).

**Proof.** Radial solutions \( g(\|x\|) \) of Laplace’s equation satisfy \( \Delta g = 0 \), where

\[
\Delta g = \frac{\partial^2}{\partial \rho^2} + \frac{N-1}{\rho} \frac{\partial}{\partial \rho}.
\]

We define

\[
g_S(\rho) = m(E) \min\{\psi_N(\rho), \psi_N(\rho)\} \quad (\rho > 0),
\]

and choose \( c_N > 0 \) such that \( -\Delta U_{B(1)} = 2Nc_N \) on \( B(1) \). Next, let \( f_A : (0, \infty) \to \mathbb{R} \) denote the largest convex function of \( \psi_N(\rho) \) satisfying \( f_A(\rho) \leq g_S(\rho) + c_N \rho^2 \), and define \( g_A(\rho) = f_A(\rho) - c_N \rho^2 \). Clearly \( g_A \leq g_S \). To see that the set \( \{ \rho > 0 : g_A(\rho) < g_S(\rho) \} \) is bounded, let

\[
\sigma = \chi \left\{ \frac{m(E) \max\{N-2, 1\}}{2\epsilon} \right\},
\]

where \( \epsilon \in (0, c_N) \) is chosen small enough to ensure that

\[
\sigma > r \quad \text{and} \quad m(E)\psi_N(\sigma) + \epsilon \sigma^2 < m(E)\psi_N(r).
\]

Then the function defined by

\[
g(\rho) = \begin{cases} 
m(E)\psi_N(\sigma) + \epsilon(\sigma^2 - \rho^2) & (0 < \rho \leq \sigma) \\
m(E)\psi_N(\rho) & (\rho > \sigma)
\end{cases}
\]

is \( C^1 \), satisfies \( \Delta g(\rho) + c_N \rho^2 \geq 0 \) when \( \rho \neq \sigma \), and \( g \leq g_S \). Since \( g(\rho) = g_S(\rho) \) when \( \rho \geq \sigma \), we see that \( \{ \rho > 0 : g_A(\rho) < g_S(\rho) \} \) is bounded, as claimed. Further, this set must be of the form \( (r_1, r_2) \), where \( 0 \leq r_1 < r < r_2 \), since if \( g_A(t) = g_S(t) \) for some \( t > r \) (respectively, \( t < r \)), then maximality and the fact that \( \Delta g = 0 \) when \( \rho \neq r \) ensures that \( g_A = g_S \) on \( (t, \infty) \)}
The functions defined by \( u_A(x) = g_A(\|x\|) \) and \( u_S(x) = g_S(\|x\|) \), extended to the origin by continuity, are Newtonian (or logarithmic, if \( N = 2 \)) potentials. More precisely, \( u_A = U_{A(r_1, r_2)} \), and \( u_S \) is the potential of the uniformly distributed measure on \( S(r) \) of total mass \( m(E) \) since

\[
u_S(y) = m(E)M(h_y, r)
\]

by (7) and (9). These potentials satisfy \( u_A \leq u_S \) everywhere, and \( u_A < u_S \) on \( A(r_1, r_2) \). Further, \( m(A(r_1, r_2)) = m(E) \), since \( u_A(x) = m(E)\psi_N(\|x\|) \) on \( \mathbb{R}^N \setminus B(r_2) \).

If \( N \geq 3 \), then \( U_E = u_S \geq u_A \) almost everywhere on \( \mathbb{R}^N \setminus E \) by (8) and (10). Hence \( U_E \geq u_A \) outside \( \bar{E}^{\circ} \), and so this inequality holds everywhere, by the minimum principle applied to \( U_E - u_A \) on \( \bar{E}^{\circ} \). Since the non-negative function \( U_E - u_A \), which is superharmonic on \( \mathbb{R}^N \setminus B(r_2) \), attains the value 0 there, it follows from the minimum principle that \( U_E = u_A \) on \( \mathbb{R}^N \setminus B(r_2) \), and so \( m(E \setminus B(r_2)) = 0 \).

If \( N = 2 \), then we instead argue as in the final paragraph of the proof of Theorem 3 (with \( u = U_E - u_A + m(E)M(h_z, r) \) and \( B(r_2) \) in place of \( B \)) to see that \( U_E^2 \geq u_A + m(E)M(h_z, r) \) on \( \mathbb{R}^2 \) and again \( m(E \setminus B(r_2)) = 0 \). It follows that \( U_E \) is finite, so \( U_E^2 = U_E - U_E(z) \), and hence

\[
U_E(x) - U_E(z) = m(E)(M(h_x, r) - M(h_z, r)) \quad \text{a.e. outside } E,
\]

by (8). Letting \( \|x\| \to \infty \), we see that \( U_E(z) = m(E)M(h_z, r) \), and so

\[
U_E(x) = m(E)M(h_x, r) \quad \text{a.e. outside } E.
\]

If \( r_1 = 0 \), then \( m(E) = m(A(r_1, r_2)) = m(B(r_2)) \) and conclusion (ii) holds.

If \( r_1 > 0 \) and \( m(E \cap B(r_1)) = 0 \), then \( m(E \setminus A(r_1, r_2)) = 0 \) and so \( m(A(r_1, r_2) \setminus E) = 0 \). Further, \( u_A(0) = u_S(0) \), so

\[
\int_{A(r_1, r_2)} h_0 \, dm = m(E)\psi_N(r) = m(A(r_1, r_2))\psi_N(r),
\]

and a straightforward calculation establishes (2). Thus conclusion (i) holds.

It remains to consider the case where \( r_1 > 0 \), whence \( u_A = u_S \) on \( B(r_1) \) and (2) holds, and where \( m(E \cap B(r_1)) > 0 \). If \( m(B(r_1) \setminus E) > 0 \), then \( U_E - u_A \) would attain its minimum value in \( B(r_1) \), contradicting the minimum principle. Hence \( B(r_1) \subset \bar{E} \). We suppose, for the sake of contradiction, that \( B(r) \setminus \bar{E} \neq \emptyset \), and choose a point \( x_0 \) in the closure of \( B(r) \setminus \bar{E} \) at minimum distance from the origin. Let \( r_0 = \|x_0\| \) and

\[
u_0(x) = \frac{\|x\|^2 - r_0^2}{\|x - x_0\|^N} \quad (x \in \mathbb{R}^N \setminus \{x_0\}).
\]
Then \( r_1 \leq r_0 < r \), \( u_0 < 0 \) on \( B(r_0) \) and \( u_0 > 0 \) on \( \mathbb{R}^N \backslash \overline{B(r_0)} \). Further, \( M(u_0, \rho) = \rho^{2-N} \) when \( \rho > r_0 \), since \( M(u_0, \rho) \) is a linear function of \( \rho^{2-N} \) on \((r_0, \infty)\) (see Theorem 3.5.6(i) of [2]) and \( \|x\|^{N-2} u_0(x) \to 1 \) as \( \|x\| \to \infty \). Hence
\[
\int_E u_0 \ dm < \int_{E \setminus B(r_0)} u_0 \ dm \leq \int_{A(r_0, r_2)} u_0 \ dm \\
= \int_{A(r_0, r_2)} \|x\|^{2-N} \ dm \leq \int_{A(r_1, r_2)} \|x\|^{2-N} \ dm \\
= m(A(r_1, r_2)) \|x\|^{2-N} = m(E) M(u_0, r), \tag{12}
\]
where the penultimate equality follows from (11) when \( N \geq 3 \), and is trivial when \( N = 2 \). However, \( U_E \) is \( C^1 \) and the function \( y \mapsto M(h_y, r) \) is constant on \( B(r) \). Thus, if \( y \in B(r) \) is in the closure of \( \mathbb{R}^N \backslash \tilde{E} \), we see from (8) that \( \int_E h_y \ dm = m(E) M(h_y, r) \) and
\[
\int_E \frac{\partial h_y}{\partial y_i} \ dm = m(E) M \left( \frac{\partial h_y}{\partial y_i}, r \right) \ (i = 1, ..., N)
\]
(this follows from Theorem 4.5.3 of [2], since \( m(E \setminus B(r_2)) = 0 \)). Since
\[
u_0(x) = \|x - x_0\|^{2-N} + \frac{2}{\max\{N-2, 1\}} (x_0, \nabla_{x_0} h_{x_0}(x)),
\]
it follows that \( \int_E u_0 \ dm = m(E) M(u_0, r) \), contradicting (12). Hence \( B(r) \subset \tilde{E} \), and thus
\[
\frac{1}{m(E)} \int_E h_x \ dm = M(h_x, r) = h_x(0) \ (x \in \mathbb{R}^N \setminus \tilde{E}).
\]
We now see from Theorem 3 that conclusion (ii) holds. 

\section{3 Deduction of Theorems 1 and 2}

\textbf{Lemma 5} Let \( \Omega \) be an open set such that \( S(r) \subset \Omega \), where \( r > 0 \).
(i) If \( \tilde{\Omega} = A(r_1, r_2) \) and (4) holds, then \( \tilde{\Omega} = A(r_1, r_2) \).
(ii) If \( \tilde{\Omega} = B(r_2) \) and (4) holds, then either \( \tilde{\Omega} = A(0, r_2) \), or \( \tilde{\Omega} = B(r_2) \setminus T \) where \( T \subset S(r_0) \) for some \( r_0 \in (0, r) \).

\textbf{Proof.} Let \( v(y) = m(\Omega) M(h_y, r) - U_\Omega(y) \), whence \( v \in C^1(\mathbb{R}^N \setminus S(r)) \) and \( v = 0 \) on \( \mathbb{R}^N \setminus \Omega \), by (4).
(i) If \( \tilde{\Omega} = A(r_1, r_2) \), then \( r_1 > 0 \). We claim that \( v \neq 0 \) on \( \tilde{\Omega} \setminus S(r) \). To see this, suppose first that \( v(y_0) = 0 \) where \( y_0 \in A(r, r_2) \). Then \( v = 0 \) on \( \partial A(\|y_0\|, r_2) \) by rotational symmetry, and \( \Delta v = 2 N C_N > 0 \) on \( A(\|y_0\|, r_2) \). Hence \( v < 0 \) on \( A(\|y_0\|, r_2) \) by the maximum principle, and we arrive at
the contradictory conclusion that $||\nabla v|| > 0$ on $S(r_2)$. A similar argument applies if $y_0 \in A(r_1, r)$. Hence $\Omega = A(r_1, r_2)$.

(ii) If $\Omega = B(r_2)$, we again see that $v \neq 0$ on $A(r, r_2)$. If there exists $x_0 \in A(0, r)$ such that $v(x_0) = 0$, then $v = 0$ on $S(r_0)$, where $r_0 = ||x_0||$, and so $v < 0$ on $B(r_0)$. It follows that $\Omega \setminus \Omega \subset S(r_0)$. The remaining possibility is that $v \neq 0$ on $A(0, r)$, whence either $\Omega = A(0, r_2)$ or $\Omega = B(r_2)$. ■

**Proof of Theorem 2.** The hypotheses of Theorem 4 are satisfied, with $E = \Omega$, so $\Omega$ is either of the form $A(r_1, r_2)$, where $r_1 > 0$, or $B(r_2)$. If $0 \notin \Omega$, then it follows from Lemma 5 that $\Omega$ is of the form $A(r_1, r_2)$, where $0 \leq r_1 < r_2$, and from (11) that (2) holds. Otherwise, the lemma shows that $\Omega = B(r_2) \setminus T$ where $T \subset S(r_0)$ for some $r_0 \in (0, r)$, as required. ■

**Proof of Theorem 1.** In view of Theorem 2, it remains to consider the case where $\Omega = B(r_2) \setminus T$ and $T \subset S(r_0)$ for some $r_0 \in (0, r)$. If there exists $x_0 \in T$, then we can adapt (12) to see that

$$\int_{\Omega} u_0 \, dm < \int_{A(r_0, r_2)} u_0 \, dm = \int_{A(r_0, r_2)} ||x||^{2-N} \, dm(x) = \int_{A(r_0, r_2)} ||x - x_0||^{2-N} \, dm(x) < \int_{\Omega} ||x - x_0||^{2-N} \, dm(x) = m(\Omega)r^{2-N} = m(\Omega)M(u_0, r),$$

where the penultimate equality follows by applying (3) to the function $h_{x_0}$ when $N \geq 3$ and is trivial when $N = 2$. This contradicts (3). Hence $T = \emptyset$ and so $\Omega = B(r_2)$. ■

**References**


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