A characterization of annular domains by quadrature identities

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Abstract

This note verifies a conjecture of Armitage and Goldstein, that annular domains may be characterized as quadrature domains for harmonic functions with respect to a uniformly distributed measure on a sphere.

1 Introduction

Let B be an open ball of centre 0 in $\mathbb{R}^N$ $(N \geq 2)$ and m denote volume measure on $\mathbb{R}^N$. Then

$$\frac{1}{m(B)} \int_B u \, dm = u(0) \tag{1}$$

for any integrable harmonic function u on B. Further, the following theorem of Kuran [8] shows that this property actually characterizes balls. (Its original formulation required Ω to be connected, but this hypothesis is redundant.) It belongs to a long tradition of results that are surveyed in Netuka amd Veselý [10].

Theorem A *Let Ω be an open set such that $m(\Omega) < \infty$ and $0 \in \Omega$. If*

$$\frac{1}{m(\Omega)} \int_\Omega u \, dm = u(0) \quad \textit{for any integrable harmonic function } u \textit{ on } \Omega,$$

then Ω is a ball of centre 0.

If $r > 0$, then let $S(r) = \partial B(r)$, where $B(r) = \{x \in \mathbb{R}^N : \|x\| < r\}$, and $M(u, r)$ denote the mean value of an integrable function u over $S(r)$ with respect to surface area measure. For annular domains of the form

$$A(r_1, r_2) = \{x \in \mathbb{R}^N : r_1 < \|x\| < r_2\} \quad (0 \leq r_1 < r_2),$$

it is known (see, for example, Corollary 2.1 in [3]) that

$$\frac{1}{m(A(r_1, r_2))} \int_{A(r_1,r_2)} u \, dm = M(u, r)$$

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for any integrable harmonic function u on $A(r_1, r_2)$, where

$$r = \begin{cases} \left(\frac{2}{N} \frac{r_2^N - r_1^N}{r_2^2 - r_1^2} \right)^{1/(N-2)} & (N \geq 3) \\ \exp\left(\frac{r_2^2 \log r_2 - r_1^2 \log r_1}{r_2^2 - r_1^2} - \frac{1}{2} \right) & (N = 2) \end{cases} . \tag{2}$$

(The necessity of (2) is clear from consideration of the function $x \mapsto \|x\|^{2-N}$ when $N \geq 3$, or $x \mapsto \log \|x\|$ when $N = 2$. Further, the strict convexity of $t \mapsto t^{N/2}$ if $N \geq 3$, or of $t \longmapsto t \log t$ if $N = 2$, ensures that $r_1 < r < r_2$.)

Sakai [11] used an argument based on holomorphic functions to show that the above quadrature identity characterizes annuli among multiply connected planar domains that contain $S(r)$. In higher dimensions Armitage and Goldstein [3] subsequently showed that a similar quadrature identity for an open set Ω implies that $\overline{\Omega}$ is of the form $\overline{A(r_1, r_2)}$, where (2) holds. They asked, in Problem 3.35 of [5] (see also [9]), whether annular domains themselves could be characterized in this way, having pointed out errors in an earlier paper of Avci [4] on this problem (see pp.142,145 of [3]). We answer their question affirmatively below.

Theorem 1 *Let Ω be an open set such that $m(\Omega) < \infty$ and $S(r) \subset \Omega$. If*

$$\frac{1}{m(\Omega)} \int_\Omega u \, dm = M(u, r) \text{ for any integrable harmonic function } u \text{ on } \Omega, \tag{3}$$

then either
(i) Ω is of the form $A(r_1, r_2)$, where $0 \leq r_1 < r_2$ and (2) holds, or
(ii) Ω is an open ball centred at 0.

We define $h_y(x) = \psi_N(\|x - y\|)$, where $\psi_N(t) = t^{2-N}$ when $N \geq 3$ and $\psi_2(t) = -\log t$.

Theorem 2 *Let Ω be an open set such that $m(\Omega) < \infty$ (or Ω is bounded, if $N = 2$) and $S(r) \subset \Omega$. If*

$$\frac{1}{m(\Omega)} \int_\Omega h_y \, dm = M(h_y, r) \qquad (y \in \mathbb{R}^N \backslash \Omega), \tag{4}$$

then either
(i) Ω is of the form $A(r_1, r_2)$, where $0 \leq r_1 < r_2$ and (2) holds, or
(ii) Ω is of the form $B \backslash T$, where B is a ball centred at 0 and $T \subset S(r_0)$ for some $r_0 \in (0, r)$. (The set T may be empty.)

In connection with part (ii) of the above result we note that the identity (4) holds for $\Omega = B(r_2)\backslash S(r_0)$, where

$$r_0 = \begin{cases} r_2\sqrt{\dfrac{N/2-(r_2/r)^{N-2}}{N/2-1}} & (N \geq 3) \\ r_2\sqrt{2\log(r/r_2)+1} & (N=2) \end{cases},$$

provided that r_0 exists and $0 < r_0 < r$.

2 An intermediate result

Let $E \subset \mathbb{R}^N$ be Lebesgue measurable, where $0 < m(E) < \infty$, and $U_E(y) = \int_E h_y\ dm$. Since this potential may not be finite when $N = 2$, we define $U_E^z(y) = \int_E (h_y - h_z)\ dm$, where z is some fixed point of $\mathbb{R}^N\backslash E$. We also define

$$\widetilde{E} = E \cup \{x \in \mathbb{R}^N : m(B_x\backslash E) = 0 \text{ for some ball } B_x \text{ centred at } x\},$$

whence $E \subset \widetilde{E}$ and $m(\widetilde{E}\backslash E) = 0$,

The main result of Hansen and Netuka [7] is the following analogue of Theorem A. (Its converse is immediate from (1).) We give below a short alternative proof of it and then establish an analogue for annular regions.

Theorem 3 *Let B be the open ball of centre 0 such that $m(B) = m(E)$. If*

$$\frac{1}{m(E)}\int_E (U_C - U_D)\ \ dm = (U_C - U_D)(0) \tag{5}$$

whenever C and D are compact subsets of $\mathbb{R}^N\backslash E$ and $U_C - U_D$ is bounded, then $m(B\backslash E) = 0$.

Proof. We first consider the case where $N \geq 3$. Let y be a Lebesgue point of $\mathbb{R}^N\backslash(E \cup \{0\})$. We choose a sequence (C_n) of (non-negligible) compact sets satisfying

$$C_n \subset \{x \in \mathbb{R}^N\backslash E : \|x-y\| < n^{-1}\} \text{ and } \frac{m(B(n^{-1}))}{m(C_n)} \to 1 \quad (n \to \infty). \tag{6}$$

We note that $U_{C_n} \leq m(B(n^{-1}))h_y$, and

$$\int_E \frac{U_{C_n}}{m(C_n)}\ dm = m(E)\frac{U_{C_n}(0)}{m(C_n)},$$

by (5). Since $m(E) < \infty$ we can use dominated convergence to conclude that $U_E(y) = m(E)h_0(y)$. Hence $U_E = m(E)h_0$ almost everywhere outside E.

Since $U_B \leq m(B)h_0 = m(E)h_0$ on $\mathbb{R}^N$, it follows by continuity that $U_B \leq U_E$ outside $(\widetilde{E})^\circ$. This inequality extends to $\mathbb{R}^N$, by the minimum principle applied to $U_E - U_B$ on $(\widetilde{E})^\circ$. (Although we have not assumed that E is bounded, we know that $U_E - U_B \geq -U_B \to 0$ at infinity.) Since the non-negative function $U_E - U_B$ is superharmonic on $\mathbb{R}^N \backslash \overline{B}$ and attains the value 0 there, it follows from the minimum principle that $U_E = U_B$ on $\mathbb{R}^N \backslash \overline{B}$. Hence $m(E \backslash \overline{B}) = 0$, and so $m(B \backslash E) = 0$, as required. (We note, in passing, that the argument in this paragraph provides a short proof of the main result of [1].)

When $N = 2$ we choose a further Lebesgue point z of $\mathbb{R}^2 \backslash (E \cup \{0\})$ and then two sequences $(C_n), (D_n)$ of compact sets satisfying (6) and

$$D_n \subset \{x \in \mathbb{R}^N \backslash E : \|x - z\| < n^{-1}\} \quad \text{and} \quad m(D_n) = m(C_n).$$

Since $|U_{C_n}/m(C_n) - h_0| \leq \log 2$ outside $B(2 + 2\|y\|)$, we see that

$$|U_{C_n} - U_{D_n}| / m(C_n) \leq 2\log 2 \text{ outside } B(R), \text{ where } R = 2+2\max\{\|y\|, \|z\|\}.$$

On $B(R)$ we have

$$\begin{aligned} |U_{C_n}(x)| &\leq \int_{\{\|t-y\|<n^{-1}\}} \log \frac{2R}{\|x-t\|} dm(t) + m(B(n^{-1})) \log(2R) \\ &\leq m(B(n^{-1})) \left(|h_y(x)| + 2\log(2R)\right), \end{aligned}$$

and so

$$|U_{C_n} - U_{D_n}| \leq m(B(n^{-1})) \left(|h_y| + |h_z| + 4\log(2R)\right).$$

We can now use (5) and dominated convergence as before to see that $U_E^z = m(E)(h_0 - h_0(z))$ almost everywhere outside E.

Let $u = U_E^z - U_B + m(E)h_0(z)$ and

$$u_n = \frac{m(E)}{m(E \cap B(n))} U_{E \cap B(n)}^z - U_B + m(E)h_0(z) \quad (n \in \mathbb{N}).$$

Then $u_n \to u$ locally uniformly on $\mathbb{R}^2$. In particular, there exists $c > 0$ such that $|u_n| \leq c$ on $S(1)$ for all n. Each function u_n is superharmonic outside $\overline{B}$ and tends to 0 at infinity, so $u_n \geq -c$ on $\mathbb{R}^2 \backslash \overline{B}$ by the minimum principle. Hence $u \geq -c$ on $\mathbb{R}^2 \backslash \overline{B}$. Since $u \geq 0$ outside $(\widetilde{E})^\circ$ and $\{\infty\}$ is polar, we can argue as before to see that $u \geq 0$ on $\mathbb{R}^2$, and then that $m(B \backslash E) = 0$. ■

It is clear from the above proof that, in Theorem 3, we might as well replace (5) by the requirement that $U_E = m(E)h_0$ almost everywhere outside E when $N \geq 3$, or that $U_E^z = m(E)(h_0 - h_0(z))$ almost everywhere outside E when $N = 2$, where z is a Lebesgue point of $\mathbb{R}^2 \backslash (E \cup \{0\})$. The analogous result for annular domains is given below. Its proof combines an argument of Kuran [8] with ideas from the technique of partial balayage as expounded in [6]. We note, for future reference, that

$$M(h_y,\rho)=\min\{\psi_N(\rho),\psi_N(\|y\|)\}\qquad(\rho>0)\tag{7}$$

(see Example 4.2.9 in [2]).

Theorem 4 *Let $r>0$. If*

$$\begin{array}{ll} U_E(x)=m(E)M(h_x,r)\ \ \text{a.e. outside } E & (N\geq 3)\\ U_E^z(x)=m(E)M(h_x-h_z,r)\ \ \text{a.e. outside } E & (N=2)\end{array},\tag{8}$$

where z is a Lebesgue point of $\mathbb{R}^2\backslash E$, then either
(i) there exist r_1,r_2 satisfying $0<r_1<r_2$ and (2), such that $m(A(r_1,r_2))=m(E)$ and $m(A(r_1,r_2)\backslash E)=0$, or
(ii) there exists $r_2\geq r$ such that $m(B(r_2))=m(E)$ and $m(B(r_2)\backslash E)=0$.

Proof. Radial solutions $g(\|x\|)$ of Laplace's equation satisfy $\Delta_\rho g=0$, where

$$\Delta_\rho=\frac{d^2}{d\rho^2}+\frac{N-1}{\rho}\frac{d}{d\rho}.$$

We define

$$g_S(\rho)=m(E)\min\{\psi_N(r),\psi_N(\rho)\}\qquad(\rho>0),\tag{9}$$

and choose $c_N>0$ such that $-\Delta U_{B(1)}=2Nc_N$ on $B(1)$. Next, let $f_A:(0,\infty)\to\mathbb{R}$ denote the largest convex function of $\psi_N(\rho)$ satisfying $f_A(\rho)\leq g_S(\rho)+c_N\rho^2$, and define $g_A(\rho)=f_A(\rho)-c_N\rho^2$. Clearly $g_A\leq g_S$. To see that the set $\{\rho>0:g_A(\rho)<g_S(\rho)\}$ is bounded, let

$$\sigma=\sqrt[N]{\frac{m(E)\max\{N-2,1\}}{2\varepsilon}},$$

where $\varepsilon\in(0,c_N)$ is chosen small enough to ensure that

$$\sigma>r\ \text{ and }\ m(E)\psi_N(\sigma)+\varepsilon\sigma^2<m(E)\psi_N(r).$$

Then the function defined by

$$g(\rho)=\begin{cases} m(E)\psi_N(\sigma)+\varepsilon(\sigma^2-\rho^2) & (0<\rho\leq\sigma)\\ m(E)\psi_N(\rho) & (\rho>\sigma)\end{cases}$$

is C^1, satisfies $\Delta_\rho(g(\rho)+c_N\rho^2)\geq 0$ when $\rho\neq\sigma$, and $g\leq g_S$. Since $g(\rho)=g_S(\rho)$ when $\rho>\sigma$, we see that $\{\rho>0:g_A(\rho)<g_S(\rho)\}$ is bounded, as claimed. Further, this set must be of the form (r_1,r_2), where $0\leq r_1<r<r_2$, since if $g_A(t)=g_S(t)$ for some $t>r$ (respectively, $t<r$), then maximality and the fact that $\Delta_\rho g_S(\rho)=0$ when $\rho\neq r$ ensures that $g_A=g_S$ on (t,∞)

(respectively, on $(0,t)$). Maximality also ensures that $-\Delta_\rho g_A = 2Nc_N$ on (r_1, r_2).

The functions defined by $u_A(x) = g_A(\|x\|)$ and $u_S(x) = g_S(\|x\|)$, extended to the origin by continuity, are Newtonian (or logarithmic, if $N = 2$) potentials. More precisely, $u_A = U_{A(r_1,r_2)}$, and u_S is the potential of the uniformly distributed measure on $S(r)$ of total mass $m(E)$ since

$$u_S(y) = m(E)M(h_y, r) \tag{10}$$

by (7) and (9). These potentials satisfy $u_A \le u_S$ everywhere, and $u_A < u_S$ on $A(r_1, r_2)$. Further, $m(A(r_1, r_2)) = m(E)$, since $u_A(x) = m(E)\psi_N(\|x\|)$ on $\mathbb{R}^N \backslash B(r_2)$.

If $N \ge 3$, then $U_E = u_S \ge u_A$ almost everywhere on $\mathbb{R}^N \backslash E$ by (8) and (10). Hence $U_E \ge u_A$ outside $(\widetilde{E})^\circ$, and so this inequality holds everywhere, by the minimum principle applied to $U_E - u_A$ on $(\widetilde{E})^\circ$. Since the non-negative function $U_E - u_A$, which is superharmonic on $\mathbb{R}^N \backslash \overline{B(r_2)}$, attains the value 0 there, it follows from the minimum principle that $U_E = u_A$ on $\mathbb{R}^N \backslash \overline{B(r_2)}$, and so $m(E \backslash B(r_2)) = 0$.

If $N = 2$, then we instead argue as in the final paragraph of the proof of Theorem 3 (with $u = U_E^z - u_A + m(E)M(h_z, r)$ and $B(r_2)$ in place of B) to see that $U_E^z \ge u_A - m(E)M(h_z, r)$ on $\mathbb{R}^2$ and again $m(E \backslash B(r_2)) = 0$. It follows that U_E is finite, so $U_E^z = U_E - U_E(z)$, and hence

$$U_E(x) - U_E(z) = m(E)\left(M(h_x, r) - M(h_z, r)\right) \quad \text{a.e. outside } E,$$

by (8). Letting $\|x\| \to \infty$, we see that $U_E(z) = m(E)M(h_z, r)$, and so

$$U_E(x) = m(E)M(h_x, r) \quad \text{a.e. outside } E.$$

If $r_1 = 0$, then $m(E) = m(A(r_1, r_2)) = m(B(r_2))$ and conclusion (ii) holds.

If $r_1 > 0$ and $m(E \cap B(r_1)) = 0$, then $m(E \backslash A(r_1, r_2)) = 0$ and so $m(A(r_1, r_2) \backslash E) = 0$. Further, $u_A(0) = u_S(0)$, so

$$\int_{A(r_1,r_2)} h_0 \, dm = m(E)\psi_N(r) = m(A(r_1, r_2))\psi_N(r), \tag{11}$$

and a straightforward calculation establishes (2). Thus conclusion (i) holds.

It remains to consider the case where $r_1 > 0$, whence $u_A = u_S$ on $B(r_1)$ and (2) holds, and where $m(E \cap B(r_1)) > 0$. If $m(B(r_1) \backslash E) > 0$, then $U_E - u_A$ would attain its minimum value in $B(r_1)$, contradicting the minimum principle. Hence $B(r_1) \subset \widetilde{E}$. We suppose, for the sake of contradiction, that $B(r) \backslash \widetilde{E} \ne \emptyset$, and choose a point x_0 in the closure of $B(r) \backslash \widetilde{E}$ at minimum distance from the origin. Let $r_0 = \|x_0\|$ and

$$u_0(x) = \frac{\|x\|^2 - r_0^2}{\|x - x_0\|^N} \quad (x \in \mathbb{R}^N \backslash \{x_0\}).$$

Then $r_1 \leq r_0 < r$, $u_0 < 0$ on $B(r_0)$ and $u_0 > 0$ on $\mathbb{R}^N \backslash \overline{B(r_0)}$. Further, $M(u_0, \rho) = \rho^{2-N}$ when $\rho > r_0$, since $M(u_0, \rho)$ is a linear function of ρ^{2-N} on (r_0, ∞) (see Theorem 3.5.6(i) of [2]) and $\|x\|^{N-2} u_0(x) \to 1$ as $\|x\| \to \infty$. Hence

$$\begin{aligned} \int_E u_0 \, dm &< \int_{E \backslash B(r_0)} u_0 \, dm \leq \int_{A(r_0, r_2)} u_0 \, dm \\ &= \int_{A(r_0, r_2)} \|x\|^{2-N} \, dm \leq \int_{A(r_1, r_2)} \|x\|^{2-N} \, dm \\ &= m(A(r_1, r_2)) r^{2-N} = m(E) M(u_0, r), \end{aligned} \tag{12}$$

where the penultimate equality follows from (11) when $N \geq 3$, and is trivial when $N = 2$. However, U_E is C^1 and the function $y \mapsto M(h_y, r)$ is constant on $B(r)$. Thus, if $y \in B(r)$ is in the closure of $\mathbb{R}^N \backslash \widetilde{E}$, we see from (8) that $\int_E h_y \, dm = m(E) M(h_y, r)$ and

$$\int_E \frac{\partial h_y}{\partial y_i} \, dm = m(E) M \left(\frac{\partial h_y}{\partial y_i}, r \right) \qquad (i = 1, ..., N)$$

(this follows from Theorem 4.5.3 of [2], since $m(E \backslash B(r_2)) = 0$). Since

$$u_0(x) = \|x - x_0\|^{2-N} + \frac{2}{\max\{N-2, 1\}} \langle x_0, \nabla_{x_0} h_{x_0}(x) \rangle ,$$

it follows that $\int_E u_0 \, dm = m(E) M(u_0, r)$, contradicting (12). Hence $B(r) \subset \widetilde{E}$, and thus

$$\frac{1}{m(E)} \int_E h_x \, dm = M(h_x, r) = h_x(0) \qquad (x \in \mathbb{R}^N \backslash \widetilde{E}).$$

We now see from Theorem 3 that conclusion (ii) holds. ■

3 Deduction of Theorems 1 and 2

Lemma 5 *Let Ω be an open set such that $S(r) \subset \Omega$, where $r > 0$.*
(i) If $\widetilde{\Omega} = A(r_1, r_2)$ and (4) holds, then $\Omega = A(r_1, r_2)$.
(ii) If $\widetilde{\Omega} = B(r_2)$ and (4) holds, then either $\Omega = A(0, r_2)$, or $\Omega = B(r_2) \backslash T$ where $T \subset S(r_0)$ for some $r_0 \in (0, r)$.

Proof. Let $v(y) = m(\Omega) M(h_y, r) - U_\Omega(y)$, whence $v \in C^1(\mathbb{R}^N \backslash S(r))$ and $v = 0$ on $\mathbb{R}^N \backslash \Omega$, by (4).

(i) If $\widetilde{\Omega} = A(r_1, r_2)$, then $r_1 > 0$. We claim that $v \neq 0$ on $\widetilde{\Omega} \backslash S(r)$. To see this, suppose first that $v(y_0) = 0$ where $y_0 \in A(r, r_2)$. Then $v = 0$ on $\partial A(\|y_0\|, r_2)$ by rotational symmetry, and $\Delta v = 2Nc_N > 0$ on $A(\|y_0\|, r_2)$. Hence $v < 0$ on $A(\|y_0\|, r_2)$ by the maximum principle, and we arrive at the contradictory conclusion that $\|\nabla v\| > 0$ on $S(r_2)$. A similar argument applies if $y_0 \in A(r_1, r)$. Hence $\Omega = A(r_1, r_2)$.

(ii) If $\widetilde{\Omega} = B(r_2)$, we again see that $v \neq 0$ on $A(r, r_2)$. If there exists $x_0 \in A(0, r)$ such that $v(x_0) = 0$, then $v = 0$ on $S(r_0)$, where $r_0 = \|x_0\|$, and so $v < 0$ on $B(r_0)$. It follows that $\widetilde{\Omega} \backslash \Omega \subset S(r_0)$. The remaining possibility is that $v \neq 0$ on $A(0, r)$, whence either $\Omega = A(0, r_2)$ or $\Omega = B(r_2)$. ■

Proof of Theorem 2. The hypotheses of Theorem 4 are satisfied, with $E = \Omega$, so $\widetilde{\Omega}$ is either of the form $A(r_1, r_2)$, where $r_1 > 0$, or $B(r_2)$. If $0 \notin \Omega$, then it follows from Lemma 5 that Ω is of the form $A(r_1, r_2)$, where $0 \leq r_1 < r_2$, and from (11) that (2) holds. Otherwise, the lemma shows that $\Omega = B(r_2) \backslash T$ where $T \subset S(r_0)$ for some $r_0 \in (0, r)$, as required. ■

Proof of Theorem 1. In view of Theorem 2, it remains to consider the case where $\Omega = B(r_2) \backslash T$ and $T \subset S(r_0)$ for some $r_0 \in (0, r)$. If there exists $x_0 \in T$, then we can adapt (12) to see that

$$\begin{aligned} \int_{\Omega} u_0 \, dm &< \int_{A(r_0, r_2)} u_0 \, dm = \int_{A(r_0, r_2)} \|x\|^{2-N} \, dm(x) \\ &= \int_{A(r_0, r_2)} \|x - x_0\|^{2-N} \, dm(x) < \int_{\Omega} \|x - x_0\|^{2-N} \, dm(x) \\ &= m(\Omega) r^{2-N} = m(\Omega) M(u_0, r), \end{aligned}$$

where the penultimate equality follows by applying (3) to the function h_{x_0} when $N \geq 3$ and is trivial when $N = 2$. This contradicts (3). Hence $T = \emptyset$ and so $\Omega = B(r_2)$. ■

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