Extension results for harmonic functions which vanish on cylindrical surfaces

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Abstract

The Schwarz reflection principle applies to a harmonic function which continuously vanishes on a relatively open subset of a planar or spherical boundary surface. It yields a harmonic extension to a predefined larger domain and provides a simple formula for this extension. Although such a point-to-point reflection law is unavailable for other types of surface in higher dimensions, it is natural to investigate whether similar harmonic extension results still hold. This article describes recent progress on such results for the particular case of cylindrical surfaces, and concludes with several open questions.

1 Introduction

Throughout this article $\mathcal{H}(U)$ will denote the collection of all harmonic functions on an open set U in the complex plane $\mathbb{C}$ or Euclidean space $\mathbb{R}^N$.

To begin with, let $\Omega \subset \mathbb{C}$ be open, and let $U = \Omega \cap \{\operatorname{Im} z > 0\}$ and $h \in \mathcal{H}(U)$, where h continuously vanishes on $\Omega \cap \{\operatorname{Im} z = 0\}$. Then the Schwarz reflection principle tells us that $h^* \in \mathcal{H}(U^*)$, where U^* is the union of $\Omega \cap \{\operatorname{Im} z \geq 0\}$ and the reflected set $\{z : \overline{z} \in U\}$, and

$$h^*(z) = \begin{cases} h(z) & (z \in U) \\ 0 & (z \in \Omega, \operatorname{Im} z = 0) \\ -h(\overline{z}) & (\overline{z} \in U) \end{cases} .$$

An analogous formula holds for harmonic extension through a circular arc, provided that we use inversion in place of reflection. Indeed, such results even hold for extension across analytic arcs (see Chapter 9 of [7]).

In higher dimensions the Schwarz reflection formula readily generalizes to give harmonic extension across a relatively open subset of a hyperplane or a sphere. However, this is as far as we can go. For, when $N = 3$, Ebenfelt and Khavinson [3] have shown that a point-to-point reflection law can only hold when the containing real analytic surface is either a hyperplane or a

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sphere. Thus, for other surfaces, more elaborate arguments are required to investigate whether such harmonic extension to some specific enlarged domain is still guaranteed.

An important particular case concerns cylindrical surfaces, since a cylinder is the Cartesian product of a line and a sphere, each of which separately admits Schwarz reflection. Indeed, for this case, the existence of a point-to-point reflection formula was originally disproved by Khavinson and Shapiro [9]. Khavinson subsequently asked at various conferences whether a function which is harmonic on a circular cylinder in $\mathbb{R}^3$ and vanishes on the boundary must automatically have a harmonic extension to the whole of space. This issue has particular significance for the study of the Dirichlet problem on cylindrical domains with entire boundary data f. It was recently shown in [8] that, provided f has order less than 1, this problem has a solution that is also entire. Of course, given the unbounded nature of the domain, this problem does not have a unique solution. However, a positive answer to the harmonic extension question would imply that all solutions to this Dirichlet problem are necessarily entire.

This article describes an affirmative answer to the above question (see Theorem 1), along with several other recent extension results for harmonic functions which vanish on cylindrical surfaces. It concludes with some related open questions.

A typical point of $\mathbb{R}^N = \mathbb{R}^{N-1} \times \mathbb{R}$ $(N \geq 3)$ will be denoted by $x = (x', x_N)$, and the unit ball of $\mathbb{R}^{N-1}$ will be denoted by B'. The first result below is taken from [4].

Theorem 1 *Let $a > 0$. If h is harmonic on the finite cylinder $B' \times (-a, a)$ and continuously vanishes on $\partial B' \times (-a, a)$, then h has a harmonic extension $\widetilde{h}$ to the infinite strip $\mathbb{R}^{N-1} \times (-a, a)$.*

It is natural to ask whether such harmonic extension remains possible if the function h is merely harmonic near the curved boundary rather than on the whole cylinder. The next result comes from [5].

Theorem 2 *Let $a > 0$ and $\phi : [-a, a] \to [0, 1)$ be continuous. If h is harmonic on the set*

$$\{(x', x_N) : |x_N| < a \quad and \quad \phi(x_N) < \|x'\| < 1\} \tag{1}$$

and continuously vanishes on $\partial B' \times (-a, a)$, then h extends to a harmonic function on the set

$$\{(x', x_N) : |x_N| < a \quad and \quad \phi(x_N) < \|x'\| < 2 - \phi(x_N)\}. \tag{2}$$

A striking aspect of the above result is that, although we know that no point-to-point reflection principle applies to the function h, the domain of the harmonic extension is nevertheless formed by reflection along the radii of the cylinder. Further, this result is sharp: if $N = 4$ and the function $\phi : [-a, a] \to [0, 1)$ is continuous, then there is a harmonic function on the domain (1) which continuously vanishes on $\partial B' \times (-a, a)$ and does not have a harmonic extension beyond the domain (2). (See [5] for details.)

Theorems 1 and 2 both concern outward harmonic extension through a cylindrical surface. Given the lack of symmetry in higher dimensions, we can ask if inward extension is also possible. This question turns out to raise new technical challenges. We pose some questions of this nature in Section 3, and present here just one result which involves simultaneous inward and outward extension for harmonic functions which vanish on two coaxial cylindrical surfaces. Let Ω_b denote the infinite annular cylinder $A'_b \times \mathbb{R}$, where

$$A'_b = \{x' \in \mathbb{R}^{N-1} : 1 < \|x'\| < b\} \quad (b > 1).$$

The following result was recently established in [6].

Theorem 3 *If $h \in \mathcal{H}(\Omega_b)$ and h continuously vanishes on $\partial\Omega_b$, then h has a harmonic extension to $(\mathbb{R}^{N-1} \backslash \{0'\}) \times \mathbb{R}$.*

In the next section we will give an overview of the methods used to establish the above results.

2 Methods of proof

2.1 The use of series expansions

Let $G_U(\cdot, y)$ denote the Green function for U with pole at $y \in U$. Then $-\Delta G_U(\cdot, y) = C(N)\delta_y$ in the sense of distributions, where $C(N) > 0$ is a dimensional constant and δ_y is the unit mass concentrated at y. Further, $G_U(\cdot, y)$ continuously vanishes on ∂U, except possibly at a polar set. For example, if U is the half-space $\mathbb{R}^{N-1} \times (0, \infty)$, where $N \geq 3$, then

$$G_U(x, y) = \frac{1}{\|x - y\|^{N-2}} - \frac{1}{\|x - \overline{y}\|^{N-2}}, \quad \text{where } \overline{y} = (y_1, ..., y_{N-1}, -y_N).$$

This formula bears an obvious relationship with the Schwarz reflection principle. Indeed, the Green function is both a test case and a building block for the above harmonic extension questions. As usual, we write

$$G_U\mu(x) = \int_U G_U(x, z)\ d\mu(z)$$

for the potential of a suitable measure μ on U. The following useful lemma, taken from [5], is not difficult to prove. It connects the Green function for the N-dimensional cylinder $B' \times \mathbb{R}$ with the Green function for the $(N-1)$-dimensional ball B'. The obvious analogue for cylindrical domains of more general cross-section is also valid.

Lemma 4 *Let $\Omega = B' \times \mathbb{R}$ and $y' \in B'$, and let $d\mu(z', 0) = G_{B'}(z', y')dz'$ on $B' \times \{0\}$. Then*

$$G_\Omega(\cdot, (y', 0)) = C(N) \frac{\partial^2}{\partial x_N^2} G_\Omega \mu.$$

Let T_n denote the usual Chebyshev polynomial of degree n, given by the formula $T_n(t) = \cos(n \cos^{-1} t)$ when $|t| \leq 1$. Then the Green function for the unit disc $\mathbb{D}$ has the known expansion

$$G_{\mathbb{D}}(x', y') = -\log \|x'\| + \sum_{n=1}^{\infty} \frac{1}{n} T_n \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \|y'\|^n \left\{ \|x'\|^{-n} - \|x\|^n \right\} \quad (3)$$

when $\|y'\| < \|x'\| < 1$. The Bessel function J_n of the first kind, of order n, satisfies the differential equation

$$t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + (t^2 - n^2) y = 0$$

(see Watson [10]). Its positive zeros are denoted by $j_{n,1} < j_{n,2} < ....$ When $n > 0$ let $g_{n,m}(s, t)$ denote the summand in the following classical Fourier-Bessel expansion (see Carslaw [2]):

$$4n \sum_{m=1}^{\infty} \frac{J_n(j_{n,m} s) J_n(j_{n,m} t)}{j_{n,m}^2 \{J_{n+1}(j_{n,m})\}^2} = \left\{ \begin{array}{ll} t^n (s^{-n} - s^n) & (0 \leq t < s) \\ s^n (t^{-n} - t^n) & (s \leq t \leq 1) \end{array} \right. . \quad (4)$$

Further, let $g_{0,m}(s, t)$ denote the summand in the analogous expansion

$$2 \sum_{m=1}^{\infty} \frac{J_0(j_{0,m} s) J_0(j_{0,m} t)}{j_{0,m}^2 \{J_1(j_{0,m})\}^2} = \left\{ \begin{array}{ll} -\log s & (0 \leq t < s) \\ -\log t & (s \leq t \leq 1) \end{array} \right. . \quad (5)$$

If $\Omega = \mathbb{D} \times \mathbb{R}$, then (3) - (5) and Lemma 4 together lead to the expansion

$$\begin{aligned} G_\Omega(x, (y', 0)) &= 2 \sum_{m=1}^{\infty} g_{0,m}(\|x'\|, \|y'\|) j_{0,m} e^{-j_{0,m}|x_3|} \\ &+ 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} T_n \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) g_{n,m}(\|x'\|, \|y'\|) j_{n,m} e^{-j_{n,m}|x_3|}. \end{aligned} \quad (6)$$

This formula for the Green function of a three-dimensional cylinder has long been known (see [2]). It has a generalization to all dimensions which uses ultraspherical polynomials in place of Chebyshev polynomials.

The next step is to examine the convergence of this expansion when x lies outside Ω. Using a variety of estimates for Bessel functions and their zeros, this series was shown in [4] to converge when $x' \in \mathbb{R}^{N-1}$ and $x_N \neq y_N$.

This yields an extension result for the Green function. It can, in turn, be used to establish the extension of more general harmonic functions in finite cylinders that vanish on the curved part of the boundary, thus leading to a proof of Theorem 1.

Before leaving Theorem 1 we remark that the harmonic extension $\widetilde{h}$ of h to $\mathbb{R}^{N-1} \times (-a, a)$ decays near infinity. More precisely, we have the following result.

Theorem 5 *Let h and $\widetilde{h}$ be as in Theorem 1. Then, for any $b \in (0, a)$, there is a constant c, depending on a, b, N and h, such that*

$$\left|\widetilde{h}(x)\right| \leq c \left\|x'\right\|^{1-N/2} \qquad (x' \in \mathbb{R}^{N-1} \backslash B', |x_N| < b).$$

2.2 Domain reflection

Although point-to-point reflection formulae are unavailable for cylindrical surfaces, Theorem 2 still exhibits a domain reflection property. A key tool in proving this is the following result of Wimp and Colton [11], which was established using the theory of Volterra integral equations.

Theorem A. *Let $q, y_m \in C[-\delta, \delta]$ and $c_m \in \mathbb{R} \backslash \{0\}$ be such that*

$$\frac{d^2 y_m}{dt^2} + \left(c_m^2 - q(t)\right) y_m = 0, \quad y_m(0) = 0, \quad y_m'(0) = c_m \qquad (m \in \mathbb{N}).$$

If

$$\sum_m b_m y_m(t) = 0 \qquad (0 \leq t \leq \delta), \tag{7}$$

where $\sum |b_m| < \infty$, then the equality in (7) holds when $-\delta \leq t \leq \delta$.

The relevance of this result to the present context lies in the well known fact that, if we define $y(t) = \sqrt{t} J_n(kt)$, where k is a non-zero constant, then

$$\frac{d^2 y}{dt^2} + \left(k^2 + \frac{\frac{1}{4} - n^2}{t^2}\right) y = 0 \qquad (t > 0).$$

Thus, if we define (for fixed $n > 0$)

$$y_{n,m}(t) = \sqrt{t} J_n(j_{n,m} t), \quad c_m = \sqrt{j_{n,m}^2 + 1}, \quad q(t) = \frac{n^2 - \frac{1}{4}}{t^2} + 1 \qquad (m \geq 1, t > 0),$$

we find that

$$\frac{d^2 y_{n,m}}{dt^2} + \left(c_m^2 - q(t)\right) y_{n,m} = \frac{d^2 y_{n,m}}{dt^2} + \left(j_{n,m}^2 + \frac{\frac{1}{4} - n^2}{t^2}\right) y_{n,m} = 0$$

and $y_{n,m}(1) = 0$.

Recalling from above that

$$4n\sum_{m=1}^{\infty}\frac{J_n(j_{n,m}s)J_n(j_{n,m}t)}{j_{n,m}^2\{J_{n+1}(j_{n,m})\}^2}=s^n(t^{-n}-t^n)\qquad(s<t\leq 1),$$

we can now use Theorem A to extend the validity of this equation to the range $s<t<2-s$ for any given $s\in(0,1)$. We can then substitute $s=\|y'\|$ and $t=\|x'\|$ into this identity and combine it with (6), (4) and reasoning related to Lemma 4 to show that the series expansion of the Green function $G_\Omega(\cdot,y)$, where $\Omega=B'\times\mathbb{R}$, converges even on the annular cylinder

$$\{x'\in\mathbb{R}^{N-1}:\left\|y'\right\|<\left\|x'\right\|<2-\left\|y'\right\|\}\times\mathbb{R}.$$

We already knew from Theorem 1 that $G_\Omega(\cdot,y)$ extends to $\mathbb{R}^{N-1}\times(\mathbb{R}\backslash\{y_N\})$. This, in turn, leads to the domain extension result for harmonic functions stated in Theorem 2.

2.3 Extension through two coaxial cylindrical surfaces

Now we consider the case of a function h that is harmonic on the finite annular cylinder $A_b'\times(-a,a)$ and continuously vanishes on the two curved parts of the boundary. The proof of Theorem 1 can be adapted to show that h extends outwards to a harmonic function on $(\mathbb{R}^{N-1}\backslash\overline{B'})\times(-a,a)$. Inward extension requires a more delicate analysis.

In Theorems 1 and 2 we used the fact that $J_n(j_{n,m}\|x'\|)$ vanishes, like h, on $\partial B'\times\mathbb{R}$. In the current situation, h vanishes on $\partial A_b'\times\mathbb{R}$. In place of $J_n(j_{n,m}\|x'\|)$ we now need to consider cross-product terms of the form

$$J_n(\rho\left\|x'\right\|)Y_n(\rho b)-J_n(\rho b)Y_n(\rho\left\|x'\right\|),$$

where Y_n is the Bessel function of the second kind of order n. These functions clearly vanish when $\|x'\|=b$. To make them vanish also when $\|x'\|=1$, we need to replace ρ by the positive zeros $(\rho_{n,m})_{m\geq 1}$ of the above expression when $\|x'\|=1$. Strong uniform estimates for $\rho_{n,m}$ of the form

$$\rho_{n,m+1}-\rho_{n,m}>\frac{\pi}{2b-1}\qquad(n\geq 1,m\geq 2)$$

were established in [6], and then used to show that h can be extended to the domain

$$\left\{(x',x_N):|x_N|<a,\left\|x'\right\|>e^{(|x_N|-a)/b}\right\}.$$

The conclusion of Theorem 3 follows on letting $a\to\infty$.

3 Open questions

Two questions left open above concern whether Theorems 1 and 2 have analogues for inward extension.

Question 1. If h is harmonic on the set $(\mathbb{R}^{N-1}\backslash\overline{B}') \times (-a, a)$ and continuously vanishes on $\partial B' \times (-a, a)$, does h then have a harmonic extension to $(\mathbb{R}^{N-1}\backslash\{0'\}) \times (-a, a)$?

Question 2. Let $\phi : [-a, a] \to (1, \infty)$ be continuous. If h is harmonic on the set

$$\{(x', x_N) : |x_N| < a \text{ and } 1 < \|x'\| < \phi(x_N)\}$$

and continuously vanishes on $\partial B' \times (-a, a)$, does h then have a harmonic extension to a set of the form

$$\{(x', x_N) : |x_N| < a \text{ and } \psi(x_N) < \|x'\| < \phi(x_N)\},$$

where $\psi : [-a, a] \to (0, 1)$ is continuous and independent of the function h?

Another open question is whether Theorem 3 can be strengthened to give a direct analogue of Theorem 1 for finite annular cylinders:

Question 3. If $h \in \mathcal{H}(A'_b \times (-a, a))$ and $h = 0$ on $\partial A'_b \times (-a, a)$, does h then have a harmonic extension to $(\mathbb{R}^{N-1}\backslash\{0'\}) \times (-a, a)$?

A further line of enquiry concerns ellipsoidal cylinders: it was conjectured in [8] that Theorem 1 should remain true in this context (at least when $a = \infty$).

It is also natural to ask whether Theorems 1–3 have analogues for surfaces other than cylinders. In particular, an obvious next step would be to consider conical surfaces. In this regard we pose some further questions below. Let

$$\mathcal{C}(\alpha) = \{(x', x_N) : x_N > \alpha \|x\|\} \quad (0 < \alpha < 1)$$

and the N-dimensional annular set

$$A_b = \{x \in \mathbb{R}^N : 1 < \|x\| < b\} \quad (b > 1).$$

Question 4. If h is harmonic on the set $\mathcal{C}(\alpha) \cap A_b$ and continuously vanishes on $\partial\mathcal{C}(\alpha) \cap A_b$, does h then have a harmonic extension to $A_b\backslash L$, where $L = \mathbb{R}^{N-1} \times (-\infty, 0]$?

Question 5. Let $\phi : [1, b] \to (\alpha, 1)$ be continuous. If h is harmonic on the set

$$\left\{x : \alpha < \frac{x_N}{\|x\|} < \phi(\|x\|) \text{ and } 1 < \|x\| < b\right\}$$

and continuously vanishes on $\partial\mathcal{C}(\alpha) \cap A_b$, does h then have a harmonic extension to some set of the form

$$\left\{ x : \psi(\|x\|) < \frac{x_N}{\|x\|} < \phi(\|x\|) \text{ and } 1 < \|x\| < b \right\},$$

where $\psi : [1, b] \to (-1, \alpha)$ is continuous and independent of the function h?

Question 6. Let $0 < \alpha < \beta < 1$. If h is harmonic on the set $\mathcal{C}(\alpha)\backslash\overline{\mathcal{C}(\beta)}$ and continuously vanishes on $\partial\mathcal{C}(\alpha) \cup \partial\mathcal{C}(\beta)$, does h then have a harmonic extension to $(\mathbb{R}^{N-1}\backslash\{0'\}) \times \mathbb{R}$?

In connection with the last three questions we note that, when $N = 3$, Carslaw [2] has obtained double series expansions for the Green function of domains with conical boundaries.

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