Modelling weighted signed networks

Modellazione di reti segnate pesate

Alberto Caimo and Isabella Gollini

Abstract In this paper we introduce a new modelling approach to analyse weighted signed networks by assuming that their generative process consists of two models: the interaction model which describes the overall connectivity structure of the relations in the network without taking into account neither the weight nor the sign of the dyadic relations; and the conditional weighted signed network model describes how the edge signed weights form given the interaction structure. We then show how this modelling approach can facilitate the interpretation of the overall network process. Finally, we adopt a Bayesian inferential approach to illustrate the new methodology by modelling the Sampson's influence network.

Abstract *In questo articolo, introduciamo un nuovo approccio modellistico per analizzare reti segnate pesate supponendo che il loro processo generativo sia costituito da due modelli: il modello d'interazione che descrive la struttura connettiva generale della rete senza tenere in considerazione né il peso né il segno degli archi; e il modello per la rete segnata pesata che descrive come gli archi segnati pesati si creino condizionatamente alla struttura d'interazione. Mostriamo quindi come questo approccio modellistico possa facilitare l'interpretazione del processo relazionale generale. Per concludere, adottiamo un approccio inferenziale bayesiano per illustrare la nuova metodologia attraverso la modellazione della rete di Sampson sulle relazioni d'influenza.*

Key words: Signed networks, weighted networks, exponential-family network models, Bayesian inference.

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1 Exponential random graph models

Relations between actors on social networks often consist of positive and negative interactions. And typically these interactions have weights assigned to them. The relational structure of a network graph is represented by an adjacency matrix $\boldsymbol{y}$ whose elements $\boldsymbol{y}_{i,j}$ are defined by a value corresponding to the intensity of the interaction between any pair of nodes.

Exponential random graph models (ERGMs) are a particular class of discrete linear exponential families [8, 17] which represent the probability distribution of a network $\boldsymbol{Y}$ on a fixed set of nodes as:

$$p(\boldsymbol{y}|\theta) = h(\boldsymbol{y})\exp\{\theta's(\boldsymbol{y}) - \psi(\theta)\}, \tag{1}$$

where $h(\boldsymbol{y})$ is a reference distribution [10] specifying the model for the data before any network effect is considered; $s(\boldsymbol{y})$ is a known vector of p network statistics measuring the quantity of some selected sub-graph configurations in the network [14], $\theta \in \mathbb{R}^p$ is the parameter vector associated to the vector of network statistics, and $\psi(\theta)$ is a normalising constant which is typically computationally difficult to evaluate for all but trivially small networks [12]. The dependence hypothesis at the basis of the ERGMs is that the observed network structure is the result of a generative process in which edges self organise into sub-network configurations. There is a wide range of possible network configurations which gives the flexibility to adapt ERGMs to various different contexts. A positive parameter value for θ_i results in a tendency for the certain configuration corresponding to $s_i(\boldsymbol{y})$ to be observed in the data than would otherwise be expected by chance.
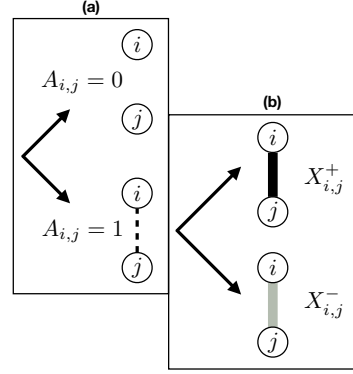
2 ERGMs for weighted signed networks

A weighted signed network graph between N nodes can be described by $N \times N$ adjacency matrix $\boldsymbol{Y}$ where:

$$Y_{i,j} = \begin{cases} y_{i,j} \neq 0, & i \text{ connected to } j; \\ y_{i,j} = 0, & i \text{ not connected to } j. \end{cases}$$

The connection value of $y_{i,j}$ represents the weight of positive and negative edges between nodes.

ERGMs have recently been generalised to binary signed networks [9]. In this paper we adopt a new modelling approach for weighted signed networks by assuming the existence of two distinct processes: the interaction process determining the presence or absence of an interaction between the nodes (see also [11]), and the conditional weighted signed process which is describing the joint structure of the positive and negative weight relations given the interaction process. We distinguish between an interaction variable $\boldsymbol{A}$ and a weighted signed variable $\boldsymbol{X}$ by assuming that the prob-

Fig. 1 Structure of the model:
(a) interaction process; **(b)**
conditional weighted signed
network process.

ability of Y represents the joint probability of A and X:

$$\Pr(Y = y) = \Pr(X = x, A = a) = \Pr(X = x \mid A = a) \times \Pr(A = a).$$

In particular, we assume that the overall weighted signed network process can be jointly modelled by a joint conditional weighted ERGM process for the positive weighted edges (X^+) and the negative weighted edges (X^-), so that for any dyad (i, j) we have:

$$\begin{cases} \Pr(Y_{i,j} = y_{i,j} \mid Y_{i,j} > 0) = \Pr(X_{i,j}^+ = x_{i,j}^+ \mid A_{i,j} = 1, \theta_X^+) \times \Pr(A_{i,j} = 1 \mid \theta_A); \\ \Pr(Y_{i,j} = y_{i,j} \mid Y_{i,j} < 0) = \Pr(X_{i,j}^- = x_{i,j}^- \mid A_{i,j} = 1, \theta_X^-) \times \Pr(A_{i,j} = 1 \mid \theta_A), \end{cases}$$

where:

$$A_{i,j} = \begin{cases} a_{i,j} = 1, & i \text{ connected to } j; \\ a_{i,j} = 0, & i \text{ not connected to } j. \end{cases}$$

It is important to notice that positive and negative processes are not conditionally independent given A as the two signed structures X^+ and X^- are mutually exclusive given A.

We therefore propose to model the interaction process assuming that $A \mid \theta_A \sim ERGM(\theta_A)$ and the weighted network process assuming that X is modelled by two joint weighted signed ERGM processes: one for positive edge relations and one for negative edge relations with parameters θ_X^+ and θ_X^-, respectively. Figure 1 shows the structure of the interaction/weighted signed modelling framework proposed.

The conditional weighted ERGM processes can be defined according to specific forms of weighted network model. ERGM modelling approaches for weighted networks include the multi-valued curved ERGMs [18], generalised ERGMs for inference on networks with continuous edge values [6]; Geometric/Poisson reference ERGMs for ordinal/count networks [10]; and the hierarchical multilayer ERGM approach for polytomous networks [4].

3 Bayesian inference

Bayesian methods are becoming increasingly popular as techniques for modelling social networks. Following the Bayesian paradigm, prior distribution is assigned to θ. The posterior distribution of θ given the observed network data $\boldsymbol{y}$ is:

$$p(\theta|\boldsymbol{y}) = \frac{p(\boldsymbol{y}|\theta)p(\theta)}{p(\boldsymbol{y})}.$$

From an ERGM viewpoint, direct evaluation of $p(\theta|\boldsymbol{y})$ requires the calculation of both the likelihood $p(\boldsymbol{y}|\theta)$ and the marginal likelihood or model evidence $p(\boldsymbol{y})$ which are typically intractable. According to our modelling framework, the parameter posterior distribution defined in Section 2 can be written as:

$$p(\theta_X^+, \theta_X^-, \theta_A \mid \boldsymbol{x}^+, \boldsymbol{x}^-, \boldsymbol{a}) \propto p(\boldsymbol{x}^+, \boldsymbol{x}^- \mid \boldsymbol{a}, \theta_X^+, \theta_X^-) p(\theta_X^+, \theta_X^-) p(\boldsymbol{a} \mid \theta_A) p(\theta_A). \quad (2)$$

where:

- $p(\boldsymbol{a} \mid \theta_A) \propto h(\boldsymbol{a}) \, \exp\{\theta_A s(\boldsymbol{a})\}$ is the interaction ERGM likelihood;
- $p(\boldsymbol{x}^+, \boldsymbol{x}^- \mid \boldsymbol{a}, \theta_X^+, \theta_X^-)$ is the joint weighted signed ERGM likelihood conditional on the interaction relations;
- $p(\theta_X^+, \theta_X^-)$ and $p(\theta_A)$ are the prior parameter distributions.

To estimate the parameter posterior density defined in Equation 2, we adapt the approximate exchange algorithm for Bayesian ERGMs [1, 2] implemented in the Bergm package [3] for R [13].

4 Application

Sampson's monk directed network [15] contains ratings between monks related to a crisis in a cloister in New England (USA). In particular, we focus on the positive/negative influence between monks and we want to use the ERGM generative process defined above to describe the connectivity structure of the weighted signed directed network.

4.1 Model specification

We include the following network statistics for the binary undirected interaction ERGM model:

- Edge statistic (`edges`) is the number of edges in the network capturing the network density effect.

Fig. 2 Graph structure of the Sampson's monk network. Weighted black edges correspond to positive influence relations; gray edges correspond to negative influence relations.
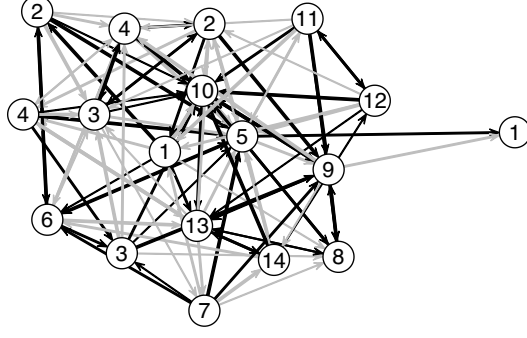
- Geometrically weighted edgewise shared partner statistic (`gwesp`) captures the tendency towards transitivity, i.e., the tendency of edges to be connected through multiple triadic relations simultaneously [16]. We fix the decay parameter of the `gwesp` statistic to be equal to 2.

We assume that the conditional weighted signed network model follows a constrained ERGM process with a uniform/truncated geometric reference distribution [10, 7] so that $h(\boldsymbol{x}^+, \boldsymbol{x}^-; \boldsymbol{a}) = 1$.
We include the following directed network statistics for the joint conditional weighted signed ERGM model:

- Weighted-sum statistic (`sum`) is the sum of the edge values capturing the weighted density effect.
- Mutual-min statistic (`mutual(min)`) is the sum of the minimum weighted mutual edge value capturing the weighted network reciprocity effect.
- Transitive-weights statistic (`transitiveweights`) captures the tendency towards transitive clustering in the weighted network.
- Cyclical-weights statistic (`cyclicalweights`) captures the tendency towards cyclical clustering in the weighted network.

It is important to emphasise that the set of the network statistics to include for describing the positive and negative structures is not necessarily the same so that we can formulate different connectivity hypothesis for the positive and negative weighted network processes.

4.2 Prior specification

We specify vague prior distribution for all the parameters in both interaction and conditional weighted signed network model:

$$\theta_X^+ \sim \mathcal{N}_p(\mu = 0, \Sigma = 10\,\boldsymbol{I}_p); \; \theta_X^- \sim \mathcal{N}_q(\mu = 0, \Sigma = 10\,\boldsymbol{I}_q); \; \theta_A \sim \mathcal{N}_r(\mu = 0, \Sigma = 10\,\boldsymbol{I}_r).$$

where I is the identity matrix. In this example, the model specification for the joint weighted signed process consist of the same set of network statistics for both the positive and the negative weighted structure.

4.3 Results

The posterior density estimates displayed in Table 1 show that the interaction network process $A|\theta_A$ is sparse (negative value for the `edges` parameter) and the interactions tend to organise into triadic clusters (positive value for `gwesp(2)`). The positive `mutual(min)` parameter estimate in the conditional positive weighted signed ERGM process $X^+|X^-,A,\theta_X^+$ explains the tendency towards reciprocation of positive influence relations. It is important to notice that the positive tendency towards clustering that we generally expect in positive weighted networks is mainly captured by the `gwesp(2)` effect in the interaction model. The negative triadic parameter (`transitiveweights` and `cyclicalweights`) estimates for the conditional negative weighted signed ERGM process $X^-|X^+,A,\theta_X^-$ confirm the assumption of the structural balance hypothesis of "the enemy of an enemy is a friend" or "the friend of an enemy is an enemy" [5].

Table 1 Parameter posterior estimates for the interaction / weighted signed network model.

	Network statistic	Mean	Std. Err.
Interaction	`edges`	-1.42	0.51
	`gwesp(2)`	0.19	0.06
Positive weights	`sum`	-0.11	0.21
	`mutual(min)`	0.82	0.38
	`transitiveweights`	0.03	0.20
	`cyclicalweights`	0.01	0.16
Negative weights	`sum`	0.37	0.21
	`mutual(min)`	0.22	0.40
	`transitiveweights`	-1.56	0.35
	`cyclicalweights`	-0.64	0.17

4.4 Model assessment

A way to examine the fit of the data to the estimated posterior distribution of the parameters is to implement a graphical Bayesian goodness-of-fit procedure. In the Bayesian context, simulated networks are simulated from a sample of 100 parameter values randomly drawn from the estimated posterior distribution and compared to the observed data according to some network statistics. The plots in Figure 3
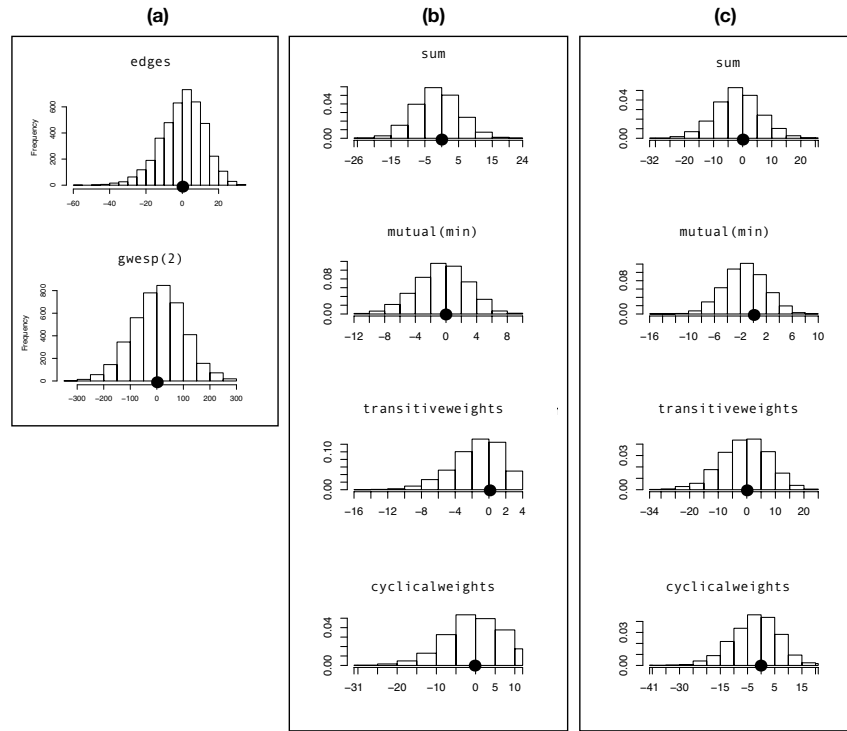
(a) **(b)** **(c)**

Fig. 3 Goodness of fit diagnostics plots showing the distribution of a set of networks simulated from the estimated posterior density of Table 1 centred on the observed network statistics for **(a)** Interaction model; **(b)** Positive weighted signed model; **(c)** Negative weighted signed model.

suggest that the proposed model is a reasonable fit to the observed data as most of the probability mass of the simulated network statistics is concentrated around the observed network statistics.

5 Discussion

We have presented a flexible model able to capture the dependence structure of weighted signed networks. In particular, given the interaction between nodes in the network we have proposed to model the weighted signed network dependencies by introducing a weighted signed ERGM processes for joint modelling the structure of negative and positive edges and adopting a Bayesian approach. As demonstrated in the illustration, the model is able to facilitate the interpretation of the complex dependence structure of weighted signed networks by making use of interpretable network effects and the assessment of structural balance in signed networks.

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