Analytic aspects of evolution algebras

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Abstract

We prove that every evolution algebra A is a normed algebra, for an l_1-norm defined in terms of a fixed natural basis. We further show that a normed evolution algebra A is a Banach algebra if, and only if, $A = A_1 \oplus A_0$, where A_1 is finite dimensional and A_0 is a zero product algebra. In particular, every non-degenerate Banach evolution algebra must be finite dimensional and the completion of a normed evolution algebra is therefore not, in general, an evolution algebra. We establish a sufficient condition for continuity of the evolution operator, L_B, of A with respect to a natural basis B and show that L_B need not be continuous. Moreover, if A is finite dimensional and $B = \{e_1, ..., e_n\}$ then L_B is given by L_e, where $e = \sum_i e_i$ and L_a is the multiplication operator $L_a(b) = ab$, for $b \in A$. We establish necessary and sufficient conditions for convergence of $(L_a^n(b))_n$, for all $b \in A$, in terms of the multiplicative spectrum, $\sigma_m(a)$, of a. Namely, $(L_a^n(b))_n$ converges, for all $b \in A$, if and only if $\sigma_m(a) \subseteq \Delta \cup \{1\}$ and $\nu(1, a) \leq 1$, where $\nu(1, a)$ denotes the index of 1 in the spectrum of L_a.

1 Introduction

The use of algebraic techniques to study genetic inheritance dates from Mendel in 1856 [21], with subsequent works [10, 11, 12, 14, 25] by various authors over the next four decades culminating in the algebraic formulation of Mendel's laws in terms of non-associative algebras [10, 11]. Since then many algebras, generally referred to as genetic algebras (Mendelian, gametic, and zygotic algebras, to name but a few) have provided a mathematical framework for studying various types of inheritance. On the other hand, certain genetic phenomena such as, for example, the case of incomplete dominance, systems of multiple alleles and asexual inheritance, do not follow Mendel's laws and evolution algebras were

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introduced by Tian and Vojtechovsky [28] in 2006, partly as an attempt to study such non-Mendelian behaviour. Evolution algebras are highly non-associative in general (they are not even power associative) although they are commutative. For a recent study of evolution algebras in infinite dimensions see [1]. Other aspects of evolution algebras have been considered in [2, 3, 4, 5, 6, 8, 9, 13, 15, 16, 17, 24, 29].

Recall that an algebra is a vector space A over $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$) provided with a bilinear map $A \times A \to A$, $(a,b) \to ab$, referred to as the multiplication of A (which, here, is not assumed to be either associative or commutative). When an algebra A is provided with a basis $B = \{e_i : i \in \Lambda\}$, such that $e_i e_j = 0$ if $i \neq j$, then we say that A is an evolution algebra and B is a natural basis of A. In Section 2 we study the existence or otherwise of algebra norms and complete algebra norms on an evolution algebra. Recall that A is a normed algebra if A has a norm $\|\cdot\|$ such that $\|ab\| \leq \|a\| \, \|b\|$, for every $a, b \in A$ and A is a Banach algebra if it has a complete algebra norm. We prove that every evolution algebra A is a normed algebra, for an l_1-norm defined in terms of a fixed natural basis and also show that a normed evolution algebra A is a Banach algebra if, and only if, $A = A_1 \oplus A_0$, where A_1 is finite dimensional and A_0 is a zero product algebra. In particular, every non-degenerate Banach evolution algebra must be finite dimensional and the completion of a normed evolution algebra is not, in general, itself an evolution algebra.

For evolution algebra A and basis $B = \{e_i : i \in \Lambda\}$ as above, the unique linear map $L_B : A \to A$ satisfying $L_B(e_i) = e_i^2$, for all $i \in \Lambda$, is known as the evolution operator on A associated to B. This is postulated in [27] as being central to the dynamics of A. In Section 3 we study the continuity of the evolution operator, giving a sufficient condition for its continuity and an example to show that it is not necessarily continuous.

In particular, if $\dim A < \infty$ and $B = \{e_1, \cdots, e_n\}$ then L_B is the multiplication map L_e, for $e = \sum_{i=1}^{n} e_i$ (of course, L_B is then automatically continuous). For $b \in A$ and $m \in \mathbb{N}$, the element $L_B^m(b)$ has biological meaning, and a typical question in this framework is to study possible accumulation points of $(L_B^m(b))_m$. Section 4 tackles this topic, and in the light of results in Section 2, we assume that A is finite dimensional and thus $L_B = L_e$. On the other hand, for $\lambda \in \mathbb{K}\backslash\{0\}$, $\tilde{e} := \lambda e$ is another evolution element (corresponding to basis $\widetilde{B} = \{\lambda e_1, \cdots, \lambda e_n\}$) with $L_{\tilde{e}}^m = \lambda^m L_e^m$. Clearly then $(L_{\tilde{e}}^m)_m$ may not converge even if $(L_e^m)_m$ does. In other words, the role of the evolution element (even assuming norm 1) is not central and we study instead convergence of $L_a^m(b)$, for arbitrary a, b in A. To this end, we employ the multiplicative spectrum, $\sigma_m(a)$, of a, as introduced in [20]. Section 4 then proves that $(L_a^m(b))_m$ converges for all $b \in A$ if, and only if,

$$\sigma_m(a) \subseteq \Delta \cup \{1\} \text{ and } \nu(1,a) \leq 1,$$

where $\nu(1,a)$ is the index of 1 as an eigenvalue of L_a, and Δ is the open unit disc in $\mathbb{C}$. Alternative formulations of this are given, in Corollaries 34 and 35, for example, $(L_a^m(b))_m$ converges for all $b \in A$ if, and only if, $L_a = P + S$, for linear maps $P, S \in L(A)$ satisfying $P = P^2$, $PS = SP = 0$ and $\rho(S) < 1$. Moreover, we show that if $(L_a^m(b))_m$ converges for all $b \in A$, then $P := \lim_m L_a^m$ is projection onto the subspace $A_a = \ker(L_a - I)$ and $P = 0$ if, and only if $\nu(1, a) = 0$. Theorem 38 and Corollary 39 examine cases where the dynamical system $L_a^m(b)$ displays recurrent states.

2 Evolution algebras as Banach algebras

While finite dimensional evolution algebras were introduced in [28] and evolution algebras with a countable basis are studied in [27], the first general algebraic study of evolution algebras of arbitrary dimension is presented in [1]. As the definition there generalises the earlier ones, we use it throughout this paper.

Definition 1 An evolution algebra is an algebra A provided with a basis $B = \{e_i : i \in \Lambda\}$, such that $e_i e_j = 0$ for $i, j \in \Lambda$ with $i \neq j$, where Λ is an arbitrary (possibly uncountable) non-empty set of indices. Such a basis B is said to be a natural basis of A. The product of A is then determined by the equalities $e_i^2 = \sum\limits_{k \in \Lambda} \omega_{ki} e_k$, for all $i \in \Lambda$, and, for fixed $k \in \Lambda$, we note that ω_{ki} is non-zero for only a finite number of indices.

The map : $\Lambda \times \Lambda \to \mathbb{K}$ such that $(i, j) \to \omega_{ij}$ encodes the algebra structure of A with respect to B. It is therefore useful to represent this map as a $\Lambda \times \Lambda$ 'matrix' which we denote by $M_A(B) = (\omega_{ij})_{i,j}$ and refer to it as the evolution matrix of A with respect to B.

In this section we are primarily interested in what happens when an evolution algebra A is endowed with an algebra norm (that is, a norm making the product continuous). When A is provided with such a norm we will say that A is a normed evolution algebra and when that norm is also complete we will say that A is a Banach evolution algebra.

Of course, all finite dimensional normed evolution algebras are automatically Banach evolution algebras since all norms are then complete. In what follows, we show that the concept of an infinite dimensional Banach evolution algebra is not as straightforward as one might expect. In fact, an immediate consequence of the Baire category theorem is that an infinite dimensional Banach space cannot have a countable basis and hence an infinite dimensional Banach evolution algebra can not have a countable natural basis. In particular, this means that infinite dimensional evolution algebras with countable basis in the sense of [27, Definition 3] are never Banach algebras.

We first show that every evolution algebra is a normed evolution algebra.

Definition 2 *If $B = \{e_i : i \in \Lambda\}$ is a natural basis of an evolution algebra A then the l_1* norm with respect to B *is the norm $\|\cdot\|_1$ defined as follows:*

$$\|a\|_1 = \sum_{i \in \Lambda_a} |\alpha_i|$$

whenever $a = \sum_{i\in\Lambda} \alpha_i e_i = \sum_{i\in\Lambda_a} \alpha_i e_i$, *and* $\Lambda_a := \{i \in \Lambda : \alpha_i \neq 0\}$ *is a finite subset of* Λ.

Proposition 3 *Let* A *be an evolution algebra,* $B = \{e_i : i \in \Lambda\}$ *be a natural basis and* $\|\cdot\|_1$ *be the* l_1 *norm with respect to* B*. Then* $\|\cdot\|_1$ *is an algebra norm on* A *if, and only if,* $\|e_i^2\|_1 \leq 1$*, for every* $i \in \Lambda$.

Proof. If $\|\cdot\|_1$ is an algebra norm on A then $\|e_i^2\|_1 \leq \|e_i\|_1^2 = 1$. Conversely, if $\|e_i^2\|_1 \leq 1$ for every $i \in \Lambda$ then, for $a = \sum_{i\in\Lambda_a} \alpha_i e_i$ and $b = \sum_{i\in\Lambda_b i} \beta_i e_i$, we have

$$\|ab\|_1 = \left\| \sum_{i\in\Lambda_a\cap\Lambda_b} \alpha_i\beta_i e_i^2 \right\|_1 \leq \sum |\alpha_i\beta_i| \leq \left(\sum_{i\in\Lambda_a} |\alpha_i|\right)\left(\sum_{i\in\Lambda_b} |\beta_i|\right) = \|a\|_1 \|b\|_1 ,$$

namely, $\|\cdot\|_1$ is an algebra norm on A. ■

This contrasts with [27, Section 3.3.1], where algebra norms are not considered. Proposition 3 also motivates the following.

Definition 4 Let A be an evolution algebra and let $B = \{e_i : i \in \Lambda\}$ be a natural basis. We say that B is a normalized natural basis if $\|e_i^2\|_1 = 1$ for every $i \in \Lambda$ such that $e_i^2 \neq 0$.

It is easy to check that every evolution algebra A has a normalized natural basis. In fact, given a natural basis $B = \{u_i : i \in \Lambda\}$ of A, for $i \in \Lambda$, define $e_i := \frac{1}{\sqrt{\|u_i^2\|_1}} u_i$ if $u_i^2 \neq 0$ and $e_i = u_i$ otherwise. Then $\{e_i : i \in \Lambda\}$ is a normalized natural basis which we call the normalized natural basis derived from B.

The following is now immediate from Proposition 3.

Corollary 5 *Every evolution algebra* A *is a normed evolution algebra, namely, if* B *is a normalized natural basis then the* l_1 *norm with respect to* B *is an algebra norm on* A.

Definition 6 Let $\|\cdot\|$ be an algebra norm on an evolution algebra A and $B = \{e_i : i \in \Lambda\}$ be a natural basis. We say that B is unital if $\|e_i\| = 1$, for every $i \in \Lambda$.

We may assume, without loss of generality, that for a given algebra norm the natural basis B is unital.

The following example shows that the completion of a normed evolution algebra is not, in general, itself an evolution algebra (for the same underlying product).

Example 7 Let c_{00} be the space of infinite sequences of finite support endowed with the product given by $e_n^2 = e_n$ and $e_n e_m = 0$ if $n \neq m$, for the standard (natural) basis $B = \{e_n : n \in \mathbb{N}\}$. Proposition 3 above implies that the l_1 norm is an algebra norm on c_{00} since $\left\|e_n^2\right\|_1 = \|e_n\|_1 = 1$. The completion of c_{00} with respect to this norm is the Banach space l_1. Suppose now that l_1 is an evolution algebra with natural basis given by $\overline{B} = \{u_i : i \in \Lambda\}$. From earlier, we know that Λ must be uncountable. For every $j \in \mathbb{N}$ there exists $m \in \mathbb{N}$ (depending on j), elements $u_{j_1},..., u_{j_m} \in \overline{B}$ and scalars $\gamma_1, ..., \gamma_m$ such that

$$e_j = \gamma_1 u_{j_1} + ... + \gamma_m u_{j_m}.$$

Then $\overline{B}_{00} := \cup_{j\in\mathbb{N}}\{u_{j_1},..., u_{j_m}\}$ is a countable subset of $\overline{B}$. Because Λ is not countable, there exists $u_{i_0} \in \overline{B}\backslash\overline{B}_{00}$ and it follows that $e_j u_{i_0} = 0$, for every $j \in \mathbb{N}$. Fix $k_0 \in \mathbb{N}$. If $u_{i_0} = \sum\limits_{k\in\mathbb{N}} \gamma_k e_k$ with $\sum\limits_{k\in\mathbb{N}} |\gamma_k| < \infty$ then

$$0 = e_{k_0} u_{i_0} = e_{k_0} \sum_{k\in\mathbb{N}} \gamma_k e_k = \gamma_{k_0} e_{k_0}^2 = \gamma_{k_0} e_{k_0}.$$

In other words, $\gamma_{k_0} = 0$ and therefore $u_{i_0} = 0$. Since this is impossible, it follows that l_1 has no natural basis and is therefore not an evolution algebra.

Lemma 8 *Let A be a Banach evolution algebra for norm $\|\cdot\|$ and natural basis $B = \{e_i : i \in \Lambda\}$. Then the set $\Lambda_B := \{i \in \Lambda : e_i^2 \neq 0\}$ is finite.*

Proof. We may assume, without loss of generality, that B is a unital natural basis so that, if $\|\cdot\|_1$ denotes the corresponding l_1-norm associated to B as above, then $\|a\| \leq \|a\|_1$, for all $a \in A$. Suppose now that Λ_B is infinite. It is well known (via the axiom of choice and axiom of countable choice) that every infinite set has a countably infinite subset, so let $\{e_i : i \in \mathbb{N}\} \subseteq \Lambda_B$. Choose non-zero scalars α_n such that $\sum_{n\in\mathbb{N}} |\alpha_n| < \infty$. Let $u_n := \sum\limits_{k=1}^{n} \alpha_k e_k$. Then $(u_n)_n$ is a $\|\cdot\|_1$ -Cauchy sequence and hence, since B is unital, it is therefore also $\|\cdot\|$ -Cauchy and consequently $\|\cdot\|$-convergent, so that the $\|\cdot\|$-limit, $u = \lim_n u_n$ exists in A. On the other hand, since B is a basis

$$u = \beta_1 e_{\gamma_1} + ... + \beta_k e_{\gamma_k}, \tag{1}$$

for some $k \in \mathbb{N}$, non-zero scalars $\beta_1, ..., \beta_k$, and indices $\gamma_1, ..., \gamma_k \in \Lambda$. Fix now $j \in \mathbb{N}$ such that $e_j \notin \{e_{\gamma_1}, \cdots, e_{\gamma_k}\}$. Since $\lim_n \|u - u_n\| = 0$ and the product is $\|\cdot\|$ continuous, we have

$$\begin{aligned} 0 &= \lim_n \|e_j(u - u_n)\| \\ &= \lim_n \|e_j(\beta_1 e_{\gamma_1} + ... + \beta_k e_{\gamma_k} - u_n)\| \\ &= \lim_n \|e_j(u_n)\| = \|\alpha_j e_j^2\| = |\alpha_j|\|e_j^2\|. \end{aligned}$$

Since $j \in \Lambda_B$ then $e_j^2 \neq 0$. In particular then $\alpha_j = 0$. Since the scalars α_n were chosen to be non-zero this contradiction proves that Λ_B must be finite. ∎

Theorem 9 *Let* $(A, \|\cdot\|)$ *be a Banach evolution algebra. Then* $A = A_0 \oplus A_1$*, where* A_1 *is a finite-dimensional evolution algebra and* A_0 *is a zero product subalgebra.*

Proof. Let $B = \{e_i : i \in \Lambda\}$ be a natural basis. By Lemma 8 the set $\Lambda_B := \{i \in \Lambda : e_i^2 \neq 0\}$ is finite. For $i \in \Lambda$, if $e_i^2 = \sum\limits_{k\in\Lambda} \omega_{ki} e_k$ let

$$\widehat{\Lambda}_i := \{k \in \Lambda : \omega_{ki} \neq 0\} \bigcup \{i\}.$$

Let $\Lambda_1 := \bigcup_{i\in\Lambda_B} \widehat{\Lambda}_i$ and $\Lambda_0 := \Lambda \backslash \Lambda_1$. Then for $A_0 = lin\{e_i : i \in \Lambda_0\}$ and $A_1 = lin\{e_i : i \in \Lambda_1\}$ we have $A = A_0 \oplus A_1$, where A_0 is (a possibly infinite-dimensional) zero product subalgebra and A_1 is a finite-dimensional evolution subalgebra of A. ∎

This motivates the following, originally introduced in [28].

Definition 10 We say that an evolution algebra A is non-degenerate if for some natural basis $B = \{e_i : i \in \Lambda\}$ then $e_i^2 \neq 0$ for every $i \in \Lambda$.

One sees easily that definition 10 is independent of the choice of natural basis, for suppose that $B = \{e_i : i \in \Lambda\}$ and $\tilde{B} = \{u_i : i \in \Omega\}$ are two natural bases of A and suppose that $e_{i_0}^2 = 0$, for some $i_0 \in \Lambda$. Then $e_j e_{i_0} = 0$ for all $j \in \Lambda$ and hence $ae_{i_0} = 0$ for all $a \in A$. There is a finite subset $\Omega_0 \subset \Omega$ such that $e_{i_0} = \sum_{j\in\Omega_0} \alpha_j u_j$, with $\alpha_j \neq 0$ for $j \in \Omega_0$. For $k \in \Omega_0$ we then have $0 = u_k e_{i_0} = \alpha_k u_k^2$. In other words, $u_k^2 = 0$, for all $k \in \Omega_0$, giving the required independence. The independence can also be seen as a consequence of [1, Corollary 2.19], namely, an evolution algebra is non-degenerate if, and only if, $ann(A) = 0$, where $ann(A)$ denotes the annihilator of A. The following is now immediate.

Corollary 11 *Non-degenerate Banach evolution algebras are finite-dimensional. Consequently, the completion of a non-degenerate infinite-dimensional normed evolution algebra is not an evolution algebra.*

If A is a degenerate normed evolution algebra then its completion $\widehat{A}$ is an evolution algebra only when $\widehat{A}$ is an algebra of the type described in Theorem 9, in which case A must also be of the same type.

The above corollary answers in the negative a question raised in [27, p. 18] as to whether or not infinite-dimensional evolution algebras can be Banach algebras.

3 Continuity of the evolution operator

We continue to study the continuty of the evolution operator, defined as in [27].

Definition 12 Let A be an evolution algebra and $B = \{e_i : i \in \Lambda\}$ be a natural basis. The evolution operator of A associated to B is the unique linear map $L_B : A \to A$ such that $L(e_i) = e_i^2$.

Remark 13 *If* $\dim A < \infty$ and $B = \{e_1, \cdots, e_n\}$ is a natural basis of A then, for $a \in A$, $L_B(a) = ea$, where $e = \sum_{i=1}^{n} e_i$. In other words, L_B is the multiplication operator L_e. Of course, in infinite-dimensions L_B is well defined even when $\sum_{i \in \Lambda} e_i$ is not.

Propositions 3 and 5 guarantee that A always has an algebra norm, namely, the l_1 norm with respect to a normalised natural basis. Moreover we have the following.

Proposition 14 *Let A be an algebra provided with a norm $\|\cdot\|$. Then $\|\cdot\|$ is an algebra norm, if and only if, for every $a \in A$ the multiplication operator L_a is continuous with $\|L_a\| \leq \|a\|$.*

Proof. If $\|\cdot\|$ is an algebra norm then $\|L_a(b)\| = \|ab\| \leq \|a\| \|b\|$ so L_a is continuous and $\|L_a\| \leq \|a\|$. Conversely, if $\|L_a\| \leq \|a\|$ then,

$$\|ab\| = \|L_a(b)\| \leq \|L_a\| \|b\| \leq \|a\| \|b\|,$$

so that $\|\cdot\|$ is an algebra norm. ■

We show now that the evolution operator is not necessarily continuous for every algebra norm in the infinite dimensional case (of course, all norms are equivalent and every linear map is continuous in finite dimensions).

Proposition 15 *There exists a normed evolution algebra $(A, \|\cdot\|)$ with a natural basis such that L_B is not continuous.*

Proof. Let A be the space c_{00} of infinite sequences of finite support, as in example 7 above. Let $B := \{e_n : n \in \mathbb{N}\}$ where $e_n := (\delta_{kn})_{k \in \mathbb{N}}$. For $n, m \in \mathbb{N}$ define $e_n^2 = ne_n$ and $e_n e_m = 0$, if $n \neq m$. Then A is an evolution algebra and B is a natural basis for A. Let $\gamma : \mathbb{N} \to \mathbb{N}$ be such that $\gamma(n) \geq n$, for every $n \in \mathbb{N}$. Let $F : A \to A$ be the unique linear operator such that $F(e_k) = \gamma(k)e_k$, for $k \in \mathbb{N}$. For $a \in A$ define $\|a\| = \|F(a)\|_1$, for every $a \in A$. It is straightforward to check that this is a norm. In fact,

$$\begin{aligned}
\|ab\| = \|F(ab)\|_1 &= \left\|F(\sum \alpha_n \beta_n e_n^2)\right\|_1 = \left\|F(\sum \alpha_n \beta_n n e_n)\right\|_1 \\
&= \left\|\sum \alpha_n \beta_n n \gamma(n) e_n\right\|_1 \leq \left(\sum |\alpha_n| \gamma(n)\right) \left(\sum |\beta_n| \gamma(n)\right) \\
&= \left\|\sum \alpha_n F(e_n)\right\|_1 \left\|\sum \beta_n F(e_n)\right\|_1 = \|a\| \|b\|.
\end{aligned}$$

Obviously $\|\cdot\|$ and $\|\cdot\|_1$ are not equivalent because $\|e_n\|_1 = 1$ while $\|e_n\| = \gamma(n) \to \infty$. We claim that $L_B : A \to A$ is not $\|\cdot\|$-continuous. For $k, n \in \mathbb{N}$ let α_k be such that $\alpha_k \gamma(k) = \frac{1}{k^2}$ and define $a_n := \sum_{k=1}^{n} \alpha_k e_k$. Then,

$$\|a_n\| = \|F(a_n)\|_1 = \left\| F(\sum_{k=1}^{n} \alpha_k e_k) \right\|_1 = \left\| \sum_{k=1}^{n} \alpha_k \gamma(k) e_k \right\|_1 =$$

$$\sum_{k=1}^{n} |\alpha_k \gamma(k)| = \sum_{k=1}^{n} \frac{1}{k^2} < \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

On the other hand,

$$\begin{aligned} \|L_B(a_n)\| &= \left\| \sum_{k=1}^{n} \alpha_k e_k^2 \right\| = \left\| F(\sum_{k=1}^{n} \alpha_k e_k^2) \right\|_1 = \left\| F(\sum_{k=1}^{n} \alpha_k k e_k) \right\|_1 \\ &= \left\| \sum_{k=1}^{n} \alpha_k k \gamma(k) e_k \right\|_1 = \sum_{k=1}^{n} |\alpha_k k \gamma(k)| = \sum_{k=1}^{n} \frac{1}{k}. \end{aligned}$$

Therefore the sequence $L_B(a_n)$ is not $\| \cdot \|$-bounded, which proves the claim. ■

The next result provides a sufficient condition for continuity of L_B.

Proposition 16 *Let A be a normed evolution algebra and $B = \{e_i : i \in \Lambda\}$ be a unital natural basis. If $\sup\{\| \sum_{i \in F} e_i \| : F \subset \Lambda, F$ finite$\} < \infty$ then L_B is continuous.*

Proof. Let $M := \sup\{\| \sum_{i \in F} e_i \| : F \subset \Lambda, F \text{ finite}\}$. If $a = \sum_{i \in \Lambda_a} \alpha_i e_i$ then

$$\|L_B(a)\| = \| \sum_{i \in \Lambda_a} \alpha_i e_i^2 \| = \|(\sum_{i \in \Lambda_a} e_i) a\| \leq \| \sum_{i \in \Lambda_a} e_i\| \, \|a\| \leq M \, \|a\| \, ,$$

as desired. ■

4 Dynamics of the evolution operator

Corollary 11 above shows that non-degenerate infinite-dimensional Banach evolution algebras do not exist, so we assume henceforth that A is a finite dimensional normed evolution algebra with given algebra norm, $\| \cdot \|$.

Throughout, $L(A)$ denotes the algebra (under function composition) of all linear maps on A endowed with the usual operator norm; L_a denotes the multiplication operator, $L_a(b) = ab$, for $a, b \in A$, while $M_{n,m}$ is the space of all $n \times m$ matrices over $\mathbb{K}$ and $M_n := M_{n,n}$. Although A is non-associative in general and a^m is therefore not well defined for $a \in A$, $L(A)$ is an associative algebra and we may therefore consider the iterates of L_a, namely, $L_a^1 := L_a$ and $L_a^m := L_a \circ L_a^{m-1}$, for $m \geq 2$.

Definition 17 Let $B = \{e_1, \cdots, e_n\}$ be a fixed natural basis of A and $e := e_1 + \cdots + e_n$. We call e the evolution element of B.

Since A is finite dimensional, the evolution operator L_B of A (with respect to the basis B) is the multiplication operator L_e, cf. Remark 13.

For $b \in A$, we may postulate, to some extent, $(L_e^m(b))_m$ as a discrete time dynamical system, whose limit points may help to describe the long-term evolutionary state of b. Our goal therefore is to determine when $(L_e^m(b))_m$ converges and, more crucially, to then locate its limit. In fact, the role played by e is not so central, since any non-zero multiple of e is an evolution element for another basis. We examine therefore the more general question of the convergence or otherwise of the sequence $(L_a^m(b))_m$, and the determination of the limit where it exists, for arbitrary $a, b \in A$.

Definition 18 We say $a \in A$ *is an equilibrium generator if* $(L_a^m(b))_{m\in\mathbb{N}}$ *converges, for all* $b \in A$.

We note that since A is finite dimensional, all norms on A are equivalent so the definition is independent of the choice of norm on A.

Let $M_A(B) = (\omega_{ij})_{ij} \in M_n$ be the evolution matrix of A with respect to B, as described in section 2. It is straightforward to check that, for $a = \sum\limits_{i=1}^{n} \alpha_i e_i$, the matrix of L_a with respect to B is given by

$$W_a^B := \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n \end{pmatrix}. \tag{2}$$

We call W_a^B the evolution matrix of a (with respect to B) and write $W_a := W_a^B$ when the basis is clear from the context. We note that W_e is $M_B(A)$. As usual we write $\sigma(W_a)$ for the set of eigenvalues of W_a and $\rho(W_a)$ for its spectral radius.

We recall a concept of spectrum for non-associative algebras, introduced in [20] for general algebras and in [31] for evolution algebras and to which we refer for all details (see also [18, 19, 30]). We recall for a complex algebra E that $a \in E$ is said to be m-invertible if L_a and R_a are bijective, where R_a denotes the right multiplication map $R_a(b) = ba$, for $a, b \in A$.

Definition 19 Let E be a complex algebra with unit e. The m-spectrum of a *in* E is

$$\sigma_m^E(a) := \{\lambda \in \mathbb{C} : a - \lambda e \textit{ is not m-invertible}\}.$$

If E is a complex algebra without unit, $\sigma_m^E(a) := \sigma_m^{E_1}(a)$, where E_1 denotes the unitization of E and if E is a real algebra $\sigma_m^E(a) := \sigma_m^{E_{\mathbb{C}}}(a)$, where $E_{\mathbb{C}}$ denotes the complexification of E.

When the context is clear, we write $\sigma_m(a)$ for $\sigma_m^E(a)$. For a linear map $T : E \to E$, $\sigma(T)$ denotes its usual spectrum

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bijective}\}.$$

Then $\sigma_m(a) = \sigma(L_a) \cup \sigma(R_a)$ whenever E is unital and, otherwise, $\sigma_m(a) = \sigma(L_a) \cup \sigma(R_a) \cup \{0\}$. Thus, for commutative A, and evolution algebras in particular, we have $\sigma_m(a) = \sigma(L_a)$ if A is unital and $\sigma(L_a) \cup \{0\}$ otherwise. We recall [31, Corollary 2.12] that an evolution algebra A is unital if, and only if, A is a finite-dimensional non-zero trivial evolution algebra.

Definition 20 The m-spectral radius of $a \in E$ is $\rho(a) := \sup\{|\lambda| : \lambda \in \sigma_m(a)\}$ if $\sigma_m(a) \neq \emptyset$ and $\rho(a) := 0$ otherwise.

An m-spectral radius formula is given in [20, Proposition 2.2].

Returning now to evolution algebras we note that if E is a real evolution algebra, its complexification, $E_{\mathbb{C}}$, is also an evolution algebra and every natural basis of E is a natural basis of $E_{\mathbb{C}}$ so that L_B can also be regarded as an element of $L(E_{\mathbb{C}})$. In particular, we have the following, stated implictly in [31, Proposition 5.1 and Proposition 5.3].

Proposition 21 *Let A be a finite-dimensional evolution algebra with natural basis $B = \{e_1, ..., e_n\}$ and $a \in A$. Let W_a be the evolution matrix of a with respect to B. Then $\sigma_m(a) = \sigma(W_a)$ if A is unital and $\sigma_m(a) = \sigma(W_a) \cup \{0\}$ otherwise.*

Definition 22 *Let ϕ be the natural isomorphism from A to $\mathbb{C}^n$ given by*

$$\phi(\sum_i \beta_i e_i) = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}.$$

From (2) above we have $\phi(L_a(b)) = W_a\phi(b)$ for $a, b \in A$, or equivalently, as operators, $L_a = \phi^{-1}W_a\phi$ and hence by induction

$$L_a^m = \phi^{-1}W_a^m\phi \tag{3}$$

for $a \in A$ and $m \in \mathbb{N}$. Since the spectrum of a linear map is independent of its matrix representation $\sigma(L_a) = \sigma(W_a)$ and hence from Proposition 21

$$\rho(a) = \rho(L_a) = \rho(W_a).$$

Using the natural isomorphism ϕ, every norm on A induces a corresponding norm on $\mathbb{C}^n$ by $\|x\| := \|\phi^{-1}(x)\|$, for $x \in \mathbb{C}^n$, and for this norm on $\mathbb{C}^n$ the isomorphism ϕ then becomes an isometry. In addition, every norm on A gives a unique operator norm on $L(A)$, namely,

$$\|T\| = \sup_{\|b\|\leq 1} \|T(b)\|, \quad \text{for} \quad T \in L(A), \; b \in A.$$

In fact, since A is finite dimensional this supremum is achieved. In exactly the same way, every norm on $\mathbb{C}^n$ (and, in particular, the norm induced from A via ϕ above) gives a unique operator norm on $L(\mathbb{C}^n)$ and we may identify $L(\mathbb{C}^n)$ with M_n in the usual way.

Of course, for any algebra norm on A, we have

$$\|L_a^m\| \leq \|L_a\|^m \leq \|a\|^m, \text{ for all } m \in \mathbb{N},$$

so that $\|a\| < 1$ implies $\lim_m L_a^m = 0$. Furthermore, we get the following (for the given norm on A and the above induced norms on $\mathbb{C}^n$, $L(A)$ and M_n respectively).

Proposition 23 *Let A be a finite dimensional evolution algebra. Let W_a be the evolution matrix of $a \in A$ with respect to a fixed natural basis B. The following are equivalent.*

(i) *a is an equilibrium generator, that is, $(L_a^m(b))$ converges, for all $b \in A$;*

(ii) *$\lim_m L_a^m$ exists in $L(A)$;*

(iii) *$\lim_m W_a^m$ exists in M_n.*

(iv) *$\lim_m (W_a^m)_{ij}$ exists, for all $1 \leq i, j \leq n$ (where T_{ij} denotes the ij coordinate of $T \in M_n$).*

Proof. Equivalence of (ii) and (iii) is immediate from (3) above. The operator norm on M_n is equivalent to the norm defined co-ordinatewise by $\|T\| := \max_{1\leq i,j\leq n} |T_{ij}|$ giving (iii) equivalent to (iv). Clearly (ii) implies (i). We finish by showing (i) implies (ii). Assume therefore that $(L_a^m(b))_m$ converges, for all $b \in A$. Define $T : A \to A$ by $T(b) := \lim_m L_a^m(b)$. Clearly T is linear and hence bounded. Moreover since, for all $b \in A$, $\sup_m \|L_a^m(b)\| < \infty$, the uniform boundedness principle implies $\sup_m \|L_a^m\| < \infty$ and, in fact, $\|T\| \leq \sup_m \|L_a^m\|$. We finish with a standard compactness argument, given for completeness. Let $K = \sup_m \|L_a^m\|$. Fix ϵ arbitrary. By compactness of $D = \{x \in A : \|x\| \leq 1\}$ there exists $x_1, \ldots, x_k \in D$ such that

$$D \subset \bigcup_{j=1}^{k} B(x_j, \epsilon/K)$$

where $B(x, \alpha) = \{y \in A : \|x - y\| < \alpha\}$. For $1 \leq j \leq k$, there exists M_j such that $\|L_a^m(x_j) - T(x_j)\| < \epsilon$, for all $m \geq M_j$. Let $M := \max_j M_j$. Now take $x \in D$ and $m > M$. Then $x \in B(x_j, \epsilon/K)$ for some $1 \leq j \leq k$ and therefore

$$\begin{aligned}
\|L_a^m(x) - T(x)\| &\leq \|L_a^m(x) - L_a^m(x_j)\| + \|L_a^m(x_j) - T(x_j)\| + \|T(x_j) - T(x)\| \\
&\leq \|L_a^m\|\|x - x_j\| + \|L_a^m(x_j) - T(x_j)\| + \|T\|\|x - x_j\| \\
&\leq K(\epsilon/K) + \epsilon + K(\epsilon/K) = 3\epsilon
\end{aligned}$$

giving the result. ■

The concept of equilibrium generator is clearly independent of the natural basis chosen. As mentioned earlier however, given two evolution elements e and $\tilde{e}$ (corresponding to different bases), one may be an equilibrium generator, while the other may not, as the following example further demonstrates.

Example 24 Let A be the linear span of e_1 and e_2 with multiplication defined by $e_1e_2 = e_2e_1 = 0$ and $e_1^2 = e_2^2 = e_1$. Then A is an evolution algebra with natural basis $B = \{e_1, e_2\}$ and evolution element $e = e_1 + e_2$. Let now $\tilde{B} = \{\tilde{e}_1, \tilde{e}_2\}$, for $\tilde{e}_1 = e_1 + e_2,\ \tilde{e}_2 = e_1 - e_2$, then $\tilde{B}$ is also a natural basis with evolution element $\tilde{e} = \tilde{e}_1 + \tilde{e}_2 = 2e_1$. Then $W_e = W_e^B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $W_{\tilde{e}} = W_{\tilde{e}}^B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ are the evolution matrices of e and $\tilde{e}$ (each taken with respect to B). Clearly $\lim_m W_e^m = W_e$, while $W_{\tilde{e}}^m = \begin{pmatrix} 2^m & 0 \\ 0 & 0 \end{pmatrix}$ does not converge, so e is an equilibrium generator while $\tilde{e}$ is not.

Therefore, while the concept of equilibrium generator is independent of the basis chosen, the concept of "an evolution element (of a basis) being an equilibrium generator" is not. This suggests, in contrast to comments in [27, Section 3.2.1], that other operators apart from the evolution operator $L_B(= L_e)$ may be more relevant to the study of A. Nonetheless, we introduce the following (basis dependent) definition.

Definition 25 Let A be a finite dimensional evolution algebra with fixed natural basis B. Let e be the evolution element of B. We say that A reaches B-equilibrium if e is an equilibrium generator.

We say that $T \in L(A)$ is a projection if $T^2 = T$ and, similarly, $C \in M_n$ is a projection if $C^2 = C$. Recall that the rank of a linear map T is well-defined as the rank of any matrix representation of T.

Proposition 26 *Let A be a finite dimensional evolution algebra and $a \in A$. If a is an equilibrium generator then $P := \lim_m L_a^m$ commutes with L_a and is a projection onto the subspace $\ker(L_a - I)$. In particular, rank$(P) = \dim P(A) = \dim\ker(L_a - I)$ and if $P \neq 0$ then $1 \in \sigma_m(a)$.*

Proof. Let a be an equilibrium generator. From Proposition 23 then $P = \lim_m L_a^m$ exists in $L(A)$. The subsequence (L_a^{2m}) must then also converge to P, so that by continuity of composition in $L(A)$ we have

$$P = \lim_m L_a^{2m} = \lim_m L_a^m \circ \lim_m L_a^m = P \circ P = P^2.$$

Moreover, for $x \in A$, then

$$L_a(P(x)) = L_a((\lim_m L_a^m)(x)) = (\lim_m L_a^{m+1})(x) = P(x),$$

so that $P(A) \subseteq \ker(L_a - I)$. In particular, if $P \neq 0$ then $\ker(L_a - I) \neq \emptyset$ so $1 \in \sigma(L_a)$ and hence $1 \in \sigma_m(a)$. For $y \in ker(L_a - I)$, we have $y = L_a(y)$, so $y = L_a^m(y)$, for all $m \in \mathbb{N}$ and hence $y = P(y) \in P(A)$ so $\ker(L_a - I) \subseteq P(A)$ giving $P(A) = \ker(L_a - I)$. ■

Proposition 26 motivates the following.

Definition 27 Let A be an evolution algebra and $a \in A$ be an equilibrium generator. We then define the equilibrium subspace of a as $A_a := \ker(L_a - I)$ and we define the equilibrium rank of a as $r(a) := \dim(\ker(L_a - I))$ if $A_a \neq \{0\}$, and $r(a) = 0$ otherwise.

We note from [22] that since $L(A)$ and M_n are finite dimensional the spectral radius function is continuous.

Proposition 28 *Let A be a finite dimensional evolution algebra and $a \in A$.*

(i) *If $\rho(a) > 1$ then a is not an equilibrium generator and, in particular, $(L_a^m)_m$ has no convergent subsequences.*

(ii) $\lim_m L_a^m = 0$ *if, and only if,* $\rho(a) < 1$.

Proof. For (i) let us first suppose that a subsequence $(L_a^{m_k})_k$ converges in $L(A)$, to $\tilde{P}$, say. Then $\rho(\tilde{P}) = \lim_k \rho(L_a^{m_k})$. As A is finite dimensional, it is easy to see from the spectral radius formula that $\rho(L_a^{m_k}) = \rho(L_a)^{m_k}$ so $\rho(\tilde{P}) = \lim_k \rho(L_a^{m_k}) = \lim_k \rho(L_a)^{m_k} = \lim_k \rho(a)^{m_k}$. This is impossible if $\rho(a) > 1$ giving (i).

For (ii) let us assume that $\rho(a) < 1$. Then $\rho(L_a) < 1$ and hence $\|L_a^m\| < 1$, for all m sufficiently large (otherwise $\|L_a^{m_k}\| \geq 1$, for some subsequence $(m_k)_k$, and then $\rho(L_a) = \lim_k \|L_a^{m_k}\|^{1/m_k} \geq 1$). Then L_a^m, for m large, lies in the closed unit ball of $L(A)$ which is compact and thus every subsequence of (L_a^m) has itself a convergent subsequence. Consider the limit of any such convergent subsequence, say, $\tilde{P} := \lim_k L_a^{m_k}$. As in (i) above, $\rho(\tilde{P}) = \lim_k \rho(a)^{m_k}$ and hence $\tilde{P} = 0$. Since the limit of all such convergent subsequences of (L_a^m) is thus 0, it follows by compactness that the sequence (L_a^m) itself must converge also to 0. In other words, $\rho(a) < 1$ implies $\lim_m L_a^m = 0$. In the opposite direction, if $\lim_m L_a^m = 0$ then continuity of the spectral radius gives $0 = \rho(0) = \lim_m \rho(L_a^m) = \lim_m \rho(L_a)^m = \lim_m \rho(a)^m$ and hence $\rho(a) < 1$ and we are done. ■

It remains to examine the case $\rho(a) = 1$. To this end we use the Jordan normal form of a matrix, part of literature folklore [26], but recalled here for convenience.

Proposition 29 *For $W \in M_n$, there exists an invertible matrix Q and Jordan block matrix, J, such that $W = Q^{-1}JQ$, where*

$$J = \begin{pmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_t \end{pmatrix}.$$

Each J_i is a Jordan matrix corresponding to eigenvalue λ_i, that is, a square matrix with λ_i on the diagonal, 1 on the super-diagonal, and zeros elsewhere. Moreover, the eigenvalues of the blocks $J_1, \dots, J_t$, counting multiplicities, are precisely the eigenvalues of the matrix J and hence of W. In particular, for eigenvalue λ_i, we recall the following:

(i) *the geometric multiplicity, $m_g(\lambda_i, W) = \dim(\ker(W - \lambda_i I))$, gives the number of Jordan blocks corresponding to λ_i;*

(ii) *the algebraic multiplicty, $m_a(\lambda_i, W)$, gives the sum of the sizes of all Jordan blocks corresponding to λ_i;*

(iii) *the index, denoted $\nu(\lambda_i, W)$, gives the size of the largest Jordan block corresponding to λ_i.*

(iv) *In particular then $\nu(\lambda_i, W) = 1$ if, and only if,*

$$\dim(\ker(W - \lambda_i I)) = m_g(\lambda_i, W) = m_a(\lambda_i, W).$$

In this case, putting together all the Jordan matrices corresponding to λ_i gives $\lambda_i I_{r_i}$, a diagonal matrix of size $r_i := \dim(\ker(W - \lambda_i I))$.

Since the eigenvalues of W_a determine the multiplicative spectrum $\sigma_m(a)$ of a (Proposition 21 above), the following definitions are natural.

Definition 30 For $a \in A$ and λ an eigenvalue of W_a we define the multiplicative a-index of λ as $\nu(\lambda, a) := \nu(\lambda, W_a)$. If $\lambda \in C$ is not an eigenvalue of W we define the multiplicative a-index of λ as $\nu(\lambda, a) := 0$.

Since W_a is unique up to similarity and the Jordan form is unique up to order of its blocks, the index $\nu(\lambda, a)$ is well defined and independent of the basis.

Proposition 31 *Let A be a finite dimensional evolution algebra and $a \in A$. Let $\rho(a) = 1$. Then a is an equilibrium generator if, and only if, $\sigma_m(a) \cap \partial\Delta = \{1\}$ and $\nu(1, a) = 1$.*

Proof. Let $a \in A$ and $\rho(a) = 1$. Let W_a be the evolution matrix of a as above. Let $\lambda \in \sigma_m(a) \cap \partial\Delta$. From Proposition 21 $\lambda \in \sigma(W_a)$. Consider the Jordan normal form of W_a as above and let J be any Jordan matrix corresponding to λ. The $(1,1)$ entry of J^m is λ^m, for $m \in \mathbb{N}$. Since $\lim_m \lambda^m$ only exists if, and only if, $\lambda = 1$ we have that if $\lambda \neq 1$ then $(J^m)_{1,1}$ cannot converge and hence W_a^m

cannot converge and from Proposition 23 then a is not an equilibrium generator. If on the other hand, $\lambda = 1$ and $\nu(1, a) = \nu(1, W_a) > 1$ it means that there is a Jordan matrix J corresponding to eigenvalue 1 of size $s > 1$. Then J^m has m on its first super diagonal, so $(J^m)_m$ and hence also $(W_a^m)_m$ cannot converge and again a is not an equilibrium generator. In other words, if a is an equilibrium generator then $\sigma_m(a) \cap \partial\Delta = \{1\}$ and $\nu(1, a) = 1$.
In the opposite direction, if $\rho(a) = 1$ and $\sigma_m(a) \cap \partial\Delta = \{1\}$ and $\nu(1, a) = 1$ then, from Proposition 29 (iv), putting all Jordan blocks corresponding to eigenvalue 1 together gives the $r \times r$ identity matrix $I_r \in M_r$, where

$$r = \dim(\ker(W_a - I)) = \dim(\ker(L_a - I)) = r(a).$$

Write $R \oplus T$ for the matrix $\begin{pmatrix} R & 0 \\ 0 & T \end{pmatrix}$, for $R \in M_r$ and $T \in M_{n-r}$. Then we have $W_a = Q^{-1}(I \oplus T)Q$, for $I \in M_r, T \in M_{n-r}$ with $\rho(T) < 1$, and some invertible $Q \in M_n$. Then

$$W_a^m = Q^{-1}(I \oplus T^m)Q.$$

Since $\rho(T) < 1$ gives $\lim_m T^m = 0$ (see Proposition 28 (ii)) we then have $\lim_m W_a^m = Q^{-1}(I \oplus 0)Q$. From (3)

$$\lim_m L_a^m = \lim_m \left(\phi^{-1} \circ W_a^m \circ \phi\right) = \phi^{-1} \circ Q^{-1}(I \oplus 0)Q \circ \phi$$

and we are done. ■

We note that Propositions 28 and 31 can also be derived from Proposition 23 and known results in different formats for matrices, see for instance [23]. Propositions 26, 28 and 31 together now give the following.

Theorem 32 *Let A be a finite-dimensional evolution algebra and $a \in A$. Then a is an equilibrium generator if, and only if,*

$$\sigma_m(a) \subseteq \Delta \cup \{1\} \quad \text{and } \nu(1, a) \leq 1.$$

Moreover, if a is an equilibrium generator, then $P = \lim_m L_a^m$ is projection onto the a-equilibrium subspace $A_a = \ker(L_a - I)$, and if $\nu(1, a) = 0$ then $P = 0$.

Note that if $\rho(a) < 1$ then trivially $\nu(1, a) = 0$.

Corollary 33 *Let A be a finite-dimensional evolution algebra with evolution element e with respect to a natural basis B. Then A reaches B-equilibrium if, and only if,*

$$\sigma_m(e) \subseteq \Delta \cup \{1\} \quad \text{and } \nu(1, e) \leq 1.$$

The following two corollaries are reformulations of the above two results using the Jordan normal form and, in particular, (iv) of Proposition 29. Recall also definitions 25 and 27.

We write $I_r \oplus T$ to denote the matrix $\begin{pmatrix} I_r & 0 \\ 0 & T \end{pmatrix}$ if $r \neq 0$ and $I_r \oplus T = T$ if $r = 0$.

Corollary 34 *Let A be a finite dimensional evolution algebra and $a \in A$. Then $a \in A$ is an equilibrium generator if, and only if, its evolution matrix W_a (with respect to any basis B) is similar to a matrix of the form*

$$I_r \oplus T, \text{ where } \rho(T) < 1, \ r = r(a), \ I_r \in M_r, \ T \in M_{n-r}, \ n = \dim(A).$$

Corollary 35 *Let A be a finite dimensional evolution algebra and $a \in A$. Then a is an equilibrium generator if, and only if,*

$$L_a = P + S, \quad \text{for linear maps } P, S \text{ in } L(A)$$

satisfying $P^2 = P$, $PS = SP = 0$ and $\rho(S) < 1$.

Proof. If a is an equilibrium generator then from Corollary 34 and the proof of Theorem 31, there is an invertible matrix $Q \in M_n$ such that

$$W_a = Q^{-1} (I_r \oplus T) Q,$$

where $\rho(T) < 1, r = r(a), I_r \in M_r$, $T \in M_{n-r}$. Then, from (3),

$$L_a = \phi^{-1} Q^{-1} (I \oplus T) Q \phi.$$

Let

$$P := \phi^{-1} Q^{-1} (I_r \oplus 0) Q \phi \text{ and } S := \phi^{-1} Q^{-1} (0 \oplus T) Q \phi,$$

(recall $P = 0$ if $r = 0$).
Then $L_a = P + S$ and it is easy to see that P and S have the required properties. In the opposite direction, if $L_a = P + S$ with properties as stated then $L_a^m = P + S^m$ and, since $\rho(S) = \rho(T) < 1$, then $\lim_m S^m = 0$ giving $\lim_m L_a^m = P$ and a is an equilibrium generator. ■

We now examine the situation where a type of recurrent behaviour can arise, namely, when W_a has eigenvalues that are p-th roots of unity. Let

$$\Omega = \{e^{\frac{2\pi i}{p}} : p \in \mathbb{N}\}.$$

Lemma 36 *Let $W \in M_n$ with $\sigma(W) \subset \Delta \cup \Omega$ and*

$$\sigma(W) \cap \Omega = \{e^{\frac{2\pi i}{p_1}}, \ldots, e^{\frac{2\pi i}{p_s}}\} \neq \emptyset.$$

Let $\lambda_k := e^{\frac{2\pi i}{p_k}}$, $1 \leq k \leq s$. If

$$\nu(\lambda_k, W) = 1, \quad \text{for } 1 \leq k \leq s$$

then for any choice of i and k_i, $1 \le i \le s$, $1 \le k_i \le p_i$, there exists a subsequence $(m_l)_l$ of $\mathbb{N}$ and coefficients $\alpha_j \in \{1, \lambda_j, \ldots, \lambda_j^{p_j-1}\}$, for $j \ne i$, $1 \le j \le s$, such that

$$\lim_l W^{m_l} = \lambda_i^{k_i} \tilde{P}_i + \sum_{j \ne i} \alpha_j \tilde{P}_j$$

where $\tilde{P}_1, \ldots, \tilde{P}_s$ are mutually orthogonal projections onto the eigenspaces of W for $\lambda_1, \ldots, \lambda_s$ respectively.

Proof. Let $W \in M_n$ satisfy the conditions in the statement of the lemma. We note that the case $s = 1, p_1 = 1$ is covered by Theorem 32. Writing $R_1 \oplus \cdots \oplus R_t$ for the block diagonal matrix with blocks $R_1, \ldots, R_t$, Proposition 29 gives an invertible $Q \in M_n$ such that

$$W = Q^{-1} J Q, \quad \text{and} \quad J = J_1 \oplus \cdots \oplus J_t,$$

where each $J_1, \ldots, J_t$ is a Jordan matrix corresponding to some eigenvalue of W, $1 \le t \le n$. Let $\lambda_k := e^{\frac{2\pi i}{p_k}}$, for $1 \le k \le s$, as in the statement. Since $\nu(\lambda_k, W) = 1$, Proposition 29 (iv) implies that putting together all Jordan matrices corresponding to eigenvalue λ_k gives a diagonal matrix $\lambda_k I_{r_k}$ of size $r_k := \dim(\ker(W - \lambda_k I))$.

If $\sum_{k=1}^{s} r_k < n$ then $\sigma(W) \cap \Delta \ne \emptyset$. In this case, for $q = n - \sum_{k=1}^{s} r_k$, then putting together all Jordan matrices corresponding to eigenvalues in Δ gives a matrix $T \in M_q$ (also block diagonal) with $\rho(T) < 1$. We may therefore assume, without loss of generality, that

$$W = Q^{-1} \left(\lambda_1 I_{r_1} \oplus \cdots \oplus \lambda_s I_{r_s} \oplus T \right) Q, \quad \text{if} \quad q \ne 0$$

and

$$W = Q^{-1} \left(\lambda_1 I_{r_1} \oplus \cdots \oplus \lambda_s I_{r_s} \right) Q, \quad \text{if} \quad q = 0.$$

Fix now i and k_i, $1 \le i \le s$, $1 \le k_i \le p_i$. Then

$$(\lambda_i I_{r_i})^{k_i + m p_i} = \lambda_i^{k_i} I_{r_i}, \quad \text{for all } m \in \mathbb{N}.$$

Moreover, for $1 \le k \le s$, each of the following sets is finite

$$\{(\lambda_k I_{r_k})^m : m \in \mathbb{N}\} = \{I_{r_k}, \lambda_k I_{r_k}, \ldots, \lambda_k^{p_k - 1} I_{r_k}\}.$$

Therefore there is a subsequence $(m_l)_l$ of $(k_i + m p_i)_m$ such that for all $j \ne i$, $1 \le j \le s$, there is $\alpha_j \in \{1, \lambda_j, \ldots, \lambda_j^{p_j - 1}\}$ with

$$\left(\lambda_j I_{r_j} \right)^{m_l} = \alpha_j I_{r_j}.$$

Of course, α_j may depend on the fixed i and k_i chosen. For convenience (re-ordering if necessary), we'll assume $i = 1$. Then

$$(\lambda_1 I_{r_1} \oplus \cdots \oplus \lambda_s I_{r_s} \oplus T)^{m_l} = \lambda_1^{k_1} I_{r_1} \oplus \alpha_2 I_{r_2} \oplus \cdots \oplus \alpha_s I_{r_s} \oplus T^{m_l}, \quad \text{if } q \ne 0$$

and equals

$$\lambda_1^{k_1} I_{r_1} \oplus \alpha_2 I_{r_2} \oplus \cdots \oplus \alpha_s I_{r_s}, \ \text{ if } q = 0.$$

Let now, for $1 \leq k \leq s$,

$$\tilde{P}_k := Q^{-1}\left(0_{r_1} \oplus \cdots 0_{r_{k-1}} \oplus I_{r_k} \oplus 0_{r_{k+1}} \cdots \oplus 0_{r_s} \oplus 0_q\right) Q \in M_n, \text{ if } q \neq 0$$

and

$$\tilde{P}_k := Q^{-1}\left(0_{r_1} \oplus \cdots 0_{r_{k-1}} \oplus I_{r_k} \oplus 0_{r_{k+1}} \cdots \oplus 0_{r_s}\right) Q, \text{ if } q = 0.$$

Clearly then $\tilde{P}_1, \ldots, \tilde{P}_s$ are mutually orthogonal projections in M_n (and $\tilde{P}_k(\mathbb{C}^n)$ is exactly the λ_k-eigenspace of W). Then

$$W^{m_l} = \lambda_1^{k_1} \tilde{P}_1 + \alpha_2 \tilde{P}_2 + \cdots + \alpha_s \tilde{P}_s + Q^{-1}\left(0_{n-q} \oplus T^{m_l}\right) Q, \text{ for } l \in \mathbb{N}, \text{if } q \neq 0$$

and

$$W^{m_l} = \lambda_1^{k_1} \tilde{P}_1 + \alpha_2 \tilde{P}_2 + \cdots + \alpha_s \tilde{P}_s, \ \text{ for all } l \in \mathbb{N}, \text{ if } q = 0. \tag{4}$$

Of course, if $q \neq 0$, then $T \in M_q$ has $\rho(T) < 1$ and hence $\lim_l T^{m_l} = 0$ giving the required result. ■

Lemma 36 also covers the case where the spectrum contains only p-th roots of unity and since then $q = 0$ the next result follows immediately from (4).

Corollary 37 *Let $W \in M_n$ with $\sigma(W) \subset \Omega$ and*

$$\sigma(W) \cap \Omega = \{e^{\frac{2\pi i}{p_1}}, \ldots, e^{\frac{2\pi i}{p_s}}\}.$$

Let $\lambda_k := e^{\frac{2\pi i}{p_k}}$, $1 \leq k \leq s$. If

$$\nu(\lambda_k, W) = 1, \ \ \textit{for } \ 1 \leq k \leq s$$

then for any choice of i and k_i, $1 \leq i \leq s$, $1 \leq k_i \leq p_i$, there exists a subsequence $(m_l)_l$ of $\mathbb{N}$ and coefficients $\alpha_j \in \{1, \lambda_j, \ldots, \lambda_j^{p_j-1}\}$, for $j \neq i$, $1 \leq j \leq s$, such that

$$W^{m_l} = \lambda_i^{k_i} \tilde{P}_i + \sum_{j \neq i} \alpha_j \tilde{P}_j, \ \ \textit{for all } l \in \mathbb{N} \tag{5}$$

where $\tilde{P}_1, \ldots, \tilde{P}_s$ are mutually orthogonal projections onto the eigenspaces of W for $\lambda_1, \ldots, \lambda_s$ respectively.

Lemma 36 and Proposition 23 now give the following.

Theorem 38 *Let A be a finite-dimensional evolution algebra and $a \in A$ with $\sigma_m(a) \subset \Delta \cup \Omega$ and*

$$\sigma_m(a) \cap \Omega = \{e^{\frac{2\pi i}{p_1}}, \ldots, e^{\frac{2\pi i}{p_s}}\} \neq \emptyset.$$

Let $\lambda_k := e^{\frac{2\pi i}{p_k}},\ 1 \le k \le s$. *If*

$$\nu(\lambda_k, a) = 1, \quad \textit{for} \ \ 1 \le k \le s$$

then for any choice of i *and* k_i, $1 \le i \le s,\ 1 \le k_i \le p_i$, *there exists a subsequence* $(m_l)_l$ *of* $\mathbb{N}$ *and coefficients* $\alpha_j \in \{1, \lambda_j, \ldots, \lambda_j^{p_j-1}\}$, *for* $j \ne i,\ 1 \le j \le s$, *such that*

$$\lim_l L_a^{m_l} = \lambda_i^{k_i} P_i + \sum_{j \ne i} \alpha_j P_j$$

where $P_1, \ldots, P_s$ *are mutually orthogonal projections onto the* L_a*-eigenspaces for* $\lambda_1, \ldots, \lambda_s$ *respectively.*

Proof. Let $a \in A$ satisfy the conditions in the statement of the theorem and let W_a be its evolution matrix with respect to a fixed natural basis. Then W_a satisfies the conditions of Lemma 36. Fixing i and k_i, $1 \le i \le s, 1 \le k_i \le p_i$, Lemma 36 then yields a subsequence $(m_l)_l$ of $\mathbb{N}$, mutually orthogonal projection matrices $\tilde{P}_1, \ldots, \tilde{P}_s$ and scalars $\alpha_j \in \{1, \lambda_j, \ldots, \lambda_j^{p_j-1}\}$, for $j \ne i,\ 1 \le j \le s$, such that (5) holds, namely,

$$\lim_l W_a^{m_l} = \lambda_i^{k_i} \tilde{P}_i + \sum_{j \ne i} \alpha_j \tilde{P}_j.$$

For $1 \le k \le s$, let

$$P_k := \phi^{-1} \circ \tilde{P}_k \circ \phi, \tag{6}$$

where ϕ is the isometry in (3) above. It follows that $P_1, \ldots, P_s$ are mutually orthogonal projections onto the L_a-eigenspaces for $\lambda_1, \ldots, \lambda_s$ respectively. Proposition 23 and (3) then give

$$\lim_l L_a^{m_l} = \lambda_i^{k_i} P_i + \sum_{j \ne i} \alpha_j P_j$$

as required. ∎

If $\sigma_m(a)$ contains only p-th roots of unity the next result follows from (6) in Theorem 38 and (4) in Lemma 36 above.

Corollary 39 *Let* A *be a finite-dimensional evolution algebra and* $a \in A$ *with* $\sigma_m(a) \subset \Omega$ *and*

$$\sigma_m(a) \cap \Omega = \{e^{\frac{2\pi i}{p_1}}, \ldots, e^{\frac{2\pi i}{p_s}}\}.$$

Let $\lambda_k := e^{\frac{2\pi i}{p_k}},\ 1 \le k \le s$. *If*

$$\nu(\lambda_k, a) = 1, \quad \textit{for} \ \ 1 \le k \le s$$

then for any choice of i *and* k_i, $1 \le i \le s, 1 \le k_i \le p_i$, *there exists a subsequence* $(m_l)_l$ *of* $\mathbb{N}$ *and coefficients* $\alpha_j \in \{1, \lambda_j, \ldots, \lambda_j^{p_j-1}\}$, *for* $j \ne i,\ 1 \le j \le s$, *such that*

$$L_a^{m_l} = \lambda_i^{k_i} P_i + \sum_{j \ne i} \alpha_j \ P_j, \quad \textit{for all } l \in \mathbb{N} \tag{7}$$

where $P_1, \ldots, P_s$ are mutually orthogonal projections onto the L_a-eigenspaces for $\lambda_1, \ldots, \lambda_s$ respectively.

In Corollary 39 above, for fixed i and k_i, $1 \leq i \leq s, 1 \leq k_i \leq p_i$ then the subsequence in $L(A)$ obtained in (7) above, namely,

$$L_{a,i,k_i} := (L_a^{m_l})_l$$

is constant. In particular, this means that for all $b \in A$, the sequence $(L_a^n(b))_n$ will return to the value $L_{a,i,k_i}(b)$ infinitely often. Borrowing from the languauge of Markov processes, we would say that $L_{a,i,k_i}(b)$ is a recurrent state of the system.

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