TWO-PHASE QUADRATURE DOMAINS

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ABSTRACT. Recent work on two-phase free boundary problems has led to the investigation of a new type of quadrature domain for harmonic functions. This paper develops a method of constructing such quadrature domains based on the technique of partial balayage, which has proved to be a useful tool in the study of one-phase quadrature domains and Hele-Shaw flows.

1. INTRODUCTION

Let Ω be a bounded open set in Euclidean space $\mathbb{R}^N$ ($N \geq 2$), let μ be a positive (Radon) measure with compact support in Ω, and let λ denote Lebesgue measure on $\mathbb{R}^N$. We say that Ω is a (one-phase) quadrature domain for harmonic functions with respect to μ if

$$\int_\Omega h d\lambda = \int h d\mu \quad \text{for every integrable harmonic function } h \text{ on } \Omega. \qquad (1)$$

(Some papers allow μ to be a signed measure, but it has now been shown [8] that this does not give any greater generality.) Let $U\mu$ denote the Newtonian (or logarithmic, if $N = 2$) potential of μ, normalized so that $-\Delta U\mu = \mu$ in the sense of distributions. Then (1) is equivalent to saying that

$$U(\lambda|_\Omega) = U\mu \quad \text{and} \quad \nabla U(\lambda|_\Omega) = \nabla U\mu \quad \text{on} \quad \Omega^c,$$

where $\Omega^c = \mathbb{R}^N \backslash \Omega$. The strong connection between quadrature domains and free boundary theory becomes clear from consideration of the function $U\mu - U(\lambda|_\Omega)$. For background information on quadrature domains we refer to the survey of Gustafsson and Shapiro [11].

Recent work [14], [15] on two-phase free boundary problems has led Emamizadeh, Prajapat and Shahgholian [5] to propose the study of two-phase quadrature domains, which we define as follows.

Definition 1.1. *Let Ω^+, Ω^- be disjoint bounded open sets in $\mathbb{R}^N$, and μ^+, μ^- be positive measures with compact supports in Ω^+, Ω^- respectively. If*

$$U(\lambda|_{\Omega^+} - \lambda|_{\Omega^-}) = U(\mu^+ - \mu^-) \quad on \quad (\Omega^+ \cup \Omega^-)^c, \qquad (2)$$

then we say that the pair (Ω^+, Ω^-) is a two-phase quadrature domain for harmonic functions with respect to (μ^+, μ^-).

As will be explained in Section 3, such a pair (Ω^+, Ω^-) has the property that

$$\int h d(\mu^+ - \mu^-) = \int_{\Omega^+} h d\lambda - \int_{\Omega^-} h d\lambda$$

for every $h \in C(\overline{\Omega^+ \cup \Omega^-})$ that is harmonic on $\Omega^+ \cup \Omega^-$, $\qquad (3)$

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where $C(A)$ denotes the collection of all real-valued continuous functions on a set A. Further, any pair (Ω^+, Ω^-) of disjoint bounded open sets satisfying (3) can be modified, by the addition of a polar set to each if necessary, to form a two-phase quadrature domain.

Trivial examples of two-phase quadrature domains arise when Ω^+ and Ω^- are disjoint one-phase quadrature domains with respect to μ^+ and μ^-, respectively. If we had also required matching gradients on $(\Omega^+ \cup \Omega^-)^c$ in (2), or that the equality in (3) holds for all integrable harmonic functions on $\Omega^+ \cup \Omega^-$, the discussion would end here. However, the above definition admits more interesting examples. We denote by δ_x the unit measure concentrated at a point x.

Example 1. *In $\mathbb{R}^2$ the pair $(\{|x| < 1\}, \{1 < |x| < 2\})$ is a two-phase quadrature domain with respect to $(\pi a \delta_0, \mu_a)$, for any $a \geq 0$, where μ_a has total mass $(2 + a)\pi$ uniformly distributed on the circle $\{\log |x| = (8 \log 2 - 3)/(4 + 2a)\}$. This follows readily from the fact that the mean value of $\log |y - \cdot|$ over $\{|x| = r\}$ is given by $\max\{\log |y|, \log r\}$.*

Example 2. *Let $\mu^+ = 4\delta_p$ and $\mu^- = 4\delta_{-p}$, where $p = (0, 1) \in \mathbb{R}^2$. There is a bounded domain Ω^+ contained in the upper half-plane S^+, and a measure ν on $\overline{S^+}$, such that $\nu|_{S^+} = \lambda|_{\Omega^+}$ and $\nu|_{\partial S^+} \neq 0$, and the function $v = U\mu^+ - U\nu$ vanishes outside Ω^+. (See Section 3 for details.) We define $\Omega^- = \{(x, y) : (x, -y) \in \Omega^+\}$ and*

$$u(x, y) = \left\{ \begin{array}{ll} v(x, y) & \text{if } y \geq 0 \\ -v(x, -y) & \text{if } y < 0 \end{array} \right. .$$

Then

$$u = U(\mu^+ - \mu^-) - U(\lambda|_{\Omega^+} - \lambda|_{\Omega^-}) = 0 \quad \text{on} \quad (\Omega^+ \cup \Omega^-)^c, \qquad (4)$$

and so (Ω^+, Ω^-) is a two-phase quadrature domain with respect to (μ^+, μ^-).

Example 3. *Let $\mu^+ = 4(\delta_q + \delta_{-q})$ and $\mu^- = 4(\delta_r + \delta_{-r})$, where $q = (1, 1)$ and $r = (-1, 1)$. There is a bounded domain R^+ contained in $T^+ = \{x > 0, y > 0\}$, and a measure ν on $\overline{T^+}$ such that $\nu|_{T^+} = \lambda|_{R^+}$ and $\nu|_{\partial T^+} \neq 0$, and the function $w = U(4\delta_q) - U\nu$ vanishes outside R^+. (The measure ν is symmetric about the line $y = x$.) We define $\Omega^+ = R^+ \cup (-R^+)$ and $\Omega^- = \{(x, y) : (x, -y) \in \Omega^+\}$, and then*

$$u(x, y) = \left\{ \begin{array}{ll} w(|x|, |y|) & \text{if } xy \geq 0 \\ -w(|x|, |y|) & \text{if } xy < 0 \end{array} \right. .$$

Then (4) again holds, and so (Ω^+, Ω^-) is a two-phase quadrature domain with respect to (μ^+, μ^-).

The above examples, which have obvious analogues in higher dimensions, all involve either spheres or hyperplanes. It is far from clear how to construct more general examples. The purpose of this paper is to take up a suggestion in [5] and make a potential theoretic analysis of two-phase quadrature domains. In particular, we will provide a method for constructing such pairs (Ω^+, Ω^-) for suitable given measures (μ^+, μ^-). We will also give sufficient conditions on (μ^+, μ^-) for the existence of such quadrature domains. Our approach is inspired by the method of partial balayage that has proved very useful in the construction of one-phase quadrature domains for (sub)harmonic functions. However, significantly new arguments are required for the two-phase case, as will become clear below. We note that the paper [5] allows weighted Lebesgue measure, in place of λ, in the definition of two-phase quadrature domains. We will restrict our attention to the unweighted case for the sake of exposition.

2. Key tools

2.1. **Partial balayage.** Here we recall some basic facts about the notion of (one-phase) partial balayage, which was originally developed by Gustafsson and Sakai [10]. A recent exposition of it may be found in [9]. For an open set $D \subset \mathbb{R}^N$ and a positive measure μ with compact support in D we define

$$V_D\mu(x) = \sup\left\{ v(x) : v \text{ is subharmonic on } D \text{ and } v \le U\mu + \frac{|\cdot|^2}{2N} \text{ on } \mathbb{R}^N \right\} - \frac{|x|^2}{2N}$$

and then put $B_D\mu = -\Delta V_D\mu$. It turns out that there is a measure ν such that

$$B_D\mu = \lambda|_{\omega(D,\mu)} + \mu|_{\omega(D,\mu)^c} + \nu = \lambda|_{\Omega(D,\mu)} + \mu|_{\Omega(D,\mu)^c} + \nu, \tag{5}$$

where

$$\omega(D,\mu) = \{V_D\mu < U\mu\}$$

and

$$\Omega(D,\mu) = \bigcup\{U : U \subset D \text{ open and } B_D\mu = \lambda \text{ in } U\},$$

and these are bounded open subsets of D. (Clearly $V_D\mu = U\mu$ on D^c.) Further,

$$B_D\mu \le \lambda \quad \text{on } D \quad \text{and} \quad \nu \ge 0, \tag{6}$$

and ν is supported by $\partial D \cap \partial \omega(D,\mu)$. We note that $\omega(D,\mu) \subset \Omega(D,\mu)$ and that this inclusion may be strict, even when μ has compact support contained in $\Omega(D,\mu)$. Clearly these sets increase as D increases and as μ increases. It will be convenient to define

$$W_D\mu = U\mu - V_D\mu,$$

whence $W_D\mu$ is lower semicontinuous, $-\Delta W_D\mu \ge \mu - \lambda$ on D and $W_D\mu \ge 0$ on $\mathbb{R}^N$. Finally, if $D = \mathbb{R}^N$, we will abbreviate the above notation to $V\mu$, $B\mu$, $\omega(\mu)$, $\Omega(\mu)$, and $W\mu$, respectively. In this case, $\nu = 0$.

For the reader's convenience, we note that the other notation used in this paper is introduced at the following points:

- Section 1: $\lambda, U\mu, C(A), \delta_x$;
- Section 2.2: μ_d, μ_c;
- Section 2.3: $G_\Omega\mu, \mu^A, \widetilde{U}$;
- Section 3: $B_r(x)$;
- Section 4: $\eta(u,\mu), \tau_\mu, \tau'_\mu, \overline{W}_\mu$.

2.2. **δ-subharmonic functions.** By a δ-subharmonic function on an open set Ω we mean a function w which is representable as $w = s_1 - s_2$, where s_1 and s_2 are subharmonic on Ω. Such a function is, in general, defined only quasi-everywhere on Ω, namely outside the polar set where $s_1 = s_2 = -\infty$. We will refine this observation using the fine topology, that is, the coarsest topology which makes all superharmonic functions continuous. (An introduction to its basic properties may be found in Chapter 7 of [2].) Firstly, as a distribution, $-\Delta w$ is (locally) a signed measure μ, and we may choose the functions s_1, s_2 above so that $\Delta s_1 = \mu^-$ and $\Delta s_2 = \mu^+$, where $\mu^+ - \mu^-$ is the Jordan decomposition of μ. There is a unique decomposition of μ as a sum of signed Radon measures, $\mu = \mu_d + \mu_c$, where μ_d does not charge polar sets and μ_c is carried by a polar set (see, for example, [7]). Clearly $\mu_c^+ \perp \mu_c^-$. If $\mu_c^+ \ne 0$, then by Theorem 1.XI.4 of Doob [4] we have

$$\text{fine}\lim_{x\to y}\frac{w(x)}{U\mu_c^+(x)} = 1 \quad \text{and} \quad \text{fine}\lim_{x\to y}\frac{1}{U\mu_c^+(x)} = 0 \quad \text{for } \mu_c^+\text{-almost every } y \in \Omega.$$

Hence

$$\text{fine}\lim_{x\to y} w(x) = +\infty \quad \text{for } \mu_c^+\text{-almost every } y \in \Omega,$$

and similarly

$$\text{fine} \lim_{x \to y} w(x) = -\infty \quad \text{for } \mu_c^- \text{-almost every } y \in \Omega.$$

Thus we may use fine limits to extend w so that it is defined μ-almost everywhere, and

$$w = +\infty \quad \text{a.e. } (\mu_c^+) \quad \text{and} \quad w = -\infty \quad \text{a.e. } (\mu_c^-). \tag{7}$$

We will always assign values to a δ-subharmonic function in this way.

2.3. **Further potential theoretic background.** In this section Ω is a Greenian domain (that is, Ω has a Green function) and μ is a positive (Radon) measure on Ω such that the associated Green potential $G_\Omega \mu$ exists. By a Borel carrier of μ we mean a Borel set $B \subset \Omega$ such that $\mu(\Omega \backslash B) = 0$.

Theorem 2.1. *Suppose that μ is finite and does not charge polar sets, and let B be a Borel carrier for μ. Then, for each $\varepsilon > 0$, there is a compact set $K \subset B$ such that $\mu(B \backslash K) < \varepsilon$ and $G_\Omega (\mu|_K)$ is finite-valued and continuous.*

The above result is usually stated for the case where $G_\Omega \mu$ is finite-valued, but its proof (see Corollary 4.5.2 in [2]) requires only that $G_\Omega \mu$ is finite μ-almost everywhere, which is certainly true if μ does not charge polar sets.

Theorem 2.2. *Let u, v be positive superharmonic functions on Ω, and μ_1, μ_2 be mutually singular positive measures on Ω. If*

(i) μ_1 does not charge polar sets and $\mu_1 \leq -\Delta u|_{\{u \leq v\}}$, and
(ii) $\mu_2 \leq -\Delta u$ and $\mu_2 \leq -\Delta v$,

then $\mu_1 + \mu_2 \leq -\Delta \min\{u, v\}$.

Proof. We know that $\mu_2 \leq -\Delta \min\{u, v\}$, because

$$\min\{u, v\} = \min\{u - G_\Omega \mu_2, v - G_\Omega \mu_2\} + G_\Omega \mu_2$$

and both $u - G_\Omega \mu_2$ and $v - G_\Omega \mu_2$ are superharmonic when suitably redefined on a polar set, in view of (ii).

Since $\mu_1 \perp \mu_2$ it remains to prove that $\mu_1 \leq -\Delta \min\{u, v\}$. By (i) and Theorem 2.1 we can choose an increasing sequence (K_j) of compact subsets of $\{u \leq v\}$ such that $G_\Omega (\mu_1|_{K_j})$ is continuous for each j, and $\cup K_j$ carries μ_1. We also observe from (i) that $u - G_\Omega (\mu_1|_{K_j})$ is superharmonic on Ω, and clearly $v - G_\Omega (\mu_1|_{K_j})$ is superharmonic on $\Omega \backslash K_j$. Since $K_j \subset \{u \leq v\}$, we have

$$\liminf_{x \to y, x \in \Omega \backslash K_j} \left(v - G_\Omega (\mu_1|_{K_j})\right)(x) \geq u(y) - G_\Omega (\mu_1|_{K_j})(y) \quad (y \in \partial K_j),$$

and so the function $\min\{u, v\} - G_\Omega (\mu_1|_{K_j})$ is superharmonic on Ω, by Corollary 3.2.4 in [2]. Hence $\mu_1|_{K_j} \leq -\Delta \min\{u, v\}$, and the desired inequality follows on letting $j \to \infty$. $\square$

Theorem 2.2 provides a very short route to the following result of Brezis and Ponce [3].

Corollary 2.3 (Kato's inequality)**.** *If w is a δ-subharmonic function on an open set, then*

$$-\Delta \min\{w, 0\} \geq (-\Delta w)|_{\{w \leq 0\}}. \tag{8}$$

Proof. Since this is a local result we may assume that the given open set is Greenian and that $w = u - v$, where u and v are positive superharmonic functions, $(-\Delta w)^+ = -\Delta u$ and $(-\Delta w)^- = -\Delta v$. From (7) we see that

$$(-\Delta w)_c|_{\{w \leq 0\}} \leq 0 \quad \text{and} \quad (-\Delta w)_c|_{\{w > 0\}} \geq 0,$$

so

$$(-\Delta u)_c|_{\{w \leq 0\}} \leq (-\Delta v)_c|_{\{w \leq 0\}} \quad \text{and} \quad (-\Delta v)_c|_{\{w > 0\}} \leq (-\Delta u)_c|_{\{w > 0\}}.$$

We apply Theorem 2.2 with

$$\mu_1 = (-\Delta u)_d|_{\{w \leq 0\}} \quad \text{and} \quad \mu_2 = (-\Delta u)_c|_{\{w \leq 0\}}.$$

Then we apply it again with

$$\mu_1 = (-\Delta v)_d|_{\{w > 0\}} \quad \text{and} \quad \mu_2 = (-\Delta v)_c|_{\{w > 0\}},$$

but this time with the roles of u and v reversed. The results of these two applications can be combined to yield

$$-\Delta \min\{u, v\} \geq (-\Delta u)|_{\{w \leq 0\}} + (-\Delta v)|_{\{w > 0\}},$$

whence (8) follows. $\qquad\square$

If $A \subset \Omega$, we define the swept measure $\mu^A = -\Delta \widehat{R}^A_{G_\Omega \mu}$, where $\widehat{R}^A_v$ denotes the regularized reduction of a positive superharmonic function v relative to A in Ω. Thus, if U is an open set that is compactly contained in Ω and $x \in U$, then $\delta_x^{U^c}$ is harmonic measure for U and x. If U is a finely open set, we define $\widetilde{U} = \{x : U^c \text{ is thin at } x\}$, whence $U \subset \widetilde{U}$ and $\widetilde{U} \backslash U$ is polar.

Theorem 2.4. *If U is a finely open subset of Ω and μ is a measure such that $\mu(\widetilde{U}^c) = 0$, then μ^{U^c} is singular with respect to λ.*

This result is contained in a very general theorem of Hansen and Hueber [12]. An alternative proof of the case where $\mu = \delta_x$ and $x \in \widetilde{U}$, based on partial balayage, may be found in Theorem 10 of [9]. (The proof given there for Euclidean open sets U readily extends to the case of finely open sets.) The general case then follows from the formula

$$\mu^{U^c}(B) = \int \delta_x^{U^c}(B) d\mu(x) \quad (B \text{ is a Borel set})$$

(see Theorem 1.X.5 in [4]) and the fact that the measures $\delta_x^{U^c}$ all have the same null sets as x varies over a fine component of $\widetilde{U}$ (see 12.6, Corollary in [6]).

We will only use the notation μ^{U^c} when U is a bounded set and μ has compact support. The underlying Greenian open set Ω should then be understood to contain both $\overline{U}$ and $\text{supp}(\mu)$, but will not be specified as it does not affect μ^{U^c}. Finally, for signed measures μ, we define $\mu^A = (\mu^+)^A - (\mu^-)^A$.

3. Quadrature identities

The definition of a two-phase quadrature domain for harmonic functions requires that the function

$$u = U(\mu^+ - \lambda|_{\Omega^+}) - U(\mu^- - \lambda|_{\Omega^-}) \tag{9}$$

vanishes outside $\Omega^+ \cup \Omega^-$. We begin by justifying the quadrature identity (3) and the assertion following it. In the classical case of (one-phase) quadrature domains the natural test class for the associated identity consists of the integrable harmonic functions. However, as we explained in the introduction, this class is too large for the two-phase case, and so in (3) we used harmonic functions which are continuous up to the boundary. Use of this smaller test class means that we may need to add a polar set to the domains in question in order to make them into quadrature domains. The point here is that the functions $U\delta_y$ ($y \in \partial(\Omega^+ \cup \Omega^-)$) do not belong to our test-class, but for most choices of y we can approximate them by functions harmonic in $\Omega^+ \cup \Omega^-$ and continuous up to the boundary.

We will use $B_r(x)$ to denote the open ball in $\mathbb{R}^N$ of centre x and radius r.

Theorem 3.1. *Let Ω^+, Ω^- be disjoint bounded open sets and μ^+, μ^- be measures with compact supports in Ω^+, Ω^- respectively.*
(a) If (Ω^+, Ω^-) is a two-phase quadrature domain for harmonic functions with respect to (μ^+, μ^-), then (3) holds.
(b) If (3) holds, then there are polar sets Z_1, Z_2 such that $(\Omega^+ \cup Z_1, \Omega^- \cup Z_2)$ is a two-phase quadrature domain for harmonic functions with respect to (μ^+, μ^-).

Proof. (a) Since the function u in (9) vanishes on $(\Omega^+ \cup \Omega^-)^c$, we see that

$$(\mu^+ - \lambda|_{\Omega^+})^{(\Omega^+)^c} = (\mu^- - \lambda|_{\Omega^-})^{(\Omega^-)^c}. \tag{10}$$

Noting that, for any finite measure ν on a bounded open set Ω and any $f \in C(\Omega)$, we have

$$\int f d\nu^{\Omega^c} = \int h d\nu, \text{ where } h(x) = \int f d\delta_x^{\Omega^c}, \tag{11}$$

we deduce (3).

(b) Suppose that (3) holds, let Ω be a Greenian domain containing $\overline{\Omega^+ \cup \Omega^-}$, and let u be given by (9). It follows easily from (3), applied to the functions

$$h_y = U\delta_y - G_\Omega(y, \cdot) \quad (y \in \Omega)$$

(suitably defined at y), that

$$u = G_\Omega(\mu^+ - \lambda|_{\Omega^+}) - G_\Omega(\mu^- - \lambda|_{\Omega^-}) \quad \text{in } \Omega.$$

Let

$$E = \left\{ x : (\Omega^+ \cup \Omega^-)^c \text{ is non-thin at } x \right\}$$

and $y \in E \cap \Omega$. We will show that $G_\Omega(y, \cdot)$ can be approximated from below by potentials v_n which are continuous on Ω and harmonic on $\Omega^+ \cup \Omega^-$, whence $u(y) = 0$ by (3). From this it will follow by continuity that $u = 0$ on $\overline{E}$. Since $\overline{E}^c$ is open, contains $\Omega^+ \cup \Omega^-$, and differs from it by at most a polar set, we thus have a two-phase quadrature domain of the stated form.

To prove the desired approximation property, we choose r such that $\overline{B}_r(y) \subset \Omega$ and define $A = \overline{B}_r(y) \backslash (\Omega^+ \cup \Omega^-)$ and

$$u_n = \widehat{R}_{G_\Omega(y,\cdot)}^{A \backslash B_{r/n}(y)} \quad (n \in \mathbb{N}).$$

By Theorem 2.1 we can find a continuous potential v_n on Ω such that $v_n \le u_n$ and $-\Delta v_n \le -\Delta u_n$ on Ω, and $v_n \ge u_n - n^{-1}$ on $\{\text{dist}(x, (\Omega^+ \cup \Omega^-)^c) \ge n^{-1}\}$. Clearly v_n is harmonic on $\Omega^+ \cup \Omega^-$. Since

$$u_n \uparrow \widehat{R}_{G_\Omega(y,\cdot)}^A = G_\Omega(y, \cdot),$$

by the non-thinness of A at y, we see that $v_n \to G_\Omega(y, \cdot)$ on $\Omega^+ \cup \Omega^-$, as required. $\square$

We now introduce two special types of two-phase quadrature domains.

Definition 3.2. *Let (Ω^+, Ω^-) be a two-phase quadrature domain for harmonic functions with respect to (μ^+, μ^-), and let u be given by (9). If*

$$u \ge 0 \text{ in } \Omega^+ \text{ and } u \le 0 \text{ in } \Omega^-, \tag{12}$$

then we call (Ω^+, Ω^-) a two-phase quadrature domain for subharmonic functions with respect to (μ^+, μ^-).

If both inequalities in (12) are strict, then we call (Ω^+, Ω^-) a strong two-phase quadrature domain for subharmonic functions with respect to (μ^+, μ^-).

The open sets Ω^+, Ω^- in the above definition need not be connected. However, in contrast with the case $a = 0$ of Example 1, each component of Ω^+ must intersect $\text{supp}(\mu^+)$, and each component of Ω^- must intersect $\text{supp}(\mu^-)$. To see this, we note that if ω were a component of Ω^+ that does not intersect $\text{supp}(\mu^+)$, then u

would be strictly subharmonic on ω and valued 0 on $\partial\omega$, yielding a contradiction to (12) in view of the maximum principle.

We distinguished between the two types of quadrature domain in Definition 3.2 because the latter type is emphasized in [5], whereas the former is the natural one for quadrature inequalities, as becomes clear in the following analogue of Theorem 3.1.

Theorem 3.3. *Let Ω^+, Ω^- be disjoint bounded open sets and μ^+, μ^- be measures with compact supports in Ω^+, Ω^- respectively.*
(a) If (Ω^+, Ω^-) is a two-phase quadrature domain for subharmonic functions with respect to (μ^+, μ^-), then

$$\int sd(\mu^+ - \mu^-) \leq \int_{\Omega^+} sd\lambda - \int_{\Omega^-} sd\lambda \ \ \text{for every } s \in C(\overline{\Omega^+ \cup \Omega^-})$$

$$\text{that is subharmonic on } \Omega^+ \text{ and superharmonic on } \Omega^-. \tag{13}$$

(b) If (13) holds, then there are polar sets Z_1, Z_2 such that $(\Omega^+ \cup Z_1, \Omega^- \cup Z_2)$ is a two-phase quadrature domain for subharmonic functions with respect to (μ^+, μ^-).

Proof. (a) Let $s \in C(\overline{\Omega^+ \cup \Omega^-})$, where s is subharmonic on Ω^+ and superharmonic on Ω^-, and let $\nu = \Delta s$ on $\Omega^+ \cup \Omega^-$. Further, let h_+, h_- be the (PWB) solutions to the Dirichlet problem on Ω^+, Ω^- respectively with boundary data s. Using (11), (10), and the fact that (2), (12) imply that

$$G_{\Omega^+}(\mu^+ - \lambda|_{\Omega^+}) \geq 0 \text{ on } \Omega^+, \text{ and } G_{\Omega^-}(\mu^- - \lambda|_{\Omega^-}) \geq 0 \text{ on } \Omega^-,$$

we deduce that

$$
\begin{aligned}
\int sd(\mu^+ - \mu^-) &= \int (h_+ - G_{\Omega^+}(\nu|_{\Omega^+}))\,d\mu^+ - \int (h_- - G_{\Omega^-}(\nu|_{\Omega^-}))\,d\mu^- \\
&= \int sd(\mu^+)^{(\Omega^+)^c} - \int sd(\mu^-)^{(\Omega^-)^c} \\
&\quad - \int_{\Omega^+} G_{\Omega^+}\mu^+ d\nu + \int_{\Omega^-} G_{\Omega^-}\mu^- d\nu \\
&\leq \int sd(\lambda|_{\Omega^+})^{(\Omega^+)^c} - \int sd(\lambda|_{\Omega^-})^{(\Omega^-)^c} \\
&\quad - \int_{\Omega^+} G_{\Omega^+}(\lambda|_{\Omega^+})d\nu + \int_{\Omega^-} G_{\Omega^-}(\lambda|_{\Omega^-})d\nu \\
&= \int_{\Omega^+} (h_+ - G_{\Omega^+}(\nu|_{\Omega^+}))\,d\lambda - \int_{\Omega^-} (h_- - G_{\Omega^-}(\nu|_{\Omega^-}))\,d\lambda \\
&= \int_{\Omega^+} sd\lambda - \int_{\Omega^-} sd\lambda.
\end{aligned}
$$

(b) We know from the corresponding case of Theorem 3.1 that there are disjoint open sets D^+, D^- containing Ω^+, Ω^- respectively, such that $D^+\backslash\Omega^+, D^-\backslash\Omega^-$ are polar and the function u vanishes on $(D^+ \cup D^-)^c$. Now let $x \in \Omega^+$ and choose $n_0 \in \mathbb{N}$ such that $U\delta_x \leq n_0$ outside Ω^+. Then the function $s = -\min\{U\delta_x, n\}$ is subharmonic on Ω^+ and harmonic on Ω^- whenever $n \geq n_0$. We can thus apply (13) and let $n \to \infty$ to see that $u \geq 0$ on Ω^+, and hence on D^+. Similarly, $u \leq 0$ on D^-, so the result follows $\qquad\square$

We will now provide the promised details for Example 2. Let p, μ^+ and S^+ be as stated there, and let $\Omega^+ = \omega(S^+, \mu^+)$. Clearly $W\mu^+ \geq W_{S^+}\mu^+$ on S^+, and $W_{S^+}\mu^+$ vanishes continuously on ∂S^+. We also know that

$$-\Delta W_{S^+}\mu^+ = \mu^+ - \lambda|_{\Omega^+} - \nu_0,$$

where $\nu_0 \geq 0$ and $\operatorname{supp}(\nu_0) \subset \partial S^+$. Finally,

$$\Omega^+ \subset \omega(\mu^+) = \left\{x : |x - p| < \sqrt{4/\pi}\right\},$$

so $\lambda(\Omega^+) \leq \lambda(\omega(\mu^+) \cap S^+) < 4$. Since $\lambda(\Omega^+) + \nu_0(\partial S^+) = 4$, we see that $\nu_0(\partial S^+) > 0$, as claimed.

The details for Example 3 are similar.

4. CONSTRUCTION OF TWO-PHASE QUADRATURE DOMAINS

Let $\mu = \mu^+ - \mu^-$ be a signed measure with compact support. Below we provide a means of constructing a two-phase quadrature domain for subharmonic functions with respect to (μ^+, μ^-) provided such a quadrature domain exists. We also show the uniqueness of such quadrature domains modulo sets of λ-measure zero.

Given a Borel function $u : \mathbb{R}^N \to [-\infty, +\infty]$, we define the signed measure

$$\eta(u, \mu) = \left((\mu^+ - \lambda)^+ - (\mu^+ - \lambda)^-|_{\{u>0\}}\right) - \left((\mu^- - \lambda)^+ - (\mu^- - \lambda)^-|_{\{u<0\}}\right).$$

This definition requires only that u be defined λ-almost everywhere.

Lemma 4.1. *Let* $u, u_1, u_2 : \mathbb{R}^N \to [-\infty, +\infty]$ *be Borel measurable functions,* μ, μ_1, μ_2 *be signed measures with compact supports, and* $A \subset \mathbb{R}^N$ *be a Borel set. Then*
(a) $\eta(-u, -\mu) = -\eta(u, \mu)$,
(b) $\mu - \lambda \leq \eta(u, \mu) \leq \mu + \lambda$, *and*
(c) $\eta(u_1, \mu_1)|_A \geq \eta(u_2, \mu_2)|_A$ *provided* $u_1|_A \leq u_2|_A$ *and* $\mu_1|_A \geq \mu_2|_A$.

Proof. Part (a) is obvious, and

$$\begin{aligned}
\mu - \lambda &= (\mu^+ - \lambda) - \mu^- \\
&\leq \left((\mu^+ - \lambda)^+ - (\mu^+ - \lambda)^-|_{\{u>0\}}\right) - \left((\mu^- - \lambda)^+ - (\mu^- - \lambda)^-|_{\{u<0\}}\right) \\
&\leq \mu^+ - (\mu^- - \lambda) = \mu + \lambda,
\end{aligned}$$

so (b) holds. Part (c) follows from the observations that

$$\{u_1 > 0\} \cap A \subset \{u_2 > 0\} \cap A, \quad \{u_1 < 0\} \cap A \supset \{u_2 < 0\} \cap A,$$
$$\mu_1^+|_A \geq \mu_2^+|_A, \quad \mu_1^-|_A \leq \mu_2^-|_A,$$

whence

$$(\mu_1^+ - \lambda)^+|_A \geq (\mu_2^+ - \lambda)^+|_A, \quad (\mu_1^+ - \lambda)^-|_{\{u_1>0\}\cap A} \leq (\mu_2^+ - \lambda)^-|_{\{u_2>0\}\cap A},$$
$$(\mu_1^- - \lambda)^+|_A \leq (\mu_2^- - \lambda)^+|_A, \quad (\mu_1^- - \lambda)^-|_{\{u_1<0\}\cap A} \geq (\mu_2^- - \lambda)^-|_{\{u_2<0\}\cap A}.$$

$\square$

Let $w \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}^N)$ be such that $-\Delta w \geq \eta(w, \mu)$ and $w \geq -W\mu^-$, and let $u = w + U\mu^- - |\cdot|^2/2N$. Then

$$-\Delta u = -\Delta w + \mu^- + \lambda \geq \eta(w, \mu) + \mu^- + \lambda \geq \mu^+ \geq 0,$$

by Lemma 4.1(b). It follows that, by suitable redefinition on a λ-null set, u and w can be made superharmonic and δ-subharmonic, respectively. We now define

$$\tau_\mu := \{w : w \text{ is } \delta\text{-subharmonic}, \ -\Delta w \geq \eta(w, \mu) \text{ and } w \geq -W\mu^- \text{ on } \mathbb{R}^N\}$$

and

$$\tau'_\mu := \{w + U\mu^- - |\cdot|^2/2N : w \in \tau_\mu\},$$

where members of τ'_μ are suitably redefined on a polar set to make them superharmonic, and members of τ_μ are assigned values quasi-everywhere according to the convention explained in Section 2.2. An inequality for a δ-subharmonic function w is understood to hold wherever w is defined.

Lemma 4.2. *If* $v_1, v_2 \in \tau'_\mu$, *then* $\min\{v_1, v_2\} \in \tau'_\mu$.

Proof. Let $v_1, v_2 \in \tau'_\mu$. Then $v_i = w_i + U\mu^- - |\cdot|^2/2N$, where each w_i is δ-subharmonic, $w_i \geq -W\mu^-$ and $-\Delta w_i \geq \eta(w_i, \mu)$. Hence

$$\min\{v_1, v_2\} = \min\{w_1, w_2\} + U\mu^- - |\cdot|^2/2N,$$

and $\min\{w_1, w_2\}$ is a δ-subharmonic function which majorizes $-W\mu^-$. Finally,

$$
\begin{aligned}
\eta(\min\{w_1, w_2\}, \mu) &= \left[\begin{array}{c} \left((\mu^+ - \lambda)^+ - (\mu^+ - \lambda)^-|_{\{w_1 > 0\}}\right) \\ -\left((\mu^- - \lambda)^+ - (\mu^- - \lambda)^-|_{\{w_1 < 0\}}\right) \end{array}\right]\bigg|_{\{w_1 - w_2 \leq 0\}} \\
&\quad + \left[\begin{array}{c} \left((\mu^+ - \lambda)^+ - (\mu^+ - \lambda)^-|_{\{w_2 > 0\}}\right) \\ -\left((\mu^- - \lambda)^+ - (\mu^- - \lambda)^-|_{\{w_2 < 0\}}\right) \end{array}\right]\bigg|_{\{w_1 - w_2 > 0\}} \\
&= \eta(w_1, \mu)|_{\{w_1 - w_2 \leq 0\}} + \eta(w_2, \mu)|_{\{w_1 - w_2 > 0\}} \\
&\leq -(\Delta w_1)|_{\{w_1 - w_2 \leq 0\}} - (\Delta w_2)|_{\{w_1 - w_2 > 0\}} \\
&= -(\Delta(w_1 - w_2))|_{\{w_1 - w_2 \leq 0\}} - \Delta w_2 \\
&\leq -\Delta \min\{w_1, w_2\},
\end{aligned}
$$

by Corollary 2.3. $\qquad\square$

Theorem 4.3. *(a) Let u_1, u_2 be δ-subharmonic functions with compact supports. If $-\Delta u_1 \geq \eta(u_1, \mu)$ and $-\Delta u_2 \leq \eta(u_2, \mu)$, then $u_2 \leq u_1$.*
(b) Let u be a δ-subharmonic function with compact support.

  (i) If $-\Delta u \leq \eta(u, \mu)$, then $u \leq W\mu^+$.
  (ii) If $-\Delta u \geq \eta(u, \mu)$, then $u \geq -W\mu^-$ and so $u \in \tau_\mu$.

Proof. (a) Let $v = u_2 - u_1$. Then

$$
\begin{aligned}
-\Delta v &\leq \eta(u_2, \mu) - \eta(u_1, \mu) \\
&= (\mu^+ - \lambda)^-|_{\{u_1 > 0\}} - (\mu^+ - \lambda)^-|_{\{u_2 > 0\}} \\
&\quad + (\mu^- - \lambda)^-|_{\{u_2 < 0\}} - (\mu^- - \lambda)^-|_{\{u_1 < 0\}},
\end{aligned}
$$

so $-\Delta v|_{\{v \geq 0\}} \leq 0$. Hence $\Delta v^+ \geq 0$, by Corollary 2.3. Thus v^+, when suitably redefined on a polar set, is subharmonic. Since v has compact support, the maximum principle shows that $v^+ \equiv 0$, whence the result.

(b) The function $W\mu^+$ is non-negative, δ-subharmonic, and has compact support. Further, $\mu^+|_{\{W\mu^+ = 0\}} \leq \lambda$, and

$$
\begin{aligned}
-\Delta W\mu^+ &= (\mu^+ - \lambda)|_{\{W\mu^+ > 0\}} \\
&= (\mu^+ - \lambda)^+ - (\mu^+ - \lambda)^-|_{\{W\mu^+ > 0\}} \\
&= \eta(W\mu^+, \mu^+) \\
&\geq \eta(W\mu^+, \mu), \tag{14}
\end{aligned}
$$

by Lemma 4.1(c). If $-\Delta u \leq \eta(u, \mu)$, it now follows from part (a) that $u \leq W\mu^+$.

Finally, replacing μ by $-\mu$ in (14), we obtain

$$-\Delta(-W\mu^-) = \Delta W\mu^- \leq -\eta(W\mu^-, -\mu) = \eta(-W\mu^-, \mu),$$

by Lemma 4.1(a). If $-\Delta u \geq \eta(u, \mu)$, it thus follows from part (a) that $u \geq -W\mu^-$, and so $u \in \tau_\mu$. $\qquad\square$

Theorem 4.4. *(a) The set τ_μ contains a least element $\overline{W}\mu$, which has compact support.*
(b) The function $\overline{W}\mu + W\mu^-$ is lower semicontinuous and

$$-\Delta\overline{W}\mu = \eta(\overline{W}\mu, \mu) + \gamma,$$

where γ is a measure with compact support such that $0 \leq \gamma \leq 2\lambda$.
(c) If $U|\mu|$ is finite-valued and continuous, then so also is $\overline{W}\mu$.

(d) If w is a δ-subharmonic function with compact support and $-\Delta w = \eta(w, \mu)$, then $w = \overline{W}\mu$.

Proof. (a) Since $W\mu^+ \geq 0 \geq -W\mu^-$, we see from (14) that $W\mu^+ \in \tau_\mu$, so τ_μ is non-empty. By Lemma 4.2, τ'_μ is a down-directed family of superharmonic functions, so by Choquet's lemma there is a decreasing sequence (u_n) in τ'_μ with limit u, where $\widehat{u} = \widehat{\inf \tau'_\mu}$. Further, $\widehat{u} = u$ almost everywhere (λ). Let

$$v_n = u_n - U\mu^- + |\cdot|^2/2N \quad \text{and} \quad \overline{v} = \widehat{u} - U\mu^- + |\cdot|^2/2N.$$

The sequence $(\eta(v_n, \mu))$ is then w*-convergent to the signed measure ν given by

$$
\begin{aligned}
\nu &= (\mu^+ - \lambda)^+ - (\mu^+ - \lambda)^-|_{\cap_n\{v_n > 0\}} - (\mu^- - \lambda)^+ + (\mu^- - \lambda)^-|_{\cup_n\{v_n < 0\}} \\
&= (\mu^+ - \lambda)^+ - (\mu^+ - \lambda)^-|_{\{\overline{v} > 0\}} - (\mu^+ - \lambda)^-|_A \\
&\quad -(\mu^- - \lambda)^+ + (\mu^- - \lambda)^-|_{\{\overline{v} < 0\}} \\
&= \eta(\overline{v}, \mu) - (\mu^+ - \lambda)^-|_A,
\end{aligned}
\tag{15}
$$

where $A \subset \{\overline{v} = 0\}$. Also,

$$\langle -\Delta v_n, \varphi \rangle \to \langle -\Delta \overline{v}, \varphi \rangle \quad \text{for all} \quad \varphi \in C_c^\infty(\mathbb{R}^N).$$

Since $v_n \in \tau_\mu$, we know that $-\Delta v_n \geq \eta(v_n, \mu)$ for all n, and hence from (15) that

$$-\Delta \overline{v} \geq \eta(\overline{v}, \mu) - (\mu^+ - \lambda)^-|_A. \tag{16}$$

We know that $-W\mu^- \leq \overline{v} \leq W\mu^+$, so the set $U = \{\overline{v} \neq 0\}$ is bounded, as well as finely open. Clearly $(-\Delta \overline{v})^{U^c} = 0$, so

$$-(\Delta \overline{v})|_{\widetilde{U}^c} = \left((\Delta \overline{v})|_{\widetilde{U}}\right)^{U^c},$$

where $\widetilde{U} = \{x : U^c \text{ is thin at } x\}$, and hence $-(\Delta \overline{v})|_{\widetilde{U}^c}$ is singular with respect to Lebesgue measure, by Theorem 2.4. Thus (16) yields

$$-\Delta \overline{v} \geq \eta(\overline{v}, \mu) - (\mu^+ - \lambda)^-|_{A \cap \widetilde{U}} = \eta(\overline{v}, \mu), \tag{17}$$

since $A \cap \widetilde{U} \subset \{\overline{v} = 0\} \cap \widetilde{U}$, and the latter set is polar and so λ-null. Hence $\overline{v} \in \tau_\mu$. It is clearly the least element $\overline{W}\mu$ that we sought, and has compact support.

(b) From the above construction

$$
\begin{aligned}
\overline{W}\mu + W\mu^- &= (\widehat{u} - U\mu^- + |\cdot|^2/2N) + (U\mu^- - UB\mu^-) \\
&= \widehat{u} - UB\mu^- + |\cdot|^2/2N.
\end{aligned}
$$

Thus $\overline{W}\mu + W\mu^-$ is lower semicontinuous, since $B\mu^- \leq \lambda$. Also, $-\Delta \overline{W}\mu = \eta(\overline{W}\mu, \mu) + \gamma$ where $\gamma \geq 0$, by (17). Let $w = \overline{W}\mu - 2W(\gamma/2)$. Then

$$
\begin{aligned}
-\Delta w &= \eta(\overline{W}\mu, \mu) + \gamma + 2B(\gamma/2) - \gamma \\
&= (\mu^+ - \lambda)^+ - (\mu^+ - \lambda)^-|_{\{\overline{W}\mu > 0\}} - (\mu^- - \lambda)^+ + (\mu^- - \lambda)^-|_{\{\overline{W}\mu < 0\}} \\
&\quad + 2\lambda|_{\{W(\gamma/2) > 0\}} + \gamma|_{\{W(\gamma/2) = 0\}} \\
&\geq (\mu^+ - \lambda)^+ - (\mu^+ - \lambda)^-|_{\{w > 0\}} - (\mu^+ - \lambda)^-|_{\{\overline{W}\mu > 0, 2W(\gamma/2) \geq \overline{W}\mu\}} \\
&\quad -(\mu^- - \lambda)^+ + (\mu^- - \lambda)^-|_{\{w < 0\}} - (\mu^- - \lambda)^-|_{\{\overline{W}\mu \geq 0, \overline{W}\mu < 2W(\gamma/2)\}} \\
&\quad + 2\lambda|_{\{W(\gamma/2) > 0\}} \\
&\geq \eta(w, \mu) - ((\mu^+ - \lambda)^- - \lambda)|_{\{\overline{W}\mu > 0, 2W(\gamma/2) \geq \overline{W}\mu\}} \\
&\quad -((\mu^- - \lambda)^- - \lambda)|_{\{\overline{W}\mu \geq 0, \overline{W}\mu < 2W(\gamma/2)\}} \\
&\geq \eta(w, \mu).
\end{aligned}
$$

It follows from Theorem 4.3(b) that $w \in \tau_\mu$. Hence $W(\gamma/2) = 0$, and so $\gamma \leq 2\lambda$, as claimed.

(c) By part (b) and Lemma 4.1(b),

$$|-\Delta \overline{W}\mu| \le |\eta(\overline{W}\mu, \mu)| + 2\lambda \le |\mu| + 3\lambda,$$

so $\overline{W}\mu$ is finite-valued and continuous if $U|\mu|$ is.

(d) It follows from Theorem 4.3(b) that $w \in \tau_\mu$, and from part (b) and Theorem 4.3(a) that $w \le \overline{W}\mu$, whence $w = \overline{W}\mu$. $\qquad\square$

Below we shed some further light on the measure γ that appears in Theorem 4.4. We note from part (b) of that result that, if μ^+, μ^- have disjoint compact supports, then $\overline{W}\mu$ is everywhere defined. However, the sets Ω^+, Ω^- in (18) below need not be open, as we will see later in Example 4.

Theorem 4.5. *Suppose that μ^+, μ^- have disjoint compact supports, and let*

$$\Omega^+ = \{\overline{W}\mu > 0\} \quad and \quad \Omega^- = \{\overline{W}\mu < 0\}. \tag{18}$$

Then $\gamma(\Omega^+ \cup \Omega^-) = 0$, and so

$$-\Delta\overline{W}\mu = ((\mu^+ - \lambda) - (\mu^- - \lambda)^+)|_{\widetilde{\Omega^+}} - ((\mu^- - \lambda) - (\mu^+ - \lambda)^+)|_{\widetilde{\Omega^-}} + \nu, \tag{19}$$

where

$$\nu = -(((\mu^+ - \lambda) - (\mu^- - \lambda)^+)|_{\widetilde{\Omega^+}})^{(\Omega^+)^c} + (((\mu^- - \lambda) - (\mu^+ - \lambda)^+)|_{\widetilde{\Omega^-}})^{(\Omega^-)^c}. \tag{20}$$

Further, $\nu = 0$ if $|\mu||_{(\widetilde{\Omega^+} \cup \widetilde{\Omega^-})^c} \ll \lambda$.

Proof. Let Ω be a bounded open set containing $\overline{\Omega^+ \cup \Omega^-}$, let $\varepsilon > 0$ and suppose that $x \in \Omega^+$ satisfies $0 < \varepsilon < \overline{W}\mu(x)$. Since $0 \le \gamma \le 2\lambda$, we can choose $\delta > 0$ such that $u_\delta \le \varepsilon$, where $u_\delta = G_\Omega(\gamma|_{B_\delta(x)})$. Let

$$w = \begin{cases} \overline{W}\mu - \left(u_\delta - \widehat{R}_{u_\delta}^{\{\overline{W}\mu \le \varepsilon\}}\right) & \text{on } \Omega \\ \overline{W}\mu & \text{on } \Omega^c \end{cases},$$

where the reduction is relative to superharmonic functions on Ω. Then $\overline{W}\mu \ge w > 0$ on $\{\overline{W}\mu > \varepsilon\}$ and $w = \overline{W}\mu$ quasi-everywhere on $\{\overline{W}\mu \le \varepsilon\}$, so $\eta(\overline{W}\mu, \mu) = \eta(w, \mu)$. Also,

$$\begin{aligned} -\Delta w &= -\Delta\overline{W}\mu - \gamma|_{B_\delta(x) \cap \{\overline{W}\mu > \varepsilon\}} + \left(\gamma|_{B_\delta(x) \cap \{\overline{W}\mu > \varepsilon\}}\right)^{\{\overline{W}\mu \le \varepsilon\}} \\ &\ge -\Delta\overline{W}\mu - \gamma = \eta(\overline{W}\mu, \mu). \end{aligned}$$

Hence $-\Delta w \ge \eta(w, \mu)$ and it follows from Theorem 4.3(b) that $w \in \tau_\mu$. From the minimality of $\overline{W}\mu$ we conclude that $u_\delta = \widehat{R}_{u_\delta}^{\{\overline{W}\mu \le \varepsilon\}}$, whence $\gamma(B_\delta(x) \cap \{\overline{W}\mu > \varepsilon\}) = 0$. Therefore $\gamma(\Omega^+) = 0$, in view of the arbitrary choices of ε and x.

Similar reasoning shows that the function

$$w' = \begin{cases} \overline{W}\mu - \left(G_\Omega\gamma - \widehat{R}_{G_\Omega\gamma}^{(\Omega^-)^c}\right) & \text{on } \Omega \\ \overline{W}\mu & \text{on } \Omega^c \end{cases}$$

also belongs to τ_μ, so $\gamma(\Omega^-) = 0$.

Let

$$\nu = \left(-\Delta\overline{W}\mu\right)|_{(\widetilde{\Omega^+} \cup \widetilde{\Omega^-})^c}.$$

Then (19) holds, and (20) follows since $\overline{W}\mu = 0$ on $(\widetilde{\Omega^+} \cup \widetilde{\Omega^-})^c$.

Finally, suppose that $|\mu||_{(\widetilde{\Omega^+} \cup \widetilde{\Omega^-})^c} \ll \lambda$. Since $\nu \perp \lambda$, by Theorem 2.4, and

$$\nu = \left(\eta(\overline{W}\mu, \mu) + \gamma\right)|_{(\widetilde{\Omega^+} \cup \widetilde{\Omega^-})^c},$$

we see from Lemma 4.1(b), and the fact that $\gamma \le 2\lambda$, that $\nu = 0$. $\qquad\square$

Remark 1. *We already know from Theorem 4.4(c) that the sets Ω^+, Ω^- in (18) are open whenever $U|\mu|$ is finite-valued and continuous. More generally, $\overline{W}\mu$ is lower semicontinuous outside $\mathrm{supp}(\mu^-)$ and upper semicontinuous outside $\mathrm{supp}(\mu^+)$, so $\Omega^+\backslash\mathrm{supp}(\mu^-)$ and $\Omega^-\backslash\mathrm{supp}(\mu^+)$ are open. Thus Ω^+, Ω^- are open provided*

$$\mathrm{supp}(\mu^+) \subset \{\overline{W}\mu \geq 0\} \quad and \quad \mathrm{supp}(\mu^-) \subset \{\overline{W}\mu \leq 0\}.$$

Corollary 4.6. *If*

$$\mathrm{supp}(\mu^+) \subset \Omega^+ \quad and \quad \mathrm{supp}(\mu^-) \subset \Omega^-,$$

where Ω^+, Ω^- are given by (18), then (Ω^+, Ω^-) is a strong two-phase quadrature domain for subharmonic functions with respect to (μ^+, μ^-).

Proof. We know from Remark 1 that the disjoint sets Ω^+, Ω^- are open. By Theorem 4.5 we have

$$-\Delta \overline{W}\mu = \mu^+ - \lambda|_{\Omega^+} - \mu^- + \lambda|_{\Omega^-}.$$

Since $\overline{W}\mu$ is compactly supported and δ-subharmonic, it must coincide with the function u defined by (9), and the result follows. $\square$

We now demonstrate that the sets Ω^+, Ω^- in (18) need not be open in general.

Example 4. *Let D be a bounded domain with an irregular boundary point y such that $B_r(y)\backslash D$ is non-polar for all $r > 0$. Further, suppose that all positive superharmonic functions on D are λ-integrable. (This will be the case if, for instance, D satisfies a uniform inner ball condition: see Aikawa [1].) Now let ν be a non-zero measure with compact support in D and suppose, for the sake of contradiction, that there is a sequence (x_n) of points in D such that $nG_D\nu(x_n) \leq G_D(\lambda|_D)(x_n)$ for all n. Clearly (x_n) tends to ∂D, and we may assume that $x_n \notin \mathrm{supp}(\nu)$ for all n. The function*

$$w = \sum_n \frac{1}{n^2} \frac{G_D(\cdot, x_n)}{G_D\nu(x_n)}$$

is positive and superharmonic on D, and we arrive at the contradictory conclusion that $\int_D w\,d\lambda = \infty$. Therefore we can choose m large enough so that $G_D(\mu^+ - \lambda|_D) > 0$ on D, where $\mu^+ = m\nu$.

Now let $\mu^- = (\mu^+ - \lambda|_D)^{D^c}$. Then $\mu^- \geq 0$ and the function

$$u = U\mu^+ - U(\lambda|_D) - U\mu^- = G_D(\mu^+ - \lambda|_D)$$

satisfies

$$-\Delta u = \mu^+ - \lambda|_D - \mu^- = \eta(u, \mu),$$

where $\mu = \mu^+ - \mu^-$. It follows from Theorem 4.4(d) that $u = \overline{W}\mu$. Since $G_D\nu$ has a positive fine limit at the irregular boundary point y, we can arrange (by increasing m, if necessary) that $u(y) > 0$. However, $u = 0$ at regular boundary points of D, which occur arbitrarily close to y, so the set $\Omega^+ = \{\overline{W}\mu > 0\}$ is not open.

We will now use our construction to get uniqueness results for two-phase quadrature domains for subharmonic functions.

Theorem 4.7. *(a) If (Ω^+, Ω^-) is a two-phase quadrature domain for subharmonic functions with respect to (μ^+, μ^-), then*

$$\Omega^+ = \{\overline{W}\mu > 0\} \cup \mathrm{supp}(\mu^+) \cup Z_1 \quad and \quad \Omega^- = \{\overline{W}\mu < 0\} \cup \mathrm{supp}(\mu^-) \cup Z_2,$$

where Z_1 and Z_2 are λ-null sets. In particular, two-phase quadrature domains for subharmonic functions are unique up to λ-null sets.
(b) If (Ω^+, Ω^-) is a strong two-phase quadrature domain for subharmonic functions with respect to (μ^+, μ^-), then it is unique and

$$\Omega^+ = \{\overline{W}\mu > 0\}, \quad \Omega^- = \{\overline{W}\mu < 0\}.$$

Proof. (a) Let $U = \{u > 0\}$, where u is given by (9). Clearly $U \subset \Omega^+$. Then

$$(-\Delta u)|_{\Omega^+ \setminus U} \leq (-\Delta \min\{u, 0\})|_{\Omega^+} = 0$$

by Corollary 2.3, since $u \geq 0$ on Ω^+. Hence $(-\Delta u)|_{\Omega^+ \setminus U} \ll \lambda$, since $(-\Delta u)|_{\Omega^+} = \mu^+ - \lambda|_{\Omega^+}$. Similarly, $(-\Delta u)|_{\Omega^- \setminus V} \ll \lambda$, where $V = \{u < 0\}$. On the other hand, $(-\Delta u)^{\{u=0\}} = 0$, so $(-\Delta u)|_{\widetilde{U}^c \cup \widetilde{V}^c} \perp \lambda$ by Theorem 2.4. Hence $(-\Delta u)|_{\Omega^+ \setminus U} = 0 = (-\Delta u)|_{\Omega^- \setminus V}$. We conclude that

$$\begin{aligned}
-\Delta u &= \mu^+ - \lambda|_{\Omega^+} - (\mu^- - \lambda|_{\Omega^-}) \\
&= (\mu^+ - \lambda)^+ - (\mu^+ - \lambda)^-|_{\{u>0\}} - ((\mu^- - \lambda)^+ - (\mu^- - \lambda)^-|_{\{u<0\}}) \\
&= \eta(u, \mu),
\end{aligned}$$

so $u = \overline{W}\mu$ by Theorem 4.4(d). Since

$$0 = -\Delta u = -\lambda \quad \text{on the set} \quad Z_1 = \Omega^+ \setminus (\{\overline{W}\mu > 0\} \cup \mathrm{supp}(\mu^+)),$$

we see that $\lambda(Z_1) = 0$, as required. A similar argument applies to Ω^-.

(b) In this case we have $U = \Omega^+$, so $\Omega^+ = \{\overline{W}\mu > 0\}$, and similarly $\Omega^- = \{\overline{W}\mu < 0\}$. $\qquad \square$

5. EXISTENCE OF TWO-PHASE QUADRATURE DOMAINS

It is desirable to be able to recognize which pairs (μ^+, μ^-) give rise to a two-phase quadrature domain. A complete characterization seems an unrealistic target (even for the one-phase case), but we give below some sufficient conditions on (μ^+, μ^-) for two-phase quadrature domains to exist.

Theorem 5.1. *Let μ^+, μ^- be positive measures with disjoint compact supports in $\mathbb{R}^N$.*
(a) If

$$\overline{\Omega(\mu^-)} \cap \mathrm{supp}(\mu^+) = \emptyset, \quad \overline{\Omega(\mu^+)} \cap \mathrm{supp}(\mu^-) = \emptyset,$$

and

$$\mathrm{supp}(\mu^+) \subset \Omega(\overline{\Omega(\mu^-)}^c, \mu^+), \quad \mathrm{supp}(\mu^-) \subset \Omega(\overline{\Omega(\mu^+)}^c, \mu^-), \tag{21}$$

then there is a two-phase quadrature domain for subharmonic functions with respect to (μ^+, μ^-).
(b) If

$$\overline{\omega(\mu^-)} \cap \mathrm{supp}(\mu^+) = \emptyset, \quad \overline{\omega(\mu^+)} \cap \mathrm{supp}(\mu^-) = \emptyset, \tag{22}$$

and

$$\mathrm{supp}(\mu^+) \subset \omega(\overline{\omega(\mu^-)}^c, \mu^+), \quad \mathrm{supp}(\mu^-) \subset \omega(\overline{\omega(\mu^+)}^c, \mu^-),$$

then there is a strong two-phase quadrature domain for subharmonic functions with respect to (μ^+, μ^-).

Proof. (a) We define

$$u = W\mu^+ - W_{\overline{\Omega(\mu^+)}^c}\mu^-, \quad v = W_{\overline{\Omega(\mu^-)}^c}\mu^+ - W\mu^-.$$

Then $u \geq -W_{\overline{\Omega(\mu^+)}^c}\mu^- \geq -W\mu^-$, and

$$-\Delta u = (\mu^+ - \lambda)|_{\{u>0\}} - (\mu^- - \lambda)|_{\{u<0\}} + \nu \geq \eta(u, \mu),$$

by (5), (6) and the fact that $\mu^+ \leq \lambda$ outside $\{u > 0\}$. Hence $u \in \tau_\mu$, by Theorem 4.3(b), and so $u \geq \overline{W}\mu$. Similarly, $-\Delta v \leq \eta(v, \mu)$ and, since v has compact support, we see from Theorem 4.3(a) that $v \leq \overline{W}\mu$. Thus $\Omega^+ \subset \Omega(\mu^+)$ and $\Omega^- \subset \Omega(\mu^-)$, where $\Omega^+ = \{\overline{W}\mu > 0\}$ and $\Omega^- = \{\overline{W}\mu < 0\}$. Also, clearly

$$\Omega(\overline{\Omega(\mu^-)}^c, \mu^+) \subset \Omega(\mu^+) \quad \text{and} \quad \Omega(\overline{\Omega(\mu^+)}^c, \mu^-) \subset \Omega(\mu^-).$$

It follows that the sets

$$D^+ = \Omega(\overline{\Omega(\mu^-)}^c, \mu^+) \cup \Omega^+ \quad \text{and} \quad D^- = \Omega(\overline{\Omega(\mu^+)}^c, \mu^-) \cup \Omega^-$$

are disjoint. Since

$$\Omega(\overline{\Omega(\mu^-)}^c, \mu^+) \subset \{\overline{W}\mu \geq 0\} \quad \text{and} \quad \Omega(\overline{\Omega(\mu^+)}^c, \mu^-) \subset \{\overline{W}\mu \leq 0\}, \qquad (23)$$

we see from (21) and Remark 1 that Ω^+, Ω^- are open. Thus D^+, D^- are open sets containing the compact supports of μ^+, μ^- respectively. We also note that

$$\omega(\overline{\Omega(\mu^-)}^c, \mu^+) = \{v > 0\} \subset \Omega^+ \quad \text{and} \quad \omega(\overline{\Omega(\mu^+)}^c, \mu^-) = \{u < 0\} \subset \Omega^-,$$

so

$$D^+\backslash\Omega^+ \subset \Omega(\overline{\Omega(\mu^-)}^c, \mu^+) \setminus \omega(\overline{\Omega(\mu^-)}^c, \mu^+)$$

and

$$D^-\backslash\Omega^- \subset \Omega(\overline{\Omega(\mu^+)}^c, \mu^-) \setminus \omega(\overline{\Omega(\mu^+)}^c, \mu^-).$$

In particular, $\mu^+ = \lambda$ on $D^+\backslash\Omega^+$ and $\mu^- = \lambda$ on $D^-\backslash\Omega^-$. By Theorem 4.5 this implies that

$$\begin{aligned}
-\Delta\overline{W}\mu &= ((\mu^+ - \lambda) - (\mu^- - \lambda)^+)|_{\widetilde{\Omega^+}} - ((\mu^- - \lambda) - (\mu^+ - \lambda)^+)|_{\widetilde{\Omega^-}} \\
&= (\mu^+ - \lambda)|_{\widetilde{\Omega^+}} - (\mu^- - \lambda)|_{\widetilde{\Omega^-}} \\
&= \mu^+ - \lambda|_{D^+} - \mu^- + \lambda|_{D^-}.
\end{aligned}$$

It follows that (D^+, D^-) is a quadrature domain for subharmonic functions with respect to (μ^+, μ^-).

(b) The proof is similar, and indeed somewhat simpler, so the details are left to the reader. $\qquad\square$

The following corollary is similar to Theorem 5.1 in [5].

Corollary 5.2. *Let μ^+, μ^- be positive measure with disjoint compact supports such that (22) holds and*

$$\limsup_{r \to 0^+} \frac{\mu^+(B_r(x))}{\lambda(B_r(x))} > 2^N, \quad \limsup_{r \to 0^+} \frac{\mu^-(B_r(y))}{\lambda(B_r(y))} > 2^N$$

for all $x \in \operatorname{supp}(\mu^+)$ and $y \in \operatorname{supp}(\mu^-)$. Then there is a strong two-phase quadrature domain for subharmonic functions with respect to (μ^+, μ^-).

Proof. Let $x \in \operatorname{supp}(\mu^+)$. By assumption there is a sequence (r_n), decreasing to 0, such that $\mu^+(B_{r_n}(x)) > 2^N\lambda(B_{r_n}(x))$ for each n. If $\mu^+(\{x\}) = 0$, then there exists n such that $\omega(\mu^+|_{B_{r_n}(x)}) \subset \overline{\omega(\mu^-)}^c$. It follows from Theorem 2 of Sakai [13] and the lower bound for $\mu^+(B_{r_n}(x))$ that

$$x \in \omega(\mu^+|_{B_{r_n}(x)}) = \omega(\overline{\omega(\mu^-)}^c, \mu^+|_{B_{r_n}(x)}) \subset \omega(\overline{\omega(\mu^-)}^c, \mu^+).$$

On the other hand, if $\mu^+(\{x\}) > 0$, it is clear that again $x \in \omega(\overline{\omega(\mu^-)}^c, \mu^+)$. Hence $\operatorname{supp}(\mu^+) \subset \omega(\overline{\omega(\mu^-)}^c, \mu^+)$, and similarly $\operatorname{supp}(\mu^-) \subset \omega(\overline{\omega(\mu^+)}^c, \mu^-)$. The result now follows from Theorem 5.1(b). $\qquad\square$

We remark that the condition (22) in the above two results is certainly not necessary for the existence of two-phase quadrature domains for subharmonic functions. This can be seen from Example 2, where we could have taken an arbitrarily large constant in place of the number 4 in the definition of μ^+, μ^- and still obtained existence.

Theorem 5.3. *Let μ^+, μ^- have disjoint compact supports, let*

$$u = W\mu^+ - 2W(\mu^-/2), \quad v = 2W(\mu^+/2) - W\mu^-,$$

and suppose that

$$\mathrm{supp}(\mu^+) \subset \{v > 0\}, \quad \mathrm{supp}(\mu^-) \subset \{u < 0\}. \tag{24}$$

Then $v \le \overline{W}\mu \le u$, and $(\{\overline{W}\mu > 0\}, \{\overline{W}\mu < 0\})$ is a strong two-phase quadrature domain for subharmonic functions with respect to (μ^+, μ^-).

Proof. Since

$$\{u > 0\} \subset \omega(\mu^+), \quad \omega(\mu^+)\backslash\{u > 0\} \subset \omega(\mu^-/2),$$

and

$$\mathrm{supp}(\mu^-) \subset \{u < 0\} \subset \omega(\mu^-/2), \quad \mathrm{supp}(\mu^+) \subset \{v > 0\} \subset \omega(\mu^+/2) \subset \omega(\mu^+)$$

by (24), we see that

$$
\begin{aligned}
-\Delta u &= (\mu^+ - \lambda)|_{\omega(\mu^+)} - (\mu^- - 2\lambda)|_{\omega(\mu^-/2)} \\
&\ge (\mu^+ - \lambda)^+ - (\mu^+ - \lambda)^-|_{\{u>0\}} - \lambda|_{\omega(\mu^+)\backslash\{u>0\}} \\
&\quad -(\mu^- - \lambda)|_{\{u<0\}} + \lambda|_{\omega(\mu^-/2)\backslash\{u<0\}} + \lambda|_{\omega(\mu^-/2)} \\
&\ge (\mu^+ - \lambda)^+ - (\mu^+ - \lambda)^-|_{\{u>0\}} - (\mu^- - \lambda)|_{\{u<0\}} \\
&= \eta(u, \mu).
\end{aligned}
$$

Similarly, $-\Delta v \le \eta(v, \mu)$, and it now follows from Theorem 4.3 that $v \le \overline{W}\mu \le u$. The result now follows from Corollary 4.6. $\qquad\square$

Finally, we consider the case where μ^+, μ^- have disjoint polar compact supports.

Corollary 5.4. *Suppose that μ^+ and μ^- have disjoint compact supports, and that $U\mu^+ = \infty$ on $\mathrm{supp}(\mu^+)$ and $U\mu^- = \infty$ on $\mathrm{supp}(\mu^-)$. Then there is a strong two-phase quadrature domain for subharmonic functions with respect to (μ^+, μ^-)*

Proof. With the notation from Theorem 5.3 it is clear that $v = \infty$ on $\mathrm{supp}(\mu^+)$ and $u = -\infty$ on $\mathrm{supp}(\mu^-)$, so (24) holds. $\qquad\square$

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