Ostrowski-type theorems for harmonic functions

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Abstract

Ostrowski showed that there are intimate connections between the gap structure of a Taylor series and the behaviour of its partial sums outside the disk of convergence. This paper investigates the corresponding problem for the homogeneous polynomial expansion of a harmonic function. The results for harmonic functions display new features in the case of higher dimensions.

1 Introduction

Let $B(x_0, r)$ denote the open ball with centre x_0 and radius r in Euclidean space $\mathbb{R}^N$ $(N \geq 2)$. If h is a harmonic function on $B(x_0, r)$, then its multiple Taylor series does not necessarily converge on the whole of $B(x_0, r)$. However, if we group the terms of the series according to their degree, we obtain an expansion of h which does converge on all of $B(x_0, r)$. We call this grouped Taylor series the homogeneous polynomial expansion of h about x_0 and denote by $S_m(h, x_0)$ the mth partial sum. Thus

$$S_m(h, x_0)(x) = \sum_{j=0}^{m} H_j(x - x_0),$$

where H_j is a homogeneous harmonic polynomial of degree j. (See Chapter 2 of [1].) The radius of the largest ball centred at x_0 inside which the above series converges locally uniformly is called the radius of convergence of the expansion.

[0] 2010 Mathematics Subject Classification 31B05.

Keywords: harmonic functions, homogeneous polynomial expansion, Ostrowski gaps.
This research was supported by Science Foundation Ireland under Grant 09/RFP/MTH2149.

In the case of holomorphic functions, celebrated work of Ostrowski (see [8], for example) shows a deep connection between the gap structure of the Taylor series expansion and the phenomenon of overconvergence of a subsequence of partial sums outside the disk of convergence. Ostrowski's insights have found new applications in recent years to the study of universal Taylor series (see [3], [5], [6], [7]). There is a corresponding notion of universal polynomial expansions for harmonic functions, but the theory is less well developed. As Tamptse has noted in [11], one of the barriers to progress is the absence of an Ostrowski-type theory for such expansions.

The purpose of this paper is to develop such a theory. It turns out that, in the case of harmonic functions, some of the results have a significantly different form, and this difference is essential in higher dimensions.

In order to state our results we need the following definition:

Definition: Let $\sum_{j=0}^{\infty} H_j(x-x_0)$ be the homogeneous polynomial expansion of a harmonic function on an open neighbourhood of x_0 and let (p_n) and (q_n) be two sequences of natural numbers such that $1 \leq p_1 < q_1 \leq p_2 < q_2 \leq ...$
We say that the expansion possesses *Hadamard-Ostrowski gaps* (p_n, q_n) if
(i) there exists $\theta > 0$ such that $q_n \geq (1+\theta)p_n$ for all $n \in \mathbb{N}$,
(ii) $H_j \equiv 0$ for $j \in \bigcup_{n=1}^{\infty} \{p_n + 1, ..., q_n\}$.
If we replace (i) with the stronger condition
(i$'$) $\dfrac{q_n}{p_n} \to \infty \quad$ as $n \to \infty$,
then we say that the expansion possesses *Ostrowski gaps* (p_n, q_n).

Throughout this paper $\mathcal{H}(\Omega)$ denotes the set of all harmonic functions on an open set $\Omega \subset \mathbb{R}^N$. For simplicity we write S_m instead of $S_m(h, 0)$.

Our first result is an analogue of Theorem I of Ostrowski [8].

Theorem 1 *Let $h \in \mathcal{H}(B(0,1))$ and suppose that h has a harmonic extension to a neighbourhood of some point $y \in \partial B(0,1)$. If the homogeneous polynomial expansion of h about 0 has radius of convergence 1 and possesses Hadamard-Ostrowski gaps (p_n, q_n), then the subsequence (S_{p_n}) of partial sums of h converges uniformly on a neighbourhood of y.*

The conclusion of Theorem 1 remains valid if to our initial function we add a harmonic function on $B(0, 1+\varepsilon)$ for some $\varepsilon > 0$. The following example, which was suggested by Stephen Gardiner, shows that the converse is not true for harmonic functions in higher dimensions. For the purposes of this example $B'(0, r)$ denotes the ball in $\mathbb{R}^{N-1}$ centred at 0 with radius r.

Example Let $N \geq 3$. Also, let $K(\ \cdot\ , y)$ be the Poisson kernel of $B'(0, 1)$ with pole at some fixed point $y \in \partial B'(0, 1)$. We consider the function $h : B'(0, 1) \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$h(x_1, ..., x_{N-1}, x_N) = K((x_1, ..., x_{N-1}), y).$$

Then the homogeneous expansion of h has radius of convergence 1 and its partial sums (S_n) converge locally uniformly on $B'(0, 1) \times \mathbb{R}$. However, h cannot be written in the form $h = g + v$ on $B(0, 1)$, where
(i) $v \in \mathcal{H}(B(0, 1+\varepsilon))$ for some $\varepsilon > 0$,
(ii) $g \in \mathcal{H}(B(0, 1))$ and the homogeneous expansion of g possesses Hadamard-Ostrowski gaps (p_n, q_n).

Thus, in contrast to Theorem II of Ostrowski in [8], a harmonic function on $B(0, 1)$ which has a subsequence of partial sums converging uniformly on a neighbourhood of some $y \in \partial B(0, 1)$, need not be the sum of a harmonic function on a larger ball and one with Hadamard-Ostrowski gaps. However, as the following theorem shows, there is still a significant relationship between overconvergence and occurrence of Hadamard-Ostrowski gaps. We use the following notation:
If $\delta > 0$, $y \in \partial B(0, 1)$ and $a, b \in \mathbb{R}$ with $a < b$, then we write

$$\mathcal{P}(y, \delta, a, b) = \{tu\ : u \in \partial B(0, 1) \cap \overline{B(y, \delta)},\ t \in [a, b]\}.$$

Theorem 2 *Let $h \in \mathcal{H}(B(0, 1))$ with homogeneous polynomial expansion about 0 which has radius of convergence 1, and assume that there exists a subsequence (S_{λ_n}) of partial sums of h which is uniformly bounded on some ball $B(w, \rho)$, disjoint from $B(0, 1)$. Then h can be written in the form $h = g + v$, where $g, v \in \mathcal{H}(B(0, 1))$ and*
(i) the homogeneous polynomial expansion of g possesses Hadamard-Ostrowski gaps,
(ii) the homogeneous polynomial expansion of v converges locally uniformly on $B(0, 1) \cup \mathcal{P}(\frac{w}{\|w\|}, \Delta, -r, r)$ for some $\Delta > 0$, $r > 1$.

Corollary 1 *Let h be harmonic on the unit disk $D(0, 1)$ in the complex plane $\mathbb{C}$ and suppose that it has a homogeneous polynomial expansion with radius of convergence 1. If there exist $\rho > 0$ and $z_0 \in \partial D(0, 1)$ such that a subsequence (S_{λ_n}) of partial sums of h converges uniformly* *on the disk $D(z_0, \rho)$, then there exist $g \in \mathcal{H}(D(0,1))$ with homogeneous polynomial expansion which possesses Hadamard-Ostrowski gaps and $v \in \mathcal{H}(D(0, 1+\varepsilon))$, such that $h = g + v$ on $D(0,1)$.*

Finally, we prove an analogue of the third main theorem of Ostrowski concerning overconvergence (Theorem III of [8]) for expansions which fulfil a stronger gap condition.

Theorem 3 *Let $h \in \mathcal{H}(B(0,1))$ and suppose that h has a harmonic extension to a domain G, strictly containing $B(0,1)$. If the homogeneous polynomial expansion of h about 0 has radius of convergence 1 and possesses Ostrowski gaps (p_n, q_n), then the subsequence (S_{p_n}) of partial sums of h converges locally uniformly on G.*

Remark For such a function h, there exists a largest domain D, containing $B(0,1)$, to which h can be extended harmonically. The maximum principle implies that $(\mathbb{R}^N \cup \{\infty\}) \setminus D$ is connected.

We will prove Theorems 1-3 and give details of the example in Section 3 following some preliminary material below.

2 Preliminaries

For the proofs of our results we will combine methods from the holomorphic case with tools from potential theory and some new arguments. We first prove a formula for the radius of convergence of a homogeneous polynomial expansion. If $y \in \partial B(0,1)$ and $j \in \mathbb{N}$, then $J_{y,j}$ denotes the y-axial homogeneous harmonic polynomial of degree j (for details we refer to Theorem 2.3.2 of [1]). Finally, λ denotes Lebesgue measure on $\mathbb{R}^N$ and σ denotes surface area measure on a sphere.

Lemma 1 *Let h be harmonic on an open set containing $\overline{B(0,\rho)}$ and let $\sum_{j=0}^{\infty} H_j(x)$ be the homogeneous polynomial expansion of h about 0.*
(i) *For each $j \in \mathbb{N}$ and $x \in B(0,\rho)$ we have*

$$H_j(x) = \frac{1}{\sigma(\partial B(0,\rho))} \int_{\partial B(0,\rho)} J_{\frac{y}{\rho},j}\left(\frac{x}{\rho}\right) h(y) d\sigma(y).$$

(ii) *There exists a constant $C > 0$, depending only on the dimension N, such that for each $j \in \mathbb{N}$*

$$L_j \leq \frac{C(j+1)^{N-2}}{\rho^j} \max_{\|y\|=\rho} |h(y)|,$$

where $L_j = \max_{\|y\|=1} |H_j(y)|$.

(iii) *The radius of convergence* r *of the expansion is given by*

$$r = R := \left(\limsup_{j\to\infty} L_j^{1/j}\right)^{-1},$$

where we interpret R *as* $+\infty$ *when* $\limsup_{j\to\infty} L_j^{1/j} = 0$.

Proof. (i) The formula can be derived by a suitable change of variable in formula (2.4.6) in [1].

(ii) By the j-homogeneity of H_j and the maximum principle,

$$L_j = \max_{\|x\|=\rho}\{\frac{1}{\|x\|^j}|H_j(x)|\} = \frac{1}{\rho^j}\sup_{\|x\|<\rho}|H_j(x)|.$$

By Theorem 2.4.3 of [1], there is a constant C, depending only on the dimension N, such that

$$\left|J_{\frac{y}{\rho},j}\left(\frac{x}{\rho}\right)\right| \leq C(j+1)^{N-2} \quad (x \in B(0,\rho),\ y \in \partial B(0,\rho),\ j \in \mathbb{N}).$$

Combining the above with part (i), we obtain the desired inequality.

(iii) We first observe that the radius of convergence coincides with the radius of the largest ball centred at 0 inside which h has a harmonic extension. By the j-homogeneity of H_j we see that $|H_j(x)| \leq L_j\|x\|^j$ for all $x \in \mathbb{R}^N$. Since the radius of convergence of the series $\sum_{j=0}^{\infty} L_j\|x\|^j$ is R, the series $\sum_{j=0}^{\infty} H_j(x)$ converges locally uniformly on $B(0,R)$. Hence $r \geq R$. (If $R = +\infty$ then $r = +\infty$ as well.) Let $\varrho \in (0,r)$. Then h has a harmonic extension to an open set containing $\overline{B(0,\varrho)}$. If $\max_{\|y\|=\varrho}|h(y)| = 0$, then h is identically 0 and $L_j = 0$, so $r = R = +\infty$. If $\max_{\|y\|=\varrho}|h(y)| \neq 0$, then

$$\lim_{j\to\infty}\left(\frac{C(j+1)^{N-2}}{\varrho^j}\max_{\|y\|=\varrho}|h(y)|\right)^{1/j} = \frac{1}{\varrho},$$

and so $\limsup_{j\to\infty} L_j^{1/j} \leq \frac{1}{\varrho}$ by part (ii). Now, by letting $\varrho \to r^-$, we get $\limsup_{j\to\infty} L_j^{1/j} \leq \frac{1}{r}$ and so $r \leq R$, which gives the desired formula. □

As we will see, the gap structure of the homogeneous polynomial expansion of a harmonic function h forces certain subsequences of its partial sums to converge (to h) at a faster rate inside the ball of convergence. The following theorems, due to Korevaar and Meyers [4], allow us to transfer this good property to certain sets lying outside the ball of convergence.

If $u \in L^2(\partial B(w,r))$ we write

$$\|u\|_{w,r,2} = \sqrt{\frac{1}{\sigma(\partial B(w,r))} \int_{\partial B(w,r)} u^2 d\sigma}.$$

Also, if u is bounded on a set K we write

$$\|u\|_K = \sup\{|u(x)| : x \in K\}.$$

Theorem A *Let Ω be a domain in $\mathbb{R}^N$, let $\Omega_0 \subset \Omega$ be a subdomain and $E \subset \Omega$ a compact subset. Then there is a constant $a = a\,(E, \Omega_0, \Omega) \in (0,1]$ such that, for all harmonic functions u on Ω,*

$$\|u\|_E \leq \|u\|_{\Omega_0}^a \|u\|_{\Omega}^{1-a}.$$

Theorem B *Let $0 < \rho < t < R$ and $w \in \mathbb{R}^N$. Then, for all bounded harmonic functions u on $B(w,R)$,*

$$\|u\|_{w,t,2} \leq \|u\|_{w,\rho,2}^{\beta} \|u\|_{w,R,2}^{1-\beta},$$

where β is the Hadamard exponent:

$$\beta = \frac{\log\,(t/R)}{\log\,(\rho/R)}.$$

Also we will make use of the following lemma which is a consequence of the subharmonic mean value inequality.

Lemma 2 *Let (u_n) be a sequence of non-negative subharmonic functions on a ball $B(z,t)$. If*

$$\int\limits_{B(z,t)} u_n d\lambda \to 0 \qquad as \quad n \to \infty,$$

then (u_n) converges to 0 locally uniformly on $B(z,t)$.

Finally we will use the next lemma in the proof of Theorem 2. This result is a "uniform" version of Theorem 3 of Gehlen [2].

Lemma 3 *Let K be a compact set in $\mathbb{C}$ with positive logarithmic* *capacity $c(K)$, let $M > 0$ and let (λ_n) be a subsequence of the positive integers. Then, for each $\varepsilon > 0$, there exists $\nu = \nu(\varepsilon) \in (\frac{1}{2}, 1)$ and $n_0 \in \mathbb{N}$ such that for all power series $\sum_{j=0}^{\infty} a_j z^j$ satisfying*

$$\sup_{n \in \mathbb{N}} \|s_{\lambda_n}\|_K \leq M \tag{2.1}$$

(where $s_m(z) = \sum_{j=0}^{m} a_j z^j$) we have

$$\max_{\nu\lambda_n \leq j \leq \lambda_n} |a_j|^{1/j} \leq \frac{1+\varepsilon}{c(K)} \quad (n \geq n_0).$$

Proof. We adapt the argument of Gehlen. Let $\varepsilon > 0$. We choose $\delta > 0$ such that $e^{\delta} < 1 + \varepsilon$. From the definition of the Green function g_K of $\mathbb{C} \setminus K$ with pole at ∞, we can find $R_\delta > 1$ such that if $|z| \geq R_\delta$, then

$$g_K(z) \leq \log|z| - \log c(K) + \delta.$$

Let T_j denote the set of the jth coefficients a_j of all power series $\sum_{j=0}^{\infty} a_j z^j$ satisfying (2.1). By applying Bernstein's lemma (see [9]) to the partial sums s_{λ_n} , we obtain

$$|s_{\lambda_n}(z)| \leq \|s_{\lambda_n}\|_K e^{\lambda_n g_K(z)} \leq M \left(\frac{|z|}{c(K)} e^{\delta}\right)^{\lambda_n} \qquad (|z| \geq R_\delta, n \in \mathbb{N}).$$

Further, Cauchy's formula implies that, for all $j = 1, 2, ..., \lambda_n$ and $a_j \in T_j$,

$$|a_j|^{1/j} = \left| \frac{1}{2\pi i} \int\limits_{|z|=R_\delta} \frac{s_{\lambda_n}(z)}{z^{j+1}} dz \right|^{1/j} \leq M^{1/j} \left(\frac{e^{\delta}}{c(K)}\right)^{\lambda_n/j} R_\delta^{\lambda_n/j-1}.$$

In particular, if $\nu \in (0,1)$ is sufficiently close to 1, then

$$\begin{aligned} \limsup_{n\to\infty} \max_{\nu\lambda_n \leq j \leq \lambda_n} \sup\{|a_j|^{1/j} : a_j \in T_j\} &\leq \frac{e^{\delta/\nu}}{\min\{c(K), c(K)^{1/\nu}\}} R_\delta^{1/\nu - 1} \\ &< \frac{1+\varepsilon}{c(K)}. \end{aligned}$$

□

3 Proofs

Proof of Theorem 1. Let $\sum_{j=0}^{\infty} H_j(x)$ be the homogeneous polynomial expansion of h about 0. Without loss of generality, we may assume that $y = (1, 0, ..., 0) \in \mathbb{R}^N$. Then, for sufficiently small $\delta \in (0, \frac{1}{2})$, the function h has a harmonic continuation to a neighbourhood of $\overline{B(z, \frac{1}{2}+\delta)}$, where $z = (\frac{1}{2}, 0, ..., 0) \in \mathbb{R}^N$. On $\overline{B(z, \frac{1}{2}+\delta)}$ we consider the functions h_n with $h_n(x) = h(x) - S_{p_n}(x)$. We will show that h_n converges locally uniformly to 0 on $B(z, \frac{1}{2}+\varepsilon)$ for sufficiently small $\varepsilon > 0$.

Since the homogeneous polynomial expansion of h possesses Hadamard-Ostrowski gaps (p_n, q_n), there is some $\theta > 0$ such that $q_n \geq (1+\theta)p_n$ for all $n \in \mathbb{N}$ and $H_j \equiv 0$ for $j \in \bigcup_{n=1}^{\infty}\{p_n + 1, ..., q_n\}$. Let $\eta := \mu\delta$, where $\mu \in (0, \frac{1}{2})$ is chosen small enough that

$$\frac{1+\theta}{\theta}(1-\mu) - \frac{1}{\theta}(1+\mu) > 0.$$

Let $L_j = \max_{\|x\|=1} |H_j(x)|$. Lemma 1(iii) shows that $\limsup_{j\to\infty} L_j^{1/j} = 1$. Hence, there exists $c > 1$ such that $L_j \leq c(1-\eta)^{-j}$ for all $j \in \mathbb{N}$. Additionally, by the j-homogeneity of H_j, we have $|H_j(x)| \leq L_j\|x\|^j$, for all $x \in \mathbb{R}^N$.

From all the above we see that, for each $x \in \overline{B(z, \frac{1}{2}-\delta)}$ and for each $n \in \mathbb{N}$

$$\begin{aligned} |h_n(x)| &\leq \sum_{j=p_n+1}^{\infty} |H_j(x)| = \sum_{j=q_n}^{\infty} |H_j(x)| \\ &\leq \sum_{j=q_n}^{\infty} \frac{c}{(1-\eta)^j}\|x\|^j \leq c\sum_{j=q_n}^{\infty}\left(\frac{1-\delta}{1-\eta}\right)^j \\ &= c\left(1-\frac{1-\delta}{1-\eta}\right)^{-1}\left(\frac{1-\delta}{1-\eta}\right)^{q_n} \leq K\left(\frac{1-\delta}{1-\mu\delta}\right)^{(1+\theta)p_n}, \end{aligned}$$

where $K = c\dfrac{1-\mu\delta}{(1-\mu)\delta}$.

Moreover, since h has a harmonic continuation to a neighbourhood of $\overline{B(z, \frac{1}{2}+\delta)}$, the function h is bounded there by a positive constant M. Hence for each $x \in \overline{B(z, \frac{1}{2}+\delta)}$ and for each $n \in \mathbb{N}$,

$$
\begin{aligned}
|h_n(x)| &\leq |h(x)| + \sum_{j=0}^{p_n} |H_j(x)| \\
&\leq M + \sum_{j=0}^{p_n} L_j \|x\|^j \\
&\leq M + \sum_{j=0}^{p_n} \frac{c}{(1-\eta)^j}(1+\delta)^j \\
&= M + c\left(\frac{1+\delta}{1-\eta}\right)^{p_n} \sum_{j=0}^{p_n} \left(\frac{1-\eta}{1+\delta}\right)^j \\
&\leq L\left(\frac{1+\delta}{1-\mu\delta}\right)^{p_n} \qquad ,
\end{aligned}
$$

where $L = M + c\dfrac{1+\delta}{(1+\mu)\delta}$.

Let $\varepsilon \in (0,\delta)$. We apply Theorem B for the three spheres centred at z with radii $\rho = \frac{1}{2} - \delta$, $t = \frac{1}{2} + \varepsilon$, $R = \frac{1}{2} + \delta$ and the harmonic functions h_n. This tells us that, for each $n \in \mathbb{N}$,

$$
\|h_n\|_{z,t,2} \leq \|h_n\|_{z,\rho,2}^{\beta} \|h_n\|_{z,R,2}^{1-\beta}, \text{ where } \beta = \frac{\log(\frac{t}{R})}{\log(\frac{\rho}{R})} = \frac{\log(\frac{1+2\delta}{1+2\varepsilon})}{\log(\frac{1+2\delta}{1-2\delta})}.
$$

By using the above estimates for the functions h_n on the balls $\overline{B(z,\frac{1}{2}-\delta)}$ and $\overline{B(z,\frac{1}{2}+\delta)}$ we deduce that

$$
\|h_n\|_{z,t,2} \leq K^{\beta}\left(\frac{1-\delta}{1-\mu\delta}\right)^{\beta(1+\theta)p_n} L^{1-\beta}\left(\frac{1+\delta}{1-\mu\delta}\right)^{(1-\beta)p_n} \leq c'\left(A_\delta(\varepsilon)\right)^{p_n},
$$

where $c' = \max\{K, L\}$ and

$$
A_\delta(\varepsilon) = \left(\left(\frac{1-\delta}{1-\mu\delta}\right)^{(1+\theta)\log(\frac{1+2\delta}{1+2\varepsilon})} \left(\frac{1+\delta}{1-\mu\delta}\right)^{\log(\frac{1+2\varepsilon}{1-2\delta})}\right)^{1/\log(\frac{1+2\delta}{1-2\delta})}.
$$

We claim that $A_\delta(\varepsilon) < 1$ for sufficiently small ε and δ. Indeed,

$$
A_\delta(\varepsilon) \to A_\delta^{1/\log(\frac{1+2\delta}{1-2\delta})} \text{ as } \varepsilon \to 0^+,
$$

where

$$
A_\delta = \left(1 - \frac{(1-\mu)\delta}{1-\mu\delta}\right)^{(1+\theta)\log(1+2\delta)} \left(1 + \frac{(1+\mu)\delta}{1-\mu\delta}\right)^{-\log(1-2\delta)}.
$$

However, since

$$\frac{\log A_\delta}{-2\theta\delta^2} \to \frac{1+\theta}{\theta}(1-\mu) - \frac{1}{\theta}(1+\mu) > 0 \quad \text{as} \quad \delta \to 0^+,$$

we can find a sufficiently small $\delta > 0$ such that $\log A_\delta < 0$, or equivalently, $A_\delta < 1$. Thus, for a suitable choice of $\varepsilon \in (0, \delta)$, the quantity $A_\delta(\varepsilon)$ is strictly less than 1, and so $c' \, (A_\delta(\varepsilon))^{p_n} \to 0$ as $n \to \infty$. Consequently $\|h_n\|_{z,t,2} \to 0$ as $n \to \infty$.

Since h_n is harmonic on a neighbourhood of $\overline{B(z,t)}$, the function h_n^2 is subharmonic on the same neighbourhood. Therefore,

$$\frac{1}{\lambda(B(z,t))} \int_{B(z,t)} h_n^2 d\lambda \leq \frac{1}{\sigma(\partial B(z,t))} \int_{\partial B(z,t)} h_n^2 d\sigma = \|h_n\|_{z,t,2}^2.$$

Hence $\displaystyle\int_{B(z,t)} h_n^2 d\lambda \to 0$, as $n \to \infty$ and the result follows by applying Lemma 2 to the non-negative subharmonic functions (h_n^2). $\square$

Details of Example. Since $K(\,\cdot\,, y) \in \mathcal{H}(B'(0,1))$, the function h is harmonic on the cylinder $B'(0,1) \times \mathbb{R}$. Let $\sum_{j=0}^{\infty} H_j(x_1, ..., x_N)$ be the homogeneous polynomial expansion of h about the origin. Then the radius of convergence of this expansion is 1 because $h(x, 0) \to +\infty$ as $x \to y$, where $x \in B'(0,1)$. Using Theorem 2.4.3 of [1] we obtain

$$h(x_1, ..., x_{N-1}, x_N) = K((x_1, ..., x_{N-1}), y) = \sum_{j=0}^{\infty} \frac{1}{\sigma_{N-1}} J_{y,j}(x_1, ..., x_{N-1}),$$

where $J_{y,j}$ denotes the y-axial homogeneous polynomial of degree j in $\mathbb{R}^{N-1}$ and $\sigma_{N-1} = \sigma(\partial B'(0,1))$. The uniqueness of the homogeneous polynomial expansion of h in $B(0,1)$ implies $H_j(x_1, ..., x_{N-1}, x_N) = \dfrac{1}{\sigma_{N-1}} J_{y,j}(x_1, ..., x_{N-1})$ for each $j \in \mathbb{N}$ and for each $(x_1, ..., x_{N-1}, x_N) \in \mathbb{R}^N$. Since the series $\dfrac{1}{\sigma_{N-1}} \sum_{j=0}^{\infty} J_{y,j}$ converges locally uniformly on $B'(0,1)$ to $K(\,\cdot\,, y)$, the sequence (S_n) of partial sums of h converges locally uniformly on $B'(0,1) \times \mathbb{R}$ (to h). We will show that h cannot be written in the form $h = g + v$ on $B(0,1)$, where
(i) $v \in \mathcal{H}(B(0, 1+\varepsilon))$ for some $\varepsilon > 0$,
(ii) $g \in \mathcal{H}(B(0,1))$ and it has a homogeneous polynomial expansion with Hadamard-Ostrowski gaps (p_n, q_n).

For the sake of contradiction we assume that h can be written in the above form for some functions v and g. Let $\sum_{j=0}^{\infty} v_j$ and $\sum_{j=0}^{\infty} g_j$ be the homogeneous polynomial expansions of v and g respectively. Then, using again the uniqueness of the homogeneous polynomial expansion of h, we deduce that $H_j = v_j + g_j$ in $\mathbb{R}^N$. Therefore, for each $(x_1, ..., x_{N-1}, x_N) \in \mathbb{R}^N$ and for each $j \in \mathbb{N}$,

$$\frac{1}{\sigma_{N-1}} J_{y,j}(x_1, ..., x_{N-1}) = g_j(x_1, ..., x_{N-1}, x_N) + v_j(x_1, ..., x_{N-1}, x_N).$$

In particular, condition (ii) shows that, for each $(x_1, ..., x_{N-1}, x_N) \in \mathbb{R}^N$ and each $j \in I = \bigcup_{n=1}^{\infty} \{p_n + 1, ..., q_n\}$,

$$\frac{1}{\sigma_{N-1}} J_{y,j}(x_1, ..., x_{N-1}) = v_j(x_1, ..., x_{N-1}, x_N).$$

Let

$$V_j = \max\{|v_j(x_1, ..., x_{N-1}, x_N)| \ : (x_1, ..., x_{N-1}, x_N) \in \partial B(0, 1)\}.$$

Then, for each $j \in I$,

$$\begin{aligned} V_j &= \max\{\frac{1}{\sigma_{N-1}} |J_{y,j}(x_1, ..., x_{N-1})| \ : (x_1, ..., x_{N-1}, x_N) \in \partial B(0, 1)\} \\ &= \max\{\frac{1}{\sigma_{N-1}} |J_{y,j}(x_1, ..., x_{N-1})| \ : (x_1, ..., x_{N-1}) \in \partial B'(0, 1)\} \end{aligned}$$

Consequently, since $y \in \partial B'(0, 1)$,

$$\limsup_{j\to\infty, j\in I} \left| \frac{1}{\sigma_{N-1}} J_{y,j}(y) \right|^{1/j} \leq \limsup_{j\to\infty, j\in I} V_j^{1/j} \leq \limsup_{j\to\infty} V_j^{1/j}.$$

Additionally, condition (i) and Lemma 1(iii) imply that $\limsup_{j\to\infty} V_j^{1/j} < 1$, and so

$$\limsup_{j\to\infty, j\in I} \left| \frac{1}{\sigma_{N-1}} J_{y,j}(y) \right|^{1/j} < 1.$$

As a final step we will show that $\lim_{j\to\infty} \left| \frac{1}{\sigma_{N-1}} J_{y,j}(y) \right|^{1/j} = 1$, which contradicts the above estimate. Indeed, from Corollary 2.3.7 of [1], we obtain $J_{y,j}(y) = d_{j,N-1}$, where $d_{j,N-1}$ is the dimension of the space of harmonic homogeneous polynomials of degree j in $N-1$ variables. Further, Corollary 2.1.4 of [1] gives

$$\begin{aligned} d_{j,N-1} &= \binom{j+N-2}{j} - \binom{j+N-4}{j-2} \\ &= \frac{1}{(N-2)!} \{(j+N-2) \cdot ... \cdot (j+1) - (j+N-4) \cdot ... \cdot (j-1)\}. \end{aligned}$$

Thus $d_{j,N-1}$ is a polynomial in j, and so

$$\lim_{j\to\infty}\left|\frac{1}{\sigma_{N-1}}J_{y,j}(y)\right|^{1/j} = \lim_{j\to\infty} d_{j,N-1}^{1/j} = 1.$$

□

Proof of Theorem 2. Let h, (λ_n), w and ρ be as in the statement of the theorem and let $\sum_{j=0}^{\infty} H_j(x)$ be the homogeneous polynomial expansion of h about 0. For each $u \in \partial B(0,1)$, $m \in \mathbb{N}$ we define the directional complexified partial sums

$$S_m^{(u)}(z) = \sum_{j=0}^{m} H_j(u)z^j \qquad (z \in \mathbb{C}).$$

Clearly $S_m^{(u)}(t) = S_m(tu)$ for every $t \in \mathbb{R}$, $u \in \partial B(0,1)$. We observe that $\mathcal{P}(\frac{w}{\|w\|}, \Delta, \|w\|, \|w\| + \frac{\rho}{2}) \subset B(w,\rho)$ for sufficiently small $\Delta > 0$. By considering the compact sets $K_m = \overline{D(0,1-\frac{1}{m})} \cup \left[\|w\|, \|w\| + \frac{\rho}{2}\right]$ we see that $K_m \nearrow D(0,1) \cup \left[\|w\|, \|w\| + \frac{\rho}{2}\right]$ as $m \to \infty$. Thus

$$\begin{aligned} c(K_m) &\to c(D(0,1) \cup \left[\|w\|, \|w\| + \frac{\rho}{2}\right]) \\ &= c(\overline{D(0,1)} \cup \left[\|w\|, \|w\| + \frac{\rho}{2}\right]) \\ &> c(\overline{D(0,1)}) = 1, \end{aligned}$$

where $c(\cdot)$ denotes logarithmic capacity. Thus we can find $m_0 \in \mathbb{N}$ such that $c(K_{m_0}) > 1$.

<u>Claim:</u> There exists $M > 0$ such that $\left|S_{\lambda_n}^{(u)}(z)\right| \leq M$ for all $z \in K_{m_0}$, $n \in \mathbb{N}$ and $u \in T := \partial B(0,1) \cap \overline{B(\frac{w}{\|w\|}, \Delta)}$.
Proof of the claim: If $t \in \left[\|w\|, \|w\| + \frac{\rho}{2}\right]$, then $tu \in B(w,\rho)$ for every $u \in \partial B(0,1) \cap \overline{B(\frac{w}{\|w\|}, \Delta)}$, from the choice of Δ. Hence, by hypothesis, there exists $M_0 > 0$ such that, for all $t \in \left[\|w\|, \|w\| + \frac{\rho}{2}\right]$, $n \in \mathbb{N}$ and $u \in \partial B(0,1) \cap \overline{B(\frac{w}{\|w\|}, \Delta)}$,

$$|S_{\lambda_n}^{(u)}(t)| = |S_{\lambda_n}(tu)| \leq M_0.$$

If $z \in \overline{D(0,1-\frac{1}{m_0})}$, then $|z|u \in \overline{B(0,1-\frac{1}{m_0})}$ for every $u \in \partial B(0,1)$. The local Weierstrass convergence of the homogeneous polynomial expansion of h (see Theorem 2.4.4 of [1]) implies that

$$M_1 := \sum_{j=0}^{\infty} \sup\{|H_j(x)| : x \in \overline{B(0,1-\frac{1}{m_0})}\} < +\infty.$$

Hence, for all $z \in \overline{D(0, 1 - \frac{1}{m_0})}$, $n \in \mathbb{N}$ and $u \in \partial B(0,1)$,

$$\left| S_{\lambda_n}^{(u)}(z) \right| \leq \sum_{j=0}^{\lambda_n} \left| H_j(u) z^j \right| = \sum_{j=0}^{\lambda_n} |H_j(|z|u)| \leq M_1.$$

We finish the proof of the claim by setting $M = \max\{M_0, M_1\}$.

Since $c(K_{m_0}) > 1$, we can choose $\varepsilon > 0$ such that

$$\frac{1+\varepsilon}{c(K_{m_0})} < 1.$$

In view of the above claim, we can apply Lemma 3 to the Taylor polynomials $(S_m^{(u)})_m$ for all $u \in T$. Hence we find $\nu \in (\frac{1}{2}, 1)$, $\mu < 1$ and $n_0 \in \mathbb{N}$ such that

$$|H_j(u)|^{1/j} \leq \mu \tag{3.1}$$

for all $u \in T$ and $j \in S = \bigcup_{n=n_0}^{\infty} \{[\nu\lambda_n] + 1, ..., \lambda_n\}$.

Without loss of generality we may assume that $\lambda_{n+1} \geq 2\lambda_n$ (for otherwise we can choose a suitable subsequence of (λ_n)). Hence, if we set $p_n = [\nu\lambda_{n_0+n-1}]$ and $q_n = \lambda_{n_0+n-1}$, we have

$$1 \leq p_1 < q_1 \leq p_2 < q_2 \leq ... \quad \text{and} \quad \frac{q_n}{p_n} \geq \frac{1}{\nu} > 1 \text{ for all } n \in \mathbb{N}.$$

We define

$$G_j = \begin{cases} 0 & \text{if } j \in S \\ H_j & \text{if } j \in \mathbb{N} \setminus S \end{cases}$$

and

$$V_j = \begin{cases} H_j & \text{if } j \in S \\ 0 & \text{if } j \in \mathbb{N} \setminus S \end{cases}.$$

The local Weierstrass convergence of the homogeneous polynomial expansion implies that the series

$$g(x) = \sum_{j=0}^{\infty} G_j(x) \quad , \quad v(x) = \sum_{j=0}^{\infty} V_j(x)$$

have radius of convergence at least 1, and so they define harmonic functions on $B(0,1)$. Clearly g possesses Hadamard-Ostrowski gaps (p_n, q_n) and $h = g + v$ on $B(0,1)$. We choose $r \in (1, \frac{1}{\mu})$. From (3.1) we deduce that, for all $j \in \mathbb{N}$, $t \in [-r, r]$ and $u \in T = \partial B(0,1) \cap \overline{B(\frac{w}{\|w\|}, \Delta)}$,

$$|V_j(tu)| = |V_j(u)t^j| \leq \mu^j r^j.$$

Consequently, the choice of r gives

$$\sum_{j=0}^{\infty} \sup\{|V_j(x)| : x \in \mathcal{P}(\frac{w}{\|w\|}, \Delta, -r, r)\} \leq \sum_{j=0}^{\infty} (\mu r)^j < +\infty.$$

Hence, by using the Weierstrass M-test, we conclude that the expansion of v converges uniformly on $\mathcal{P}(\frac{w}{\|w\|}, \Delta, -r, r)$, which completes the proof of the theorem. $\square$

Proof of Corollary 1. From Theorem 2 we can write h in the form $h = g + v$, where $g, v \in \mathcal{H}(D(0,1))$ and
(i) the homogeneous polynomial expansion of g possesses Hadamard-Ostrowski gaps,
(ii) the homogeneous polynomial expansion of v converges locally uniformly on $D(0,1) \cup \mathcal{P}(z_0, \Delta, -r, r)$ for some $\Delta > 0$, $r > 1$.

Applying Proposition 1.4 (i) of Siciak and Kolodziej [10] to v, we see that its expansion converges locally uniformly on $D(0,r)$ and therefore h has the desired form. $\square$

Proof of Theorem 3. Let $\sum_{j=0}^{\infty} H_j(x)$ be the homogeneous polynomial expansion of h about 0. Also let E be a compact subset of G. We consider the functions h_n with $h_n(x) = h(x) - S_{p_n}(x)$ and will show that $h_n \to 0$ uniformly on E. Lemma 1(iii) and the j-homogeneity of H_j imply that there is a constant $K > 1$ such that $|H_j(x)| \leq K \left(\frac{4}{3}\right)^j \|x\|^j$ for all $x \in \mathbb{R}^N$ and $j \in \mathbb{N}$. Since the homogeneous polynomial expansion of h possesses Ostrowski gaps (p_n, q_n), for each $x \in B(0, \frac{1}{2})$ and $n \in \mathbb{N}$, we have

$$|h_n(x)| \leq \sum_{j=p_n+1}^{\infty} |H_j(x)| = \sum_{j=q_n+1}^{\infty} |H_j(x)| \leq 3K \left(\frac{2}{3}\right)^{q_n}$$

Now we choose a bounded domain Ω such that $E \cup \overline{B(0, \frac{1}{2})} \subset \Omega \subset \overline{\Omega} \subset G$. Since h is continuous on the compact set $\overline{\Omega}$, we know that $\|h\|_{\overline{\Omega}} < +\infty$. Also $\Omega \subset B(0, R)$ for some $R > 1$. Hence, for each $x \in \Omega$ and $n \in \mathbb{N}$,

$$|h_n(x)| \leq |h(x)| + \sum_{j=0}^{p_n} |H_j(x)| \leq \|h\|_{\overline{\Omega}} + K \sum_{j=0}^{p_n} \left(\frac{4}{3}R\right)^j \leq L \left(\frac{4}{3}R\right)^{p_n},$$

where $L = K \left(\sum_{j=0}^{\infty} \left(\frac{3}{4R}\right)^j + \|h\|_{\overline{\Omega}}\right) < +\infty$.

By applying Theorem A to the harmonic functions h_n and the sets E, Ω and $\Omega_0 = B(0, \frac{1}{2})$, we find a constant $a = a(E, \Omega_0, \Omega) \in (0, 1]$ such that, for every $n \in \mathbb{N}$,

$$\|h_n\|_E \leq \|h_n\|_{\Omega_0}^a \|h_n\|_{\Omega}^{1-a} .$$

If we set $c = \max\{3K, L\}$ and $M_n = \dfrac{q_n}{p_n}$, the above estimates give

$$\begin{aligned} \|h_n\|_E &\leq (3K)^a \left(\frac{2}{3}\right)^{M_n p_n a} L^{1-a} \left(\frac{4}{3}R\right)^{p_n(1-a)} \\ &\leq c\left(\left(\frac{2}{3}\right)^{aM_n}\left(\frac{4}{3}R\right)^{1-a}\right)^{p_n} \end{aligned}$$

for all $n \in \mathbb{N}$. From the definition of Ostrowski gaps, $M_n \to \infty$ as $n \to \infty$, and since $\left(\frac{2}{3}\right)^a < 1$ we deduce that $\|h_n\|_E \to 0$ as $n \to \infty$. Equivalently, (S_{p_n}) converges to h uniformly on E and the result follows from the arbitrary nature of E. □

Acknowledgements: I would like to thank Professor Stephen Gardiner for his guidance and his helpful suggestions.

References

[1] Armitage D. H., Gardiner S. J.: Classical Potential Theory. Springer, London (2001).

[2] Gehlen W.: Overconvergent power series and conformal maps, J. Math. Anal. Appl., 198, 490-505 (1996).

[3] Gehlen W., Luh W., Müller J.: On the existence of O-universal functions, Complex Variables 41, 81-90 (2000).

[4] Korevaar J., Meyers J. L. H.: Logarithmic convexity for supremum norms of harmonic functions, Bull. London Math. Soc. 26, 353-362 (1994).

[5] Luh W.: Universal approximation properties of overconvergent power series on open sets, Analysis 6, 191-207 (1986).

[6] Melas A., Nestoridis V.: Universality of Taylor series as a generic property of holomorphic functions, Adv. Math.157, 138-176 (2001).

[7] Müller J., Vlachou V., Yavrian A.: Universal overconvergence and Ostrowski-gaps, Bull. London Math. Soc. 38, 597-606 (2006).

[8] Ostrowski A.: On representation of analytical functions by power series, J. London Math. Soc. 1, 251-263 (1926).

[9] Ransford T., Potential Theory in the Complex Plane, Cambridge University Press (1995).

[10] Siciak J., Kolodziej S.: Laplace domains, Classical Analysis (Kazimierz Dolny, 1991), ed. T. Mazur, World Sci. Publ. River Edge, NJ, pp. 247-264 (1992).

[11] Tamptse I.: Approximation of harmonic functions by universal overconvergent series, Ph.D thesis, Univ. de Montreal (2008).

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