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THE FIRST POSITIVE RANK AND CRANK MOMENTS FOR OVERPARTITIONS

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Abstract. In 2003, Atkin and Garvan initiated the study of rank and crank moments for ordinary partitions. These moments satisfy a strict inequality. We prove that a strict inequality also holds for the first rank and crank moments of overpartitions and consider a new combinatorial interpretation in this setting.

1. Introduction

A partition of a non-negative integer n is a non-increasing sequence of positive integers whose sum is n. For example, the 5 partitions of 4 are

$$4, 3+1, 2+2, 2+1+1, 1+1+1+1.$$

In 1944, Dyson introduced the rank of a partition as the largest part minus the number of parts [17]. In 1988, the first author and Garvan defined the crank of a partition as either the largest part, if 1 does not occur as a part, or the difference between the number of parts larger than the number of 1's and the number of 1's, if 1 does occur [4]. These two statistics give a combinatorial explanation of Ramanujan's congruences for the partition function modulo 5, 7 and 11. Let $N(m,n)$ denote the number of partitions of n whose rank is m and $M(m,n)$ the number of partitions of n whose crank is m.

A recent development in the theory of partitions has been the study of rank and crank moments as initiated by Atkin and Garvan [5]. For $k \geq 1$, the kth rank moment $N_k(n)$ and the kth crank moment $M_k(n)$ are given by

$$N_k(n) := \sum_{m \in \mathbb{Z}} m^k N(m,n) \tag{1.1}$$

and

$$M_k(n) := \sum_{m \in \mathbb{Z}} m^k M(m,n). \tag{1.2}$$

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As $N(-m,n)=N(m,n)$ [17] and $M(-m,n)=M(m,n)$ [4], we have $N_k(n)=M_k(n)=0$ for k odd. The even moments are of considerable interest as they have been the subject of a number of works [1, 2, 6, 7, 10, 12, 13, 16, 18, 19, 20, 29]. In particular, Garvan [19] conjectured that

$$M_{2j}(n) > N_{2j}(n) \tag{1.3}$$

for all j, $n \geq 1$. In [12], (1.3) was proved for fixed j and sufficiently large n. Garvan proved (1.3) for all j and n via symmetrized rank and crank moments and Bailey pairs [20]. Recently, the first three authors gave an elementary proof of (1.3) by considering modified versions of (1.1) and (1.2). Namely, consider the positive rank and crank moments

$$N_k^+(n) := \sum_{m=1}^{\infty} m^k N(m,n)$$

and

$$M_k^+(n) := \sum_{m=1}^{\infty} m^k M(m,n).$$

In [3], it was proved that

$$M_k^+(n) > N_k^+(n) \tag{1.4}$$

for all k, $n \geq 1$ by a careful study of the decomposition of the generating function for the difference $M_k^+(n) - N_k^+(n)$. For a discussion concerning the asymptotic behavior of these moments, see [11]. Inequality (1.4) combined with the fact that $N_{2j}(n) = 2N_{2j}^+(n)$ and $M_{2j}(n) = 2M_{2j}^+(n)$ imply (1.3).

Our interest in this paper is to consider an analogue of (1.4) for overpartitions. More specifically, we will investigate the first moments for overpartitions and what is counted by the difference. Recall that an overpartition [27] is a partition in which the first occurrence of each distinct number may be overlined. For example, the 14 overpartitions of 4 are

$$4, \overline{4}, 3+1, \overline{3}+1, 3+\overline{1}, \overline{3}+\overline{1}, 2+2, \overline{2}+2, 2+1+1, \overline{2}+1+1, 2+\overline{1}+1,$$
$$\overline{2}+\overline{1}+1, 1+1+1+1, \overline{1}+1+1+1.$$

These combinatorial objects have recently played an important role in the construction of weight 3/2 mock modular forms [8], in Rogers-Ramanujan and Gordon type identities [14] and in the study of Jack superpolynomials in supersymmetry and quantum mechanics [15].

Let $\overline{N}(n,m)$ denote the number of overpartitions of n whose rank is m and $\overline{M}(n,m)$ the number of overpartitions of n whose (first residual) crank is m. Here, Dyson's rank extends easily to overpartitions and the first residual crank of an overpartition is obtained by taking the crank of the subpartition consisting of the non-overlined parts [9]. It is now natural to consider the rank and crank overpartition moments

$$\overline{N}_k(n) := \sum_{m \in \mathbb{Z}} m^k \overline{N}(m,n)$$

and

$$\overline{M}_k(n) := \sum_{m\in\mathbb{Z}} m^k \overline{M}(m,n).$$

Via the symmetries $\overline{N}(-m,n) = \overline{N}(m,n)$ [25] and $\overline{M}(-m,n) = \overline{M}(m,n)$ [9], we have $\overline{N}_k(n) = \overline{M}_k(n) = 0$ for k odd. Thus, to obtain non-trivial odd moments, we consider

$$\overline{N}_k^+(n) := \sum_{m=1}^{\infty} m^k \overline{N}(m,n)$$

and

$$\overline{M}_k^+(n) := \sum_{m=1}^{\infty} m^k \overline{M}(m,n).$$

The main result in this paper is an analogue of (1.4) for overpartitions in the case $k = 1$.

Theorem 1.1. *For all $n \geq 1$, we have*

$$\overline{M}_1^+(n) > \overline{N}_1^+(n). \tag{1.5}$$

The paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we give a combinatorial interpretation of $\overline{M}_1^+(n) - \overline{N}_1^+(n)$. In Section 4, we conclude with some remarks regarding future directions.

2. The proof of Theorem 1.1

For $k \geq 1$, we define the generating functions

$$\overline{M}_k(q) = \sum_{n=1}^{\infty} \overline{M}_k^+(n) q^n$$

and

$$\overline{R}_k(q) = \sum_{n=1}^{\infty} \overline{N}_k^+(n) q^n$$

and compute their explicit expressions for $k = 1$. Throughout, we use the standard q-hypergeometric notation,

$$(a)_n = (a;q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}),$$

valid for $n \in \mathbb{N} \cup \{\infty\}$. For convenience, we define $(a;q)_0 = 1$.

Proposition 2.1. *We have*

$$\overline{R}_1(q) = \frac{2(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{q^{n(n+1)}}{1-q^{2n}} \tag{2.1}$$

and

$$\overline{M}_1(q) = \frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{q^{n(n+1)/2}}{1-q^n}. \tag{2.2}$$

Proof. We begin with the generalized Lambert series representation of the two-variable generating function for Dyson's rank for overpartitions,

$$\begin{aligned}
\overline{R}(z,q) &:= \sum_{n=0}^{\infty} \sum_{m\in\mathbb{Z}} \overline{N}(m,n) z^m q^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)_n q^{n(n+1)/2}}{(zq)_n (q/z)_n} \\
&= \frac{(-q)_\infty}{(q)_\infty} \left(1 + 2\sum_{n=1}^{\infty} \frac{(1-z)(1-1/z)(-1)^n q^{n^2+n}}{(1-zq^n)(1-q^n/z)} \right) \\
&= \frac{(-q)_\infty}{(q)_\infty} \left(1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} - 2\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}(1-q^n)}{1+q^n} \left(\sum_{m=0}^{\infty} z^m q^{mn} + \sum_{m=1}^{\infty} z^{-m} q^{mn} \right) \right).
\end{aligned} \tag{2.3}$$

For the second and third equalities in (2.3), see the proof of Proposition 3.2 in [25]. Here, we have used the identity

$$\frac{(1-z)(1-1/z)q^n}{(1-zq^n)(1-q^n/z)} = 1 - \frac{1-q^n}{1+q^n} \left(\sum_{m=0}^{\infty} z^m q^{mn} + \sum_{m=1}^{\infty} z^{-m} q^{mn} \right)$$

for the last equality in (2.3). We now apply the differential operator $z\frac{\partial}{\partial z}$ to both sides of (2.3) to obtain

$$\begin{aligned}
&z\frac{\partial}{\partial z}\left(\overline{R}(z,q)\right) \\
&= \frac{(-q)_\infty}{(q)_\infty} \left(2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n^2}(1-q^n)}{1+q^n} \sum_{m=1}^{\infty} m z^m q^{mn} + 2\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}(1-q^n)}{1+q^n} \sum_{m=1}^{\infty} m z^{-m} q^{mn} \right).
\end{aligned} \tag{2.4}$$

Only the first term on the right side of (2.4) contributes to positive powers of z and so

$$\begin{aligned}
\overline{R}_1(q) &= \lim_{z\to 1} \frac{2(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n^2}(1-q^n)}{1+q^n} \sum_{m=1}^{\infty} m z^m q^{mn} \\
&= \frac{2(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n^2}(1-q^n)}{1+q^n} \sum_{m=1}^{\infty} m q^{mn} \\
&= \frac{2(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n+1)}}{1-q^{2n}},
\end{aligned} \tag{2.5}$$

which is (2.1). In the last equality of (2.5), we applied the identity

$$\sum_{m=1}^{\infty} m q^{mn} = \frac{q^n}{(1-q^n)^2}.$$

For the two-variable generating function for the first residual crank for overpartitions [9], we have

$$\overline{C}(z,q) := \sum_{n=0}^{\infty} \sum_{m\in\mathbb{Z}} \overline{M}(m,n) z^m q^n = (-q)_\infty C(z,q) \tag{2.6}$$

where $C(z,q)$ is the two-variable generating function for the crank for partitions. Thus, by the proof of Theorem 1 in [3], we obtain (2.2).

□

We now require the following two lemmas for the proof of Theorem 1.1.

Lemma 2.2. *If*

$$h(q) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n+1)/2}}{1-q^n},$$

then

$$h(q) = \sum_{j=1}^{\infty} q^{j^2}(1 + 2q^j + 2q^{2j} + \cdots + 2q^{j^2-j} + q^{j^2}).$$

Proof. We first note that

$$\sum_{j=1}^{\infty} \frac{q^j}{1-q^j} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} q^{jk} = \sum_{j=1}^{\infty}\sum_{k\geq j} q^{jk} + \sum_{k=1}^{\infty}\sum_{j>k} q^{jk} = \sum_{j=1}^{\infty} \frac{q^{j^2}(1+q^j)}{1-q^j}. \tag{2.7}$$

By employing a similar argument, we can also derive that

$$\sum_{j=1}^{\infty} \frac{q^{2j-1}}{1-q^{2j-1}} = \sum_{j=1}^{\infty} \frac{q^{j(j+1)/2}}{1-q^j}. \tag{2.8}$$

By expanding the summation according to the parity of n, we find that

$$\begin{aligned}
h(q) &= \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{1-q^n} - 2\sum_{n=1}^{\infty} \frac{q^{n(2n+1)}}{1-q^{2n}} \\
&= \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{1-q^n} + \sum_{n=1}^{\infty} \frac{q^{2n^2}(1+q^{2n})}{1-q^{2n}} - \left(\sum_{n=1}^{\infty} \frac{q^{2n^2}(1+q^{2n})}{1-q^{2n}} + 2\sum_{n=1}^{\infty} \frac{q^{n(2n+1)}}{1-q^{2n}}\right) \\
&= \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1-q^{2n-1}} + \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} - \sum_{n=1}^{\infty} \frac{q^{2n^2}(1+q^n)^2}{1-q^{2n}} \quad \text{by (2.7) and (2.8)} \\
&= \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} - \sum_{n=1}^{\infty} \frac{q^{2n^2}(1+q^n)}{1-q^n}
\end{aligned}$$

$$= \sum_{j=1}^{\infty} \frac{q^{j^2}(1+q^j)}{1-q^j} - \sum_{j=1}^{\infty} \frac{q^{2j^2}(1+q^j)}{(1-q^j)} \quad \text{by (2.7)}$$
$$= \sum_{j=1}^{\infty} \frac{q^{j^2}(1+q^j)(1-q^{j^2})}{1-q^j}$$
$$= \sum_{j=1}^{\infty} q^{j^2}(1+q^j)(1+q^j+\cdots+q^{j(j-1)})$$
$$= \sum_{j=1}^{\infty} q^{j^2}(1+2q^j+2q^{2j}+\cdots+2q^{j^2-j}+q^{j^2}).$$

□

Lemma 2.3.

$$h(q) - 2h(q^2) = \sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2}\left(1-2q^n+2q^{2n}-\cdots+(-1)^{n-1}2q^{n^2-n}+(-1)^n q^{n^2}\right). \quad (2.9)$$

Proof. Expanding the right side of (2.9) according to the parity of n and then separating the positive terms from the negative terms, we find that

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2}\left(1-2q^n+2q^{2n}+\cdots+(-1)^{n-1}2q^{n^2-n}+(-1)^n q^{n^2}\right)\\
&= \sum_{n=1}^{\infty} q^{(2n-1)^2}\left(1+2q^{4n-2}+2q^{8n-4}+\cdots+2q^{4n^2-6n+2}\right)\\
&\quad - \sum_{n=1}^{\infty} q^{(2n-1)^2}\left(2q^{2n-1}+2q^{6n-3}+\cdots+2q^{4n^2-8n+3}+q^{(2n-1)^2}\right) \qquad (2.10)\\
&\quad + \sum_{n=1}^{\infty} q^{(2n)^2}\left(2q^{2n}+2q^{6n}+\cdots+2q^{4n^2-2n}\right)\\
&\quad - \sum_{n=1}^{\infty} q^{(2n)^2}\left(1+2q^{4n}+\cdots+2q^{4n^2-4n}+q^{(2n)^2}\right).
\end{aligned}$$

Using Lemma 2.2, we compute a similar expansion for $h(q)$, then compare with (2.10) in order to see that it suffice to prove

$$\begin{aligned}
h(q^2) &= \sum_{n=1}^{\infty} q^{2n^2}\left(1+2q^{2n}+2q^{4n}+\cdots+2q^{2n^2-2n}+q^{2n^2}\right)\\
&= \sum_{n=1}^{\infty} q^{(2n-1)^2}\left(2q^{2n-1}+2q^{6n-3}+\cdots+2q^{4n^2-8n+3}+q^{(2n-1)^2}\right)\\
&\quad + \sum_{n=1}^{\infty} q^{(2n)^2}\left(1+2q^{4n}+\cdots+2q^{4n^2-4n}+q^{(2n)^2}\right). \qquad (2.11)
\end{aligned}$$

Subtracting $\sum_{n=1}^{\infty} q^{2n^2}\left(1+q^{2n^2}\right)$ from both sides of (2.11) and then dividing by 2, it remains to show that

$$\begin{aligned}&\sum_{n=2}^{\infty} q^{2n^2}\left(q^{2n}+q^{4n}+\cdots+q^{2n^2-2n}\right)\\&=\sum_{n=2}^{\infty} q^{(2n-1)^2}\left(q^{2n-1}+q^{6n-3}+\cdots+q^{4n^2-8n+3}\right)+\sum_{n=2}^{\infty} q^{4n^2}\left(q^{4n}+\cdots+q^{4n^2-4n}\right).\end{aligned}\tag{2.12}$$

Define $f(n,j)=q^{2n^2+2nj}$. Substituting $f(n,j)$ into the left side of (2.12) and making a change of summation index $k=n+j$, we find that

$$\begin{aligned}&\sum_{n=2}^{\infty} q^{2n^2}\left(q^{2n}+q^{4n}+\cdots+q^{2n^2-2n}\right)=\sum_{n=2}^{\infty}\sum_{j=1}^{n-1} f(n,j)=\sum_{n=2}^{\infty}\sum_{k=n+1}^{2n-1} f(n,k-n)\\&=\sum_{l=2}^{\infty}\sum_{n=l}^{2l-2} f(n,2l-1-n)+\sum_{m=2}^{\infty}\sum_{n=m+1}^{2m-1} f(n,2m-n)\\&=\sum_{l=2}^{\infty}\left(q^{4l^2-2l}+q^{4l^2+2l-2}+\cdots+q^{8l^2-12l+4}\right)+\sum_{l=2}^{\infty}\left(q^{4l^2+4l}+q^{4l^2+8l}+\cdots+q^{8l^2-4l}\right),\end{aligned}$$

where in the penultimate equality, we rearranged the order of summation and separated the terms into odd and even values of k via $k=2l-1$ and $k=2m$. We see that these are equal to the right side of (2.12) and this completes the proof.

□

We can now prove Theorem 1.1

Proof of Theorem 1.1. By Proposition 2.1, we have

$$\overline{M}_1(q)-\overline{R}_1(q)=\frac{(-q)_\infty}{(q)_\infty}\Big(h(q)-2h(q^2)\Big).\tag{2.13}$$

Thus, it suffices to prove that the right side of (2.13) has positive power series coefficients for all positive powers of q. By Lemma 2.3,

$$\begin{aligned}h(q)-2h(q^2)&=\sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2}+2\sum_{n=2}^{\infty}(-1)^n q^{n^2}\left(q^n-q^{2n}+\cdots+(-1)^{n-2}q^{n^2-n}\right)\\&\quad-2\sum_{n=1}^{\infty}q^{2(2n)^2}-\sum_{n=1}^{\infty}(-1)^{n+1}q^{2n^2}\\&=:A_1+2A_2-2A_3-A_4.\end{aligned}$$

For the sum A_1, note that

$$-\frac{1}{2}+A_1=-\frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}=-\frac{(q)_\infty}{2(-q)_\infty}.$$

Hence
$$\frac{(-q)_\infty}{(q)_\infty}A_1 = \frac{(-q)_\infty}{2(q)_\infty} - \frac{1}{2}.$$
Similarly, for the sum A_4,
$$\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}A_4 = \frac{(-q^2;q^2)_\infty}{2(q^2;q^2)_\infty} - \frac{1}{2}.$$
Therefore,
$$\frac{(-q)_\infty}{(q)_\infty}(A_1 - A_4) = \frac{(-q)_\infty}{2(q)_\infty} - \frac{1}{2} - \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\left(\frac{(-q^2;q^2)_\infty}{2(q^2;q^2)_\infty} - \frac{1}{2}\right) = \frac{(-q;q^2)_\infty}{2(q;q^2)_\infty} - \frac{1}{2},$$
which has positive power series coefficients for all positive powers of q. Next, we examine $A_2 - A_3$. We define $g(n,j) = (-1)^{n+j-1}q^{n^2+jn}$. Then
$$\begin{aligned} A_2 - A_3 &= \sum_{n=2}^{\infty}(-1)^n q^{n^2}\sum_{j=1}^{n-1}(-1)^{j-1}q^{jn} - \sum_{n=1}^{\infty} q^{2(2n)^2} \\ &= \sum_{n=2}^{\infty}\sum_{j=1}^{n-1} g(n,j) + \sum_{n=1}^{\infty} g(2n,2n). \end{aligned}$$

We now rearrange the series $A_2 - A_3$ into several sums. Note that for $j \geq 0$ and $n \geq 2j+2$,
$$\begin{aligned} &g(2n,4j+3) + g(2n+1,4j+3) + g(2n+1,4j+1) + g(2n+2,4j+1) \\ &= (-1)^{2n+4j+2}q^{4n^2+(4j+3)2n}\left(1 - q^{4n+4j+4} - q^{4j+2} + q^{4n+8j+6}\right) \\ &= q^{4n^2+(4j+3)2n}\left(1-q^{4j+2}\right)\left(1-q^{4n+4j+4}\right), \end{aligned}$$
and for $j \geq 0$ and $n \geq 2j+2$,
$$\begin{aligned} &g(2n+1,4j+4) + g(2n+2,4j+4) + g(2n+2,4j+2) + g(2n+3,4j+2) \\ &= (-1)^{2n+4j+4}q^{(2n+1)^2+(4j+4)(2n+1)}\left(1 - q^{4n+4j+7} - q^{4j+3} + q^{4n+8j+10}\right) \\ &= q^{(2n+1)^2+(4j+4)(2n+1)}\left(1-q^{4j+3}\right)\left(1-q^{4n+4j+7}\right). \end{aligned}$$
These take care of all the terms except, for all integers $n \geq 0$,
$$\begin{aligned} &g(4n+2,4n+1) + g(4n+3,4n+1) + g(4n+4,4n+1) + g(4n+2,4n+2) \\ &+ g(4n+3,4n+2) + g(4n+4,4n+2) + g(4n+5,4n+2) + g(4n+4,4n+4) \\ &= [g(4n+2,4n+1) + g(4n+3,4n+1) + g(4n+4,4n+1) + g(4n+2,4n+2) \\ &\quad + g(4n+3,4n+2) - g(4n+3,4n+4)] \\ &\quad + [g(4n+3,4n+4) + g(4n+4,4n+4) + g(4n+4,4n+2) + g(4n+5,4n+2)]. \end{aligned}$$
Note that
$$\begin{aligned} &g(4n+2,4n+1) + g(4n+3,4n+1) + g(4n+4,4n+1) + g(4n+2,4n+2) \\ &+ g(4n+3,4n+2) - g(4n+3,4n+4) \\ &= q^{(4n+2)^2+(4n+1)(4n+2)}\left[1 - q^{12n+6} + q^{24n+14} - q^{4n+2} + q^{16n+9} - q^{24n+15}\right] \end{aligned}$$

$$= q^{(4n+2)^2+(4n+1)(4n+2)}\Big[(1-q^{4n+2})(1-q^{8n+7})(1-q^{12n+6})$$
$$+ q^{12n+8}(1-q)(1-q^{4n}) + q^{8n+7}(1-q^{4n+1})(1-q^{12n+6})\Big]$$

while

$$g(4n+3,4n+4)+g(4n+4,4n+4)+g(4n+4,4n+2)+g(4n+5,4n+2)$$
$$= q^{(4n+2)^2+(4n+1)(4n+2)+24n+15}(1-q^{4n+3})(1-q^{12n+11}).$$

These sums show that

$$\begin{aligned}
&\frac{(-q)_\infty}{(q)_\infty}(A_2-A_3)\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{n=2j+2}^{\infty} q^{4n^2+(4j+3)2n}\left(1-q^{4j+2}\right)\left(1-q^{4n+4j+4}\right)\\
&\quad + \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{n=2j+2}^{\infty} q^{(2n+1)^2+(4j+4)(2n+1)}\left(1-q^{4j+3}\right)\left(1-q^{4n+4j+7}\right)\\
&\quad + \frac{(-q)_\infty}{(q)_\infty}\sum_{n=0}^{\infty} q^{(4n+2)^2+(4n+1)(4n+2)}\Big[(1-q^{4n+2})(1-q^{8n+7})(1-q^{12n+6})\\
&\quad + q^{12n+8}(1-q)(1-q^{4n}) + q^{8n+7}(1-q^{4n+1})(1-q^{12n+6})\Big]\\
&\quad + \frac{(-q)_\infty}{(q)_\infty}\sum_{n=0}^{\infty} q^{(4n+2)^2+(4n+1)(4n+2)+24n+15}(1-q^{4n+3})(1-q^{12n+11}).
\end{aligned}$$

For positive integers a, b, c and d with $b<c<d$, expressions of the form

$$\frac{(-q)_\infty}{(q)_\infty}q^a(1-q^b)(1-q^c)$$

and

$$\frac{(-q)_\infty}{(q)_\infty}q^a(1-q^b)(1-q^c)(1-q^d)$$

have nonnegative coefficients and so $\frac{(-q)_\infty}{(q)_\infty}(A_2-A_3)$ has nonnegative power series coefficients. Since $\frac{(-q)_\infty}{(q)_\infty}(A_1-A_4)$ has positive power series coefficients for all positive powers of q, we conclude that the power series expansion of $\frac{(-q)_\infty}{(q)_\infty}(h(q)-2h(q^2))$ has positive coefficients for all q^n, $n\geq 1$. This proves (1.5). $\square$

Corollary 2.4.

$$\frac{1}{(q)_\infty}(h(q)-2h(q^2))$$

has positive power series coefficients for all q^n with $n\geq 6$.

Proof. From the proof of Theorem 1.1 and by invoking the elementary identity $(-q)_\infty = 1/(q;q^2)_\infty$, we see that

$$\frac{1}{(q)_\infty}(A_1 - A_4) = \frac{1}{(-q)_\infty}\left(\frac{(-q;q^2)_\infty}{2(-q;q^2)_\infty} - \frac{1}{2}\right) = \frac{1}{2}((-q;q^2)_\infty - (q;q^2)_\infty),$$

which has positive power series coefficients for all odd positive powers of q (the terms with even powers of q vanishes). Again, from the proof of Theorem 1.1, it is easy to see that $\frac{1}{(q)_\infty}(A_2 - A_3)$ has nonnegative power series coefficients. Since one of the terms in the corresponding expression of $\frac{1}{(q)_\infty}(A_2 - A_3)$ is

$$\frac{1}{(q)_\infty} q^6(1-q^2)(1-q^6)(1-q^8) = q^6 \prod_{\substack{k=1 \\ k\neq 2,6,8}}^{\infty} {}_{\infty}\frac{1}{1-q^k},$$

the coefficients of q^n for $n \geq 6$ in the power series expansion of $\frac{1}{(q)_\infty}(A_2 - A_3)$ are all positive. $\square$

3. A combinatorial interpretation

In [3], the first three authors defined a new counting function $\mathrm{ospt}(n)$ as

$$\mathrm{ospt}(n) = M_1^+(n) - N_1^+(n)$$

and provided its combinatorial interpretation. The function $\mathrm{ospt}(n)$ is an interesting companion of $\mathrm{spt}(n)$ in sense of that

$$\mathrm{spt}(n) = M_2^+(n) - N_2^+(n).$$

Here, $\mathrm{spt}(n)$ is the number of smallest parts in the partitions of n [2]. In this section, we discuss an overpartition analogue of $\mathrm{ospt}(n)$ and its combinatorial meaning. Let us define

$$\overline{\mathrm{ospt}}(n) = \overline{M}_1^+(n) - \overline{N}_1^+(n).$$

Before giving a combinatorial interpretation for $\overline{\mathrm{ospt}}(n)$, we first recall the description of $\mathrm{ospt}(n)$. An even string in the partition λ is a sequence of the consecutive parts starting from some even number $2k+2$ where the length is an odd number greater than or equal to $2k+1$ and $2k+2$ plus the length of the string (the number of consecutive parts) do not appear as a part. An odd string in λ is a sequence of the consecutive parts starting from some odd number $2k+1$ where the length is greater than or equal to $2k+1$ such that the part $2k+1$ appears exactly once and $2k+2$ plus the length of the string does not appear as a part. By "consecutive parts", we allow repeated parts. With these notions in mind, we have the following.

Theorem 3.1. [3, Theorem 4] *For all positive integers n,*

$$\mathrm{ospt}(n) = \sum_{\lambda \vdash n} \mathrm{ST}(\lambda),$$

where the sum runs over the partitions of n and $\mathrm{ST}(\lambda)$ is the number of even and odd strings in the partition λ.

The function $\overline{\mathrm{ospt}}(n)$ now counts the number of certain strings in the overpartitions of n, but the difference is that we have a weighted count of strings. We start by defining $f_k(q)$ as

$$f_k(q) = \sum_{n=1}^{\infty} (-1)^{n+1} q^{n(n+1)/2+n(k-1)}.$$

By Proposition 2.1 and exchanging the order of summation, we have

$$\sum_{n=1}^{\infty} (\overline{M}_1^+(n) - \overline{N}_1^+(n)) q^n = \frac{(-q)_\infty}{(q)_\infty} \sum_{k=1}^{\infty} (f_k(q) - 2f_k(q^2)).$$

Note that for a fixed $k \geq 1$,

$$\begin{aligned}
&\frac{(-q)_\infty}{(q)_\infty}(f_{2k-1}(q) + f_{2k}(q) - 2f_k(q^2)) \\
&= \frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} q^{2n^2-5n+4nk-2k+2}(1-q^{2n^2-n})(1-q^{4n+2k-2}) - q^{2n^2-3n+4nk}(1-q^{2n^2+n})(1-q^{4n+2k}).
\end{aligned}$$

Now we define $A_k(n)$ (resp. $B_k(n)$) to be the number of overpartitions of n counted by the first (resp. second) sum. By noting that

$$2n^2 - 3n + 4nk = 1 + (2k-2) + 2 + (2k-2) + \cdots + (2n-1) + (2k-2) + 2n + (2k-2),$$

we define an odd string starting from $2k-1$ in an overpartition as

(1) $2k-1, 2k, \ldots, 2\ell+2k-3$ appears at least once, i.e. there are $2\ell-1$ consecutive parts starting from $2k-1$.
(2) There is no other part of size $2\ell^2 - \ell$ and $4\ell + 2k - 2$.

Similarly, we define an even string starting from $2k$ in an overpartition as

(1) $2k-1, 2k, \ldots, 2\ell+2k-2$ appears at least once, i.e. there are 2ℓ consecutive parts starting from $2k-1$.
(2) There is no other part of size $2\ell^2 + \ell$ and $4\ell + 2k$.

As with the $\mathrm{ospt}(n)$ function, $A_k(n)$ is now the number of odd strings starting from $2k-1$ along the overpartitions of n, and $B_k(n)$ is the number of even strings starting from $2k-1$ along the overpartitions of n. Then we have

$$\sum_{n=1}^{\infty} (\overline{M}_1^+(n) - \overline{N}_1^+(n)) q^n = \sum_{n=1}^{\infty} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (A_k(n) - B_k(n)) q^n = \sum_{n=1}^{\infty} \overline{\mathrm{ospt}}(n) q^n.$$

We have thus proven the following.

Theorem 3.2. *For all positive integers n, we have*

$$\overline{\mathrm{ospt}}(n) = \overline{\mathrm{ST}}_o(n) - \overline{\mathrm{ST}}_e(n),$$

where $\overline{\mathrm{ST}}_o(n)$ (resp. $\overline{\mathrm{ST}}_e(n)$) is the number of odd (resp. even) strings along the overpartitions of n.

Let us illustrate the above discussion for $n = 5$. From Table 1, we see that $\overline{\mathrm{ST}}_o(5) = 8$ and $\overline{\mathrm{ST}}_e(5) = 4$, so $\overline{\mathrm{ospt}}(5) = 4$. This matches with $\overline{M}_1^+(5) = 24$ and $\overline{N}_1^+(5) = 20$.

Overpartitions of 5	The number of odd strings	The number of even strings
5	1	0
$\overline{4}$+1	1	0
3+2	1	0
3 +$\overline{2}$	1	0
$\overline{3}$+$\overline{1}$ +1	1	0
3+$\overline{1}$+1	1	0
2+2+1	1	1
$\overline{2}$+2+1	1	1
2+1+1+1	0	1
2+$\overline{1}$+1+1	0	1

TABLE 1. The number of strings in the overpartitions of 5.

4. Concluding Remarks

We have numerically observed that

$$\overline{M}_k^+(n) > \overline{N}_k^+(n) \tag{4.1}$$

for all k, $n \geq 1$. Inequality (4.1) and the fact that $\overline{N}_{2j}(n) = 2\overline{N}_{2j}^+(n)$ and $\overline{M}_{2j}(n) = 2\overline{M}_{2j}^+(n)$ implies that a complete analogue of (1.3) should hold, namely

$$\overline{M}_{2j}(n) > \overline{N}_{2j}(n) \tag{4.2}$$

for all j, $n \geq 1$. Motivated by our present work, Jennings-Shaffer [24] has proven (4.2) using the Bailey pair techniques from [20]. See also [21] for the case $k = 1$. It would still be interesting to see if the techniques in [3] can be used to prove (4.1) (and thus (4.2)) and discover a combinatorial meaning for $\overline{M}_k^+(n) - \overline{N}_k^+(n)$. Moreover, there is an inequality of note which has a similar flavor to (1.3). If we consider the rank moment

$$\overline{N2}_k(n) := \sum_{m \in \mathbb{Z}} m^k \overline{N2}(m, n)$$

where $\overline{N2}(m, n)$ is the number of overpartitions of n with M_2-rank m [26], then Mao [28] has proven that

$$\overline{N}_{2j}(n) > \overline{N2}_{2j}(n) \tag{4.3}$$

for all $j \geq 1$, $n \geq 2$. Another proof of (4.3) using the similarly defined positive rank moment $\overline{N2}_k^+(n)$ can be found in [23]. It is still not known what $\overline{N}_k^+(n) - \overline{N2}_k^+(n)$ counts. While proving Corollary 2.4 and Theorem 3.2, we observed the following. First, it appears that for all integers $m \geq 3$.

$$\frac{1}{(q)_\infty}(h(q) - mh(q^m)) \tag{4.4}$$

has positive power series coefficients for all positive powers of q. Second, numerical computations suggest that

$$A_k(n) \geq B_k(n) \tag{4.5}$$

for all n, $k \geq 1$. Finally, asymptotic methods reveal that the inequalities (4.1) and (4.5), and the positivity of the coefficients of (4.4) are valid for large enough integers n [22, 30]. However, it is still desirable to find q-theoretic or combinatorial proofs of these result, which shows that these conjectures are true for all positive integers. We leave these questions to the interested reader.

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References

[1] G.E. Andrews, *Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks*, Invent. Math. **169** (2007), no. 1, 37–73.
[2] G.E. Andrews, *The number of smallest parts in the partitions of n*, J. Reine Angew. Math. 624 (2008), 133–142.
[3] G.E. Andrews, S.H. Chan, B. Kim, *The odd moments of ranks and cranks*, J. Combin. Theory Ser. A **120** (2013), no. 1, 77–91.
[4] G.E. Andrews, F.G. Garvan, *Dyson's crank of a partition*, Bull. Amer. Math. Soc. (N.S.) **18** (1988), no. 2, 167–171.
[5] A.O.L. Atkin, F.G. Garvan, *Relations between the ranks and cranks of partitions*, Ramanujan J. **7** (2003), no. 1-3, 343–366.
[6] K. Bringmann, *On the explicit construction of higher deformations of partition statistics*, Duke Math. J. **144** (2008), no. 2, 195–233.
[7] K. Bringmann, F. Garvan, K. Mahlburg, *Partition statistics and quasiharmonic Maass forms*, Int. Math. Res. Not. IMRN 2009, no. 1, Art. ID rnn124, 63–97.
[8] K. Bringmann, J. Lovejoy, *Overpartitions and class numbers of binary quadratic forms*, Proc. Natl. Acad. Sci. USA **106** (2009), no. 14, 5513–5516.
[9] K. Bringmann, J. Lovejoy, R. Osburn, *Rank and crank moments for overpartitions*, J. Number Theory **129** (2009), no. 7, 1758–1772.
[10] K. Bringmann, K. Mahlburg, *Inequalities between ranks and cranks*, Proc. Amer. Math. Soc. **137** (2009), no. 8, 2567–2574.
[11] K. Bringmann, K. Mahlburg, *Asymptotic inequalities for positive crank and rank moments*, Trans. Amer. Math. Soc. **366** (2014), no. 2, 1073–1094.
[12] K. Bringmann, K. Mahlburg, R. Rhoades, *Asymptotics for rank and crank moments*, Bull. Lond. Math. Soc. **43** (2011), no. 4, 661–672.
[13] K. Bringmann, K. Mahlburg, R. Rhoades, *Taylor coefficients of mock-Jacobi forms and moments of partition statistics*, Math. Proc. Camb. Phil. Soc. **157** (2014), 231–251.
[14] W.Y.C. Chen, D. Sang, D. Shi, *The Rogers-Ramanujan-Gordon theorem for overpartitions*, Proc. London. Math. Soc. (3) **106** (2013), no. 6, 1371–1393.
[15] P. Desrosiers, L. Lapointe, P. Mathieu, *Evaluation and normalization of Jack superpolynomials*, Int. Math. Res. Not. IMRN 2012, no. 23, 5267–5327.
[16] P. Diaconis, S. Janson, R. Rhoades, *Note on a partition limit theorem for the rank and crank*, Bull. London Math. Soc. **45** (2013), no. 3, 551–553.
[17] F.J. Dyson, *Some guesses in the theory of partitions*, Eureka **8** (1944), 10–15.
[18] A. Folsom, K. Ono, *The spt-function of Andrews*, Proc. Natl. Acad. Sci. USA **105** (2008), no. 51, 20152–20156.

[19] F.G. Garvan, *Congruences for Andrews' smallest parts partition function and new congruences for Dyson's rank*, Int. J. Number Theory **6** (2010), no. 2, 281–309.
[20] F. G. Garvan, *Higher order spt-functions*, Adv. Math. **228** (2011), no. 1, 241–265.
[21] F. Garvan and C. Jennings-Shaffer, *The spt-crank for overpartitions*, , Acta Arith., to appear.
[22] B. Kim, E. Kim, J. Seo, *On the number of even and odd strings along the overpartitions of n*, Arch. Math. **102** (2014), 357–368.
[23] A. Larsen, A. Rust, H. Swisher, *Inequalities for positive rank and crank moments of overpartitions*, Int. J. Number Theory **10** (2014), 2115–2133.
[24] C. Jennings-Shaffer, *Higher order spt functions for overpartitions, overparitions with smallest part even and partitions without repeated odd parts*, preprint available at `http://arxiv.org/pdf/1402.5098v1.pdf`
[25] J. Lovejoy, *Rank and conjugation for the Frobenius representation of an overpartition*, Ann. Comb. **9** (2005), 321–335.
[26] J. Lovejoy, *Rank and conjugation for a second Frobenius representation of an overpartition*, Ann. Comb. **12** (2008), no. 1, 101–113.
[27] J. Lovejoy, S. Corteel, *Overpartitions*, Trans. Amer. Math. Soc. **356** (2004), no. 4, 1623–1635.
[28] R. Mao, *Inequalities between rank moments of overpartitions*, J. Number Theory **133** (2013), no. 11, 3611–3619.
[29] R. Rhoades, *Families of quasimodular forms and Jacobi forms: the crank statistic for partitions*, Proc. Amer. Math. Soc. **141** (2013), no. 1, 29–39.
[30] J. M. Z. Rolon, *Asymptotics of higher order spt-functions for overpartitions*, Ann. Comb., to appear.

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