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<b>Authors(s)</b>	Strobl, Eric, Walsh, Frank
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# The Ambiguous Effect of Minimum Wages on Hours<sup>1</sup>

Eric Strobl and Frank Walsh\*

*Ecole Polytechnique Paris and University College Dublin*

## *Abstract*

In a competitive model we ease the assumption that efficiency units of labour are the product of hours and workers. We show that a minimum wage may either increase or decrease hours per worker and the change will have the opposite sign to the slope of the equilibrium hours hourly wage locus. Similarly, total hours worked may rise or fall. We illustrate the results throughout with a Cobb-Douglas example.

Keywords: Minimum wages, hours, employment.

JEL Classification J22, J38

\*Corresponding author: Frank Walsh, School of Economics, g216, John Henry Newman building, University College Dublin, Dublin 4 Ireland. Email [Frank.Walsh@ucd.ie](mailto:Frank.Walsh@ucd.ie). Phone: 353-1-7168697. Fax: 353-1-2830068.

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## Section I: Introduction

The general and long held view amongst economists is that a minimum wage will reduce employment in a competitive labour market.<sup>2</sup> For example, Stigler (1946) noted: “The higher the minimum wage, the greater the number of covered workers who are discharged”. Importantly, however, this view implicitly assumes a theoretical framework where the labour input can be thought of as total hours and total hours is defined the product of workers and hours per worker. One can arguably though think of a number of different reasons why firms may not take the labour input in the production function as the product of hours and workers. The most basic one is if hours per worker have a diminishing marginal product. Alternatively, firms may have different hours technologies – for example, a long haul trucking company with a couple of large trucks may want a small number of workers with long hours, while a local delivery service may be able to have a large number of workers using the same vehicle. It may also be that differences in the firm’s demand for the mix of bodies and hours come from the demand conditions facing the firm. For instance, a restaurant in an office district may be very busy for short periods and require a large number of part time workers, while a high street restaurant may be busy over longer periods that facilitate hiring a larger share of full-time workers.

In this paper we thus examine what happens in the standard competitive model when firms are able to choose the number of workers and hours per worker and pay compensating differentials for different levels of hours per worker. Our results show that in such an arguably very plausible setting the impact of minimum wages on hours per worker, the number of workers, and total hours worked is in fact ambiguous. Importantly, the ambiguity of the employment effect in our model does not arise from

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<sup>2</sup> See Neumark and Wascher (2007) for a survey of the literature on the effect of minimum wages on the labour market.

the inability to pin down substitution effects in a complex model with many heterogeneous factors, but rather comes from applying a minimum wage to a framework that is commonly used in the literature to model how firms choose combinations of hours and workers in a competitive labour market. We also show that total hours may rise or fall at the industry level when we allow for firm entry and exit.

In contrast to the competitive model, it is of course generally well known that the effect of a minimum wage on employment is not clear in non-competitive models of the labour market. For example, the theoretical possibility that minimum wages may increase employment in a monopsony model was noted as early as Stigler (1946), although he discounted the importance of such models at the time. A more recent literature argues that the Monopsony model is perhaps more relevant in modern labour markets [See Manning (2003)] and Bhaskar and To (1999), Walsh (2003) and Strobl and Walsh (2007), amongst others, present more recent versions of the monopsony model which illustrate the ambiguity of minimum wage employment effects. Also, De Fraja (1999) shows that the employment effects of a minimum wage are small in a model with heterogeneity in workers preferences over wages and working conditions, while Rebitzer and Taylor (1995) demonstrate that minimum wages may increase employment in an efficiency wage model where monitoring becomes more difficult as employment increases. We argue here that regardless of whether one takes a competitive or a non-competitive view of the labour market, the theoretically derived effect of a minimum wage on employment is ambiguous.

One should note that the model we use to demonstrate our argument is based on the notion that firms substitute between hours and workers in response to a minimum wage. However, it could be that the minimum wage graph many of us

became familiar with as undergraduates – i.e., where hours are fixed and a minimum wage unambiguously leads to some workers being fired in competitive labour markets as firms moved up the labour demand curve - is more appropriate. However, while there are differences across the literature, many empirical studies have rather found substantial movements in hours associated with changes in minimum wages, suggesting that both hours per worker and the number of workers may be affected. More precisely, summarizing the results from the earlier empirical literature on the impact of minimum wages on hours, Brown (1999) concludes that “hours per week fall when minimum wages increase, so the effect on hours worked is more pronounced than the effect on bodies employed” (p. 2156). Michl (2000) finds a reduction in hours from the widely documented 1990 New Jersey minimum wage increase and argues that this may partly explain conflicting empirical results found by others. Other examples in this regard include Neumark and Schweitzer (2000) or Couch and Wittenberg (2001) who both discover a significant reduction in hours using a panel of U.S states, where the former argue that minimum wage workers are adversely affected by a minimum wage increase. Also, for the UK Stewart and Swaffield (2006) find that the U.K minimum wage reduced hours particularly for males.

Nevertheless, while many of the more recent studies that examine the effect on hours do indeed find a reduction therein, this is not always the case. For example, Katz and Krueger (1992) discover a fall in part-time work for the U.S. Similarly, Zavodney (2000) using a panel of US states finds a fall in the number of workers but increase in hours. Connolly and Gregory (2002) found no effect on hours for women from the national minimum wage in the U.K.. The literature indicates that there is

little evidence that hours remain fixed in response to a minimum wage, and that additionally for hours per worker the direction of the impact is not clear-cut.<sup>3</sup>

The remainder of the paper is organised as follows. In the next section we outline our model in a partial equilibrium framework at the firm level. In Section III we discuss market equilibrium in the short run, in section IV we model firm entry and long run equilibrium, section V Discusses the impact of the minimum wage on worker utility and profits in long run equilibrium and the final section concludes.

## **Section II: The firm level response to a minimum wage**

The theoretical treatment of minimum wages in the literature when firms choose a combination of hours per worker and workers is rather limited. Hamermesh (1993) develops a model that deals with the firm's choice of workers and hours in a cost minimisation framework (similar to that illustrated in Figure 2 below) which includes a brief discussion of minimum wages, while Michl (2000) outlines a model where firms with a Cobb-Douglas production function over hours and workers choose workers and hours to minimise cost, assuming that the wage does not increase with hours. Other studies, such as Stewart and Swaffield (2006), Zavodney (2000), Neumark and Schweitzer (2000), and Connolly and Gregory (2002) contain some general discussions on how minimum wages are related to hours but do not build a formal theoretical model.

In this paper we apply a minimum wage to Kinoshita's (1987) model which derives the equilibrium properties of a competitive labour market. In general the equilibrium hours per worker hourly wage locus  $h(w)$  [we will refer to this as the hours wage locus from now on] is a set of tangencies where workers who wish to

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<sup>3</sup> See Neumark and Wascher (2007) page 40 for details on this literature, as well as an update of the large literature on employment effects on the number of workers and total hours.

work longer hours match up with like-minded firms. In equilibrium the supply and demand of each worker type are equal and no worker or firm can gain from deviating to another point on the locus. Compensating wage differentials are paid to workers for working a less desirable number of hours<sup>4</sup>. Firms are assumed to be able to hire as many workers as they wish at any level of hours ( $h$ ). If there are only a small number of worker types, or one type as in the example we illustrate explicitly, the workers' indifference curve is the equilibrium hours wage locus. Figure 1 illustrates a tangency between the indifference curve of a representative worker and the isoprofit curve of a firm and is based on the simulation of the Cobb-Douglas example described in Appendix 1. We will use this example throughout the paper to illustrate the results.

It may be worth emphasising that even though firms will choose the wage in the model below, this is not indicative of any market power. Firms are price takers and can hire as many workers as they wish at any given level of hours. In a competitive labour market though, the competitive wage needed to induce different levels of hours will differ and firms can choose any point on the equilibrium hours wage locus.

In this section we assume our model is partial equilibrium in that we ignore the potential impact of the policy on the equilibrium hours wage locus or cost of firm entry. The firm's profit function is:

$$\Pi(n, w) = pf[n, h(w)] - wnh(w) - kn \quad (1.1)$$

where  $h$  is hours per worker,  $n$  is the number of workers. There are fixed costs  $k$  per worker. The output price  $p$  is given to the firm and the production function

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<sup>4</sup> The models of Lewis (1969) and Rosen (1986) are the precursors to this model.

$q = f(h, n)$  satisfies  $f_n > 0, f_h > 0, f_{nn} < 0$ .<sup>5</sup> The firm's choice of  $w$  and  $n$  at an interior solution satisfies the following first order conditions:

$$\begin{aligned}\Pi_w(n, w) &= pf_h[n, h(w)]h_w(w) - wnh_w(w) - nh = 0 \\ \Pi_n(n, w) &= pf_n[n, h(w)] - wh(w) - k = 0\end{aligned}\tag{1.2}$$

One can assess the impact of a minimum wage on the number of workers by totally differentiating the first order condition on  $n$ . Evaluating this differential at the initial equilibrium where  $\varepsilon_{qhn}$  is the percentage change in the marginal product of hours ( $f_h$ ) from a percent change in the number of workers, we get:

$$\frac{dn}{dw} = -\frac{\Pi_{nw}}{\Pi_{nn}} = \frac{[\frac{f_h}{n} - f_{hn}]}{f_{nn}}]h_w = \frac{f_h}{f_{nn}n}(1 - \varepsilon_{qhn})h_w\tag{1.3}$$

*Proposition 1: A sufficient condition for the change in the number of workers employed at a firm in response to a minimum wage to have the opposite sign to the slope of the hours wage locus is that the scale effect of output on hours per worker is negative or zero.*

*Proof:*

*If the scale effect for hours per worker are negative or zero we show in Appendix 2*

*from equation A(1.30) that  $\frac{[\frac{f_h}{n} - f_{nh}]}{f_{nn}} < 0$ . This implies from equation (1.3) that  $\frac{dn}{dw}$*

*has the opposite sign to  $h_w$ .<sup>6</sup>*

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<sup>5</sup> If costs of hiring workers were convex rather than the constant  $k$  as we assume, the firm would effectively behave as a monopsonist since worker costs would increase with employment, [see Manning (2003) p34-35].

<sup>6</sup> We also see from (1.3) that  $\frac{dn}{dw}$  and  $h_w$  will have the opposite sign if:  $\varepsilon_{qhn} < 1$ .



The proposition shows that if the assumption is true a minimum wage will lead to an increase in the number of workers if the hours wage locus has a positive slope and a decrease if the locus has a negative slope. We use Figure 2 to illustrate the intuition for the proposition. In contrast to Figure 1 which graphs the isoprofit and indifference curves, Figure 2 depicts the isocost/isoquant graph for an individual firm.<sup>7</sup> The isocost curve gives the employment hours combinations that are available at a fixed level of cost:  $C_0 = wh(w)n + nk$  as defined in Appendix 1 for the Cobb-Douglas example. The isoquant shows the combinations of hours and workers that give a fixed level of output:  $H_o = H(n, h)$  where  $H$  is the aggregate labour input for the Cobb-Douglas function. The initial equilibrium graphed in Figure 2, where the isocost and isoquant curves are tangent, gives the cost minimising bundle of hours/workers that can produce the desired output. Of course the level of hours in Figure 2 corresponds to the equilibrium level of hours in Figure 1. When the minimum wage is imposed and the wage hours locus is upward sloping, the firm is forced to pay a higher wage, but can get a higher level of hours per worker in return (as we saw in Figure 1). In Figure 2 the increase in hours implies that at fixed output the firm substitutes from workers into hours moving along the isoquant. Equation A(1.32) in Appendix 2 derives this substitution effect and we see from this equation that the change in the number of workers will certainly have the opposite sign to the slope of the wage hours locus. Of course the minimum wage may also cause output to change which will in turn affect the demand for workers and hours. We could imagine the isoquant in Figure 2 shifting in response to the minimum wage which would in turn change the demand for workers and hours. Indeed from (1.3) and the production function we see that the change in firm output from a minimum wage is:

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<sup>7</sup> The graphical analysis used here draws on the analysis used in Hamermesh (1993).

$$\frac{df}{dw} = f_n \frac{dn}{dw} + f_h \frac{dh}{dw} = [f_n (\frac{f_h - f_{nh}}{f_{nn}}) + f_h] h_w \quad (1.4)$$

In equation A(1.29) in Appendix 2 we show that the term in square brackets in (1.4) will be positive/negative if hours per worker is inferior/not inferior<sup>8</sup>. This implies that if the wage hours locus has a positive sign the change in output from imposing a minimum wage will be positive unless hours are an inferior input. That is if  $h_w > 0$  and hours are not inferior the minimum wage will cause an increase in output. The assumption that scale effects on hours are zero in Proposition 1 ensures that the output effect on the number of worker can never outweigh the substitution effect given in A(1.32). While this assumption is sufficient for Proposition 1 it is easy to find examples of commonly used production functions where the change in the number of workers will be inversely related to the slope of the wage hours locus even when scale effects are positive<sup>9</sup>. The literature modeling a firm's choice of workers and hours commonly assumes that scale effects are zero<sup>10</sup>. In addition it has been noted in the literature that the intuition for hours increasing or decreasing systematically with the scale of production is not clear. For example Hamermesh (1993) argues, that "...there is no evidence that weekly hours of full-time workers at

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<sup>8</sup> In the more standard case where a firm has two inputs with separate prices (say a firm using capital and labour) a necessary condition for a factor to be inferior is that the cross partial derivative between the inputs in the production function must be negative [see Bear (1965)].

<sup>9</sup> For the constant elasticity of substitution production function  $f(n, h) = [A + \alpha h^\rho + \beta n^\rho]^{\frac{1}{\rho}}$  we

can show that the condition  $\frac{[\frac{f_h}{n} - f_{nh}]}{f_{nn}} < 0$  will always hold even though the scale effect on hours

is positive if the elasticity of substitution  $\frac{1}{1-\rho}$  is less than one. The Cobb-Douglas case is

$\rho = 0$  where scale effects are zero.

<sup>10</sup>For example Cahuc and Zylberberg (2001) and Hamermesh (1993) use a technology with this assumption while Hamermesh (1993) discusses this literature and the functional forms used.

General Motors differ substantially from hours of workers at the local steel fabricator”(p.50).

A puzzling implication of Proposition 1 is that much of the existing empirical evidence discussed earlier finds a decline in hours per worker from a minimum wage. If this were so Proposition 1 implies that affected firms should be increasing the number of workers, but would also imply that workers are on a negatively sloped hours wage locus. There is no reason that this should not be so in theory, but one may suspect that many economists would expect the contrary. For example, Hamermesh (1993) assumes the equilibrium locus has a positive slope in his treatment of the theory of hours per workers, while Michl (2000) supposes the locus is flat. Interestingly, in a monopsony model where the supply of workers to the firm depends on the utility of the hours wage combination offered by the firm, a fall in hours in response to a minimum wage does not imply a downward sloping hours wage locus [see Strobl and Walsh (2007) for example]<sup>11</sup>. That is, if one wishes to reconcile reductions in hours per worker with an upward sloping wage hours locus which violates Proposition 1, non-competitive models can easily do this. This apart, the proposition should certainly make one reluctant to conclude that one can infer whether the labour market is competitive or not by looking at the results of studies that focus on the number of workers, as some of the literature does, since theory has no clear prediction on the change in hours per worker and predicts an offsetting change in the number of workers.

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<sup>11</sup> This conclusion may not be immediately obvious from reading Strobl and Walsh (2007) but it is straightforward to show using the model simulated in that paper that there is a range of parameters where the wage hours locus is positively sloped but a minimum wage reduces hours. The range where this is true expands with the degree of monopsony power. A maple file containing this simulated model is available from the authors.

We next examine the conditions under which a minimum wage will increase total hours<sup>12</sup>. In this regard we define the elasticity of output with respect to workers ( $n$ ) and hours per worker ( $h$ ), respectively, as  $f_h \frac{h}{f} = \varepsilon_{qh}$  and  $f_n \frac{n}{f} = \varepsilon_{qn}$ . It follows from the first order conditions (1.2) that:

$$h_w = \frac{h}{w} \left[ \frac{1}{\left(1 + \frac{k}{wh}\right) \frac{\varepsilon_{qh}}{\varepsilon_{qn}} - 1} \right] \quad (1.5)$$

If one thinks of employment as total hours ( $nh$ ), then using (1.4) and (1.5) the employment effect would be:

$$\frac{d(nh)}{dw} = n \frac{dh}{dw} + h \frac{dn}{dw} = n \left[ 1 + \frac{h}{n} \left( \frac{f_h}{f_{nn}} - \frac{f_{nh}}{f_{nn}} \right) \right] h_w \quad (1.6)$$

In Appendix 3 (a) we show that the elasticity of total hours with respect to a minimum wage is positive, if:

$$\frac{1}{1 + \frac{k}{wh}} < \frac{\varepsilon_{q,h}}{\varepsilon_{q,n}} < 1 + x \frac{h}{n} \quad (1.7)$$

We show in Appendix 3(a) that  $x > 0$  if hours are not an inferior input. A feature of this result is that one does not need extreme values for the elasticities of output with respect to workers and hours for the impact on total hours to be positive. For example take the case where the elasticity of output with respect to workers exceeds the elasticity of output with respect to hours ( $\varepsilon_{qn} > \varepsilon_{qh}$ ), but these elasticities are similar. Even if the scale effect was zero ( $x=0$ ) the presence of small fixed costs ( $k > 0$ ) would ensure that the condition in (1.7) was satisfied. It is equally true that if fixed costs were zero but there were positive scale effect the condition would hold.

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<sup>12</sup> Neumark and Wascher (2007) note “..although much of the literature has focused on the employment effects of the minimum wage, the predictions of theory tend to be about overall labour input rather than employment specifically.. (p.166)”.

Another way of illustrating the cases where total hours may increase is by establishing the following proposition.

*Proposition 2: Sufficient conditions for a minimum wage to increase total hours worked are that the hours wage locus has a positive slope, that  $\varepsilon_{qn} > \varepsilon_{qh}$  and that hours are not an inferior input.*

*Proof: See Appendix 3 (a)*

This implies that if some of the combinations of assumptions often made in this literature hold, then total hours will increase. As noted earlier, it is common to assume that the wage hours locus has a positive slope, while Feldstein (1967) and Michl (2000) amongst others explicitly assume  $\varepsilon_{qn} > \varepsilon_{qh}$ . We do not wish to argue based on Proposition 2 that one should typically expect total hours to increase. Rather it is to show that when one uses the type of simple competitive models that are used to establish the intuition that minimum wages lower employment, modifying the model using the standard approaches that have been used to analyse how a firm combines workers and hours, the result becomes ambiguous. This is so for parameter assumptions that have been commonly made in the hours/worker literature.

### **Section III: Market Equilibrium**

Section II analyses the response to a minimum wage of a firm in a competitive model for given market conditions (i.e., a given hours wage locus). Of course it may well be that the equilibrium hours wage locus is affected by the minimum wage, but it would be very difficult to explicitly solve for market equilibrium in a model with many worker and firm types. For this reason we focus on a simple model with a

representative firm and assume that workers have identical preferences over hours and wages and differ only in their disutility from going to work in a way that is described below. Imposing these assumptions limits the degree of heterogeneity across workers and firms and means that we ignore any potential spill-over effects across different types of workers and firms resulting from a minimum wage. However, since the main conclusion of the paper is that employment effects are ambiguous, our contention is that it is unlikely that generalising the model to include greater heterogeneity across workers and firms would lead to an unambiguous result.

To solve fully for the impact of the minimum wage on the market equilibrium we must make assumptions about the supply of workers and about firm entry on the demand side. We will ignore firm entry until the next section. We can think of the analysis in this section as focusing on short run market equilibrium. This will tell us how remaining firms will respond to the change in equilibrium conditions brought about by the minimum wage.

We might expect that at the level of the market the number of workers willing to supply labour would increase as the utility of employment increased. To take account of this in a tractable way we proceed as follows: We assume that workers satisfy the constraints  $h=(T-l)$ , where  $h$  is hours worked and  $T$  is a time endowment which is common to all workers and  $l$  is leisure. Consumption expenditure is  $x$  (where workers satisfy the budget constraint  $wh=x$ ). Utility is increasing in consumption and leisure. Substituting for the above constraints one gets the utility function  $u[D(h, w), v]$ , which we assume is weakly separable in  $v$ .<sup>13</sup> We assume that the utility function is monotonically decreasing in the parameter  $v$  which is distributed over a mass  $M$  of workers according to the function  $G(v)$ . All workers have

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<sup>13</sup> This model of labour supply is based on Strobl and Walsh (2007).

reservation utility  $S$ . These assumptions ensure that at any equilibrium hours wage bundle  $(w^*, h^*)$  workers with low enough values of  $v$  such that their utility at  $D(h, w)$  exceeds the reservation level  $S$  will supply labour. The framework implies that if the number of workers willing to supply labour increases the value of  $v$  for the marginal worker that participates will also increase. This means that  $D(h, w)$  must increase to ensure this workers participation since the marginal worker gets utility of exactly  $S$ . Weak separability of  $v$  ensures that each worker that supplies labour is on the same indifference curve over hours and wages, which is just the indifference curve of the marginal worker. Since this indifference curve gives the rate at which each worker trades off hourly wages and hours, it is the equilibrium hours wage locus which each firm takes as given. If a policy such as a minimum wage were to lead to an equilibrium with a higher number of workers, this implies that the marginal worker must have a higher value of  $v$ . The indifference curve of this worker would now be the equilibrium hours wage locus. That is, the policy may shift the equilibrium locus and we must take account of this shift on the firm's choice of hours and workers. The equilibrium value of  $v^*$  ensures that:

$$u[D(h, w), v^*] = S \quad (1.8)$$

That is  $v^*(h, w)$  is the value of  $v$  for the marginal worker who is just willing to supply labour. We can map from the equilibrium value of  $v^*$  to get labour supply:

$$N^s(h, w) = MG[v^*] \quad (1.9)$$

Also from (1.9) once we know  $v^*$  we can solve for the equilibrium hours locus that the firms face, which is just the indifference curve of the marginal worker. The equilibrium level of utility/profit is determined where labour supply equals labour demand:

$$N^d(h, w) = n^d[h(w, v^*)]F \quad (1.10)$$

The number of firms which is fixed in the short run is  $F$  and the demand for workers at each firm is:  $n^d[w(h)]$ . Equations (1.9) and (1.10) will be equalised in market equilibrium. This is illustrated in Figure 3 for the Cobb-Douglas example outlined in Appendix 1<sup>14</sup>. For expositional purposes we will refer to relationship between  $v$  and the number of workers as inverse supply and demand curves below. We see in Figure 3 that there is a downward sloping inverse demand curve and upward sloping inverse supply curve.

To calculate the change in the demand for workers after a minimum wage we totally differentiate the first order condition on workers from equation (1.2):

$$d\Pi_n = pf_{nn}dn^d + [pf_{nh}h_w - wh_w - h]dw + [pf_{nh} - w]h_vdv = 0 \quad (1.11)$$

The first two terms in (1.11) correspond to the total derivative from the firm's problem used to derive equation (1.3). The final term in (1.11) reflects the change the market hours wage locus from a change in  $v$ . Using the first order condition on hours in (1.11) we get the change in the demand for workers resulting from a minimum wage as:

$$\frac{dN^d}{dw} = F \frac{dn^d}{dw} = F(n_w^d + n_v^d v_w) = F \left\{ \left[ -\frac{f_h - f_{nh}}{f_{nn}} \right] h_w + \frac{[w - pf_{nh}]}{pf_{nn}} h_v v_w \right\} \quad (1.12)$$

The change in the supply of workers is:

$$\frac{dN^s(h, w)}{dw} = MG_v v_w \quad (1.13)$$

From (1.12) and (1.13) we see that if the market moves to a new equilibrium after the minimum wage we should have  $F(n_w^d + n_v^d v_w) = MG_v v_w$ . Proposition 1 establishes conditions where  $n_w^d$  will have the opposite sign to the hours wage locus. We will see

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<sup>14</sup> Figure 3 also includes the long run demand curve which will be modelled in the next section.



that in many cases and indeed for the Cobb-Douglas case we analyse explicitly  $n_v^d$  will have the opposite sign to  $n_w^d$ . This means that even when market labour demand and  $v$  are negatively related in an unconstrained market as in Figure 3 there will often be a positive co-movement between labour demand and  $v$  after a minimum wage. To explain this we note that an unconstrained firm will respond to an increase in  $v$  (a shift in the hours wage locus) by adjusting the hours wage combination. On the other hand if a minimum wage is imposed a firm cannot change the wage in response to a shift in the hours wage locus and the entire adjustment is in hours. Imagine firms are on a positive wage hours locus (as in Figure 1) and a minimum wage is enforced. From Proposition 1 all firms lay-off workers causing a decrease in the market value of  $v$  (a downward shift in the hours wage locus). The decrease in  $v$  shifts the hours wage locus downward which induces firms to increase hours further if the firm is on positively sloped hours wage locus since as we show in Appendix 3(b)  $h_w$  and  $h_v$  have opposite signs under a minimum wage. In summary, the initial reduction in the number of workers caused by the minimum wage causes a decrease in  $v$  which causes a further increase in hours. This increase in hours may induce firms to lay-off more workers which explains why there may be a positive relationship between the change in  $v$  and the number of workers after a minimum wage. Even if this is so we can solve for the new equilibrium as long as the slope of the inverse labour supply curve  $MG_v$  (which is positive by assumption) is greater than the slope of the restricted inverse labour demand where the wage is fixed. Figure 4 illustrates  $n_v^d$  and  $MG_v$  for the Cobb-Douglas case we analyse in Appendix 1. Even though the unrestricted inverse labour demand curve (in Figure 3) is downward sloping in  $v$  worker space, the restricted inverse demand curve (plotted in Figure 4) slopes upward when a minimum

wage is imposed for the reasons discussed above. We see for the Cobb-Douglas example that since the restricted inverse labour demand curve in Figure 4 is flatter than the inverse labour supply curve, we can solve for equilibrium after a minimum wage but the equilibrium fall in the number of workers after a minimum wage will be greater than the initial response by firms. However if the restricted inverse labour demand curve were steeper than the inverse labour supply curve the labour market would diverge from the initial equilibrium after a minimum wage is imposed and we cannot solve for equilibrium<sup>15</sup>. The Propositions below focus on the case where we can solve for equilibrium after the minimum wage.

*Proposition 1(a): A sufficient condition for Proposition one to continue holding for the aggregate number of workers in short run market equilibrium is that the restricted market inverse demand curve in worker  $v$  space is flatter than the market supply curve after a minimum wage is imposed.*

*Proof: See Appendix 3 (c)*

*Proposition 2(a): A sufficient condition for Proposition two to continue holding for total market hours worked in short run market equilibrium is that the restricted market inverse demand curve in worker  $v$  space is flatter than the market supply curve after a minimum wage is imposed.*

*Proof: See Appendix 3 (d)*

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<sup>15</sup> This will be true if  $MG_v - n_v^d < 0$

Proposition 1(a) and 2(a) tell us that in any case where the slope of the inverse supply and demand curves are such that we can solve for labour market equilibrium after a minimum wage Proposition 1 and Proposition 2 will continue to hold.

#### **Section IV: Firm entry**

In this section we extend the analysis in section III to allow for the fact that in the long run a minimum wage may affect firm's profits and in turn the number of firms. In a competitive model where hours are held fixed after a minimum wage there would be an excess supply of workers. The impact on profits would be small as long as firms had been optimising and the minimum wage only leads to a small increase in the wage. This may no longer be the case when we allow hours to vary. The minimum wage causes substitution between hours and workers. If the supply of workers is positively related to worker utility, the effect of each firm attempting to hire more/less workers in response to a minimum wage will be to cause the hours wage locus to shift up/down as we saw in the previous section. This shift in the equilibrium hours wage locus will indeed have a first order effect on profits.<sup>16</sup> We will assume that there is free entry of firms but the fixed cost of firm entry is driven up as the number of firms  $F$  increases. In equilibrium the number of firms will be such that the fixed entry cost  $E(F)$  equals firm profits and so the mass of firms is increasing as profits increase. The minimum wage affects firm profits by shifting the hours wage locus through the parameter  $v$ . We can write:

$$n \frac{dF}{dw} = n \frac{dF}{d\Pi} \frac{d\Pi}{dv} \frac{dv}{dw} \quad (1.14)$$

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<sup>16</sup> Bhaskar and To (1999) show in a monopsony model that while the immediate impact of a small minimum wage on the profits of an optimising firm will be second order, there is a first order negative impact on profits since the firms labour supply curve depends not only on its own wage but on the relative wage of other firms which increase.

Totally differentiating the profit function we note that since we impose a small minimum wage starting at the optimum the first order conditions for the firm (1.2) will continue to hold so that  $\Pi_n = \Pi_w = 0$  and the total derivative of the profit function is:

$$\frac{d\Pi}{dv} \frac{dv}{dw} = (pf_h - wn)h_v \frac{dv}{dw} \quad (1.15)$$

From the first order condition on  $w$  we can say that  $(pf_h - wn) = \frac{hn}{h_w}$ . Using both of

these equations we get the change in profits from a minimum wage:

$$\frac{d\Pi}{dv} \frac{dv}{dw} = hn \frac{h_v}{h_w} \frac{dv}{dw} \quad (1.16)$$

We proceed to incorporate the impact on profits into the analysis of section

III. The expression for  $\frac{dn}{dw}$  below is derived in the same way as (1.12) but we also

account for firm entry using (1.16):

$$\frac{dN^d}{dw} = F \frac{dn}{dw} + n \frac{dF}{dw} = F \left[ \frac{f_h - f_{nh}}{f_{nn}} \right] h_w + \left\{ F \left[ \frac{w - pf_{nh}}{pf_{nn}} \right] h_v + n h n \frac{h_v}{h_w} \frac{dF}{d\Pi} \right\} v_w \quad (1.17)$$

*Proposition 1(b): A sufficient condition for Proposition 1(a) to continue to hold in long run market equilibrium is that the firms inverse demand curve in worker  $v$  space is flatter than the market supply curve after a minimum wage is imposed.*

*Proof: See Appendix 3 (e)*

We note that the firm entry term in (1.17) reduces the slope of the inverse demand curve in  $v$  worker space. That is the proposition is more likely to hold in long run equilibrium than in short run equilibrium. As with the short run analysis the condition that the firms inverse demand curve is flatter than the inverse supply curve implies

that we are restricting the analysis to cases where we can solve for equilibrium after a minimum wage. We can also now establish the following proposition:

*Proposition 2(b): Proposition two continues to hold for total market hours worked in long run market equilibrium if the firm's inverse demand curve in worker  $v$  space is flatter than the market supply curve after a minimum wage is imposed.*

*Proof: See Appendix 3 (f)*

It may be useful to illustrate these results in a familiar example. In the long run equilibrium section of Appendix 1 we assume a linear relationship between firm entry and the level of profit and additionally assume that the weight on workers  $b$  equals one minus  $a$  the weight on hours. Using the parameter values given in Table A1 Figure 3 graphs the inverse supply and long run inverse demand curve illustrating the unrestricted equilibrium while Figure 4 illustrates that the restricted inverse demand curve is indeed flatter than the inverse supply curve for our Cobb-Douglas example. The solutions to the firm's problem are well behaved for all the parameter values in the graph<sup>17</sup>. Figure 5 plots the elasticity of total hours with respect to a minimum wage as we change the hours intensity of the production function. One can see that when the weight on workers and firms in the production function are similar the presence of fixed costs creates a range of parameter values where total hours worked increases as we demonstrated earlier. We see in Figure 5 that in fact there is a wider range of parameter values where the elasticity of hours is positive in long run equilibrium than there is when we look at the firms initial response to the minimum wage.

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<sup>17</sup> A maple file generating the graphs and showing that the firms second order conditions are satisfied and that profits hours workers the number of firms etc. are all positive for all the examples given is available from the authors on request.

## **Section V: The impact of the minimum wage on profits and worker utility**

The analysis in the previous section modelled firm entry and we see below has some unusual results:

*Proposition 3: If Proposition 1(b) holds the change in the long run equilibrium level of firm profits resulting from a minimum wage will have the same sign as the slope of the hours wage locus.*

*Proof: See Appendix 3 (g)*

This result makes intuitive sense given the framework we have adopted. That is, if the hours wage locus has a positive slope a minimum wage often causes the firm to substitute from hours into workers as we saw in Proposition 1. If the firm is starting at the optimum any negative impact of this change on profits will be small. However, as all firms substitute from hours to workers, the demand for workers falls and as we saw in the previous section the equilibrium hours locus shifts down which causes a first order increase in firm profits and a fall in worker utility. While Proposition 3 reflects a possible positive equilibrium impact of minimum wages on profits we certainly do not conclude from this that we should typically expect profits to increase after a minimum wage. Even in the simple framework we have adopted we can easily change the model to make the impact on profits ambiguous. The previous section assumed a minimum wage was imposed just above the optimal wage that would be chosen by firms in a free market so that there was no direct reduction in profits from the minimum wage. Of course as the minimum wage increases above this level profits will indeed fall. We can see this immediately for our Cobb-Douglas example by examining Figure 1. As the minimum wage increases, the firm is forced

onto higher points on the hours wage locus which will intersect with a lower profit isoprofit curve.

We can think of many cases where minimum wages apply but where we might expect market level labour supply elasticities for low wage workers to be large. State level minimum wages in the U.S or other countries, sectoral level minimum wages, or even minimum wages in small open economies where migration is prevalent are examples. In this case we might reasonably model the inverse labour supply curve in Figure 3 as being horizontal at the level of  $v$  determined in the larger labour market. If this were so the equilibrium wage hours locus would not shift in response to a minimum wage and as the minimum wage increased firms would exit reducing employment and hours.

Another feature of our model that is necessary for tractability, but again would have important implications for the impact of a minimum wage on profits or worker utility, is the absence of heterogeneity across workers and firms. In a model with more heterogeneity the equilibrium hours wage locus would be a set of tangency points between different types of workers and firms where workers with a relative preference for longer hours would match with firms with a relatively hours intensive technology as in Kinoshito (1987). In such a model a minimum wage would potentially have different effects on the demand for different types of workers and on the profits of different types of firms. For example if the wage hours locus had a positive slope we would expect firms initially choosing a wage below the minimum to have an initial reduction in profit after a minimum wage, while we might imagine there might well be an increase in demand for workers who initially chose to work at hours wage bundles above the minimum wage.

#### **Section IV: Conclusion**

The idea that minimum wages may lead to offsetting effects on hours per worker is generally recognised in the literature. However, given the prevalence of studies that focus solely on the number of workers, we suspect that the fact that changes in hours per workers and the number of workers from a minimum wage can be either positive or negative, and that theory predicts that the two will typically be inversely related in a competitive labour market, is not yet well understood.

While we show that there are cases where firms may respond to a minimum wage by increasing total hours, we are not suggesting that one should necessarily expect this to be typically true. Having said this, the parameter values where total hours effects are positive are not implausible (the effect may be positive for small fixed effects if the elasticity of output with respect to hours and workers are similar in the production function). The model shows that the firm's response to a minimum wage depends on the firm's technology so that it may well be, for example, that the elasticity of hours with respect to output is less than the elasticity of workers with respect to output for firms in one sector but not another, so that the impact of a minimum wage on total hours may be positive in one sector but not in another. This means that while empirical analysis that focuses on a homogeneous group of low skilled workers (say in the fast food industry) will resolve a lot of estimation problems and may provide compelling results for that group of workers, the theoretical analysis implies that even if labour markets are competitive the results may not be representative of the impact of a minimum wage across all sectors.

While Strobl and Walsh (2007) show that a minimum wage has an ambiguous effect on hours and employment in a monopsony model, the results in this paper draw similar conclusions for a competitive labour market. We might conclude that while



empirical studies that measure the impact of a minimum wage on employment or hours are clearly important from a policy perspective, the results from such studies give little guidance as to the underlying structure of the labour market.

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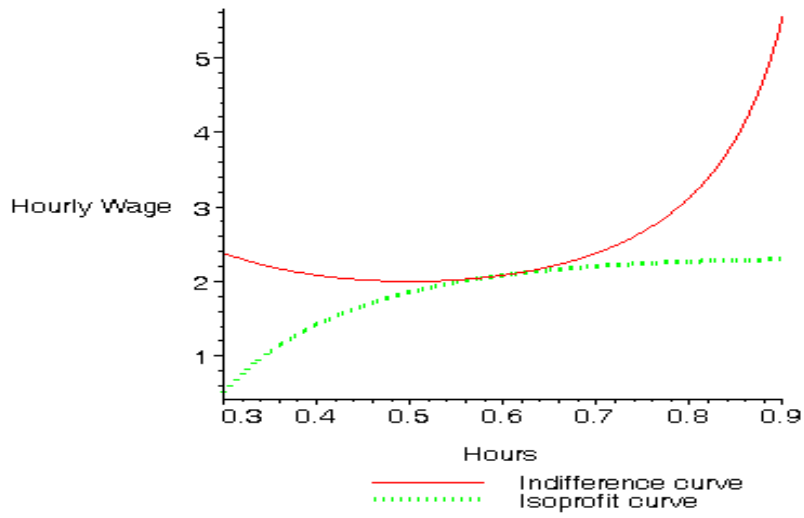
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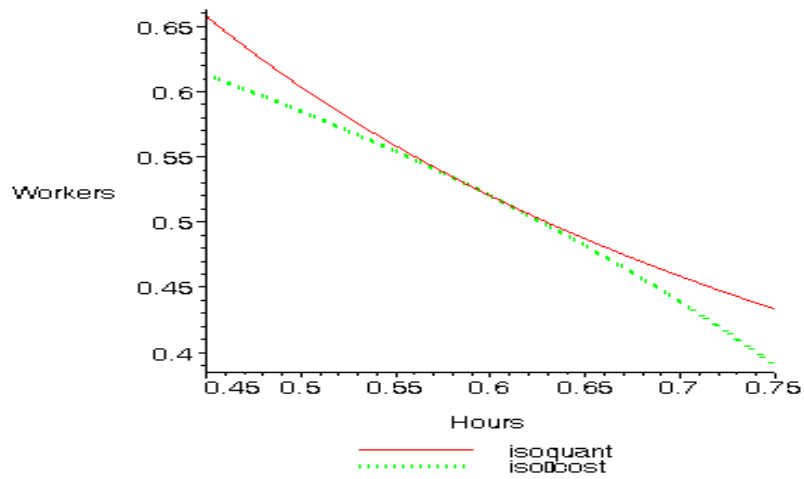
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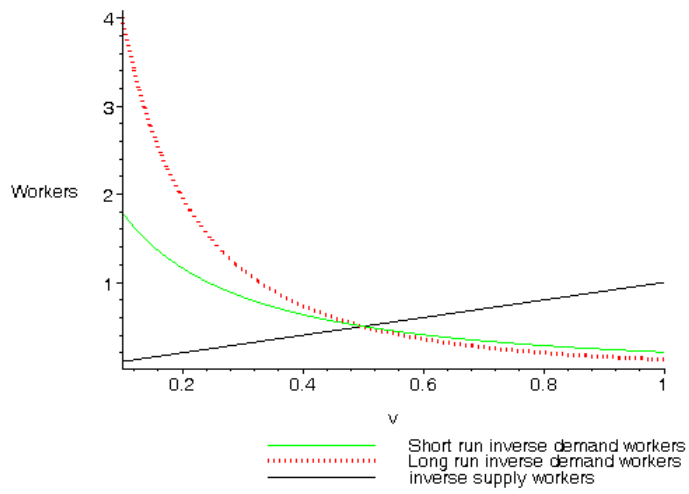
**Figure 1: equilibrium wages and hours**



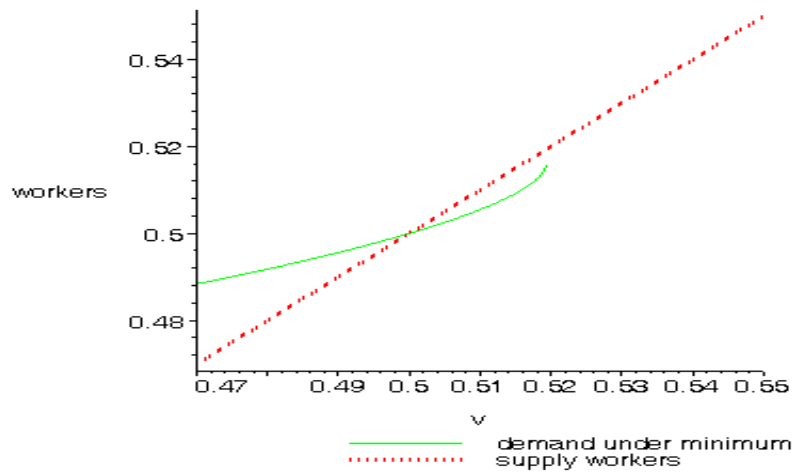
**Figure 2: The cost minimising bundle of hours/workers**



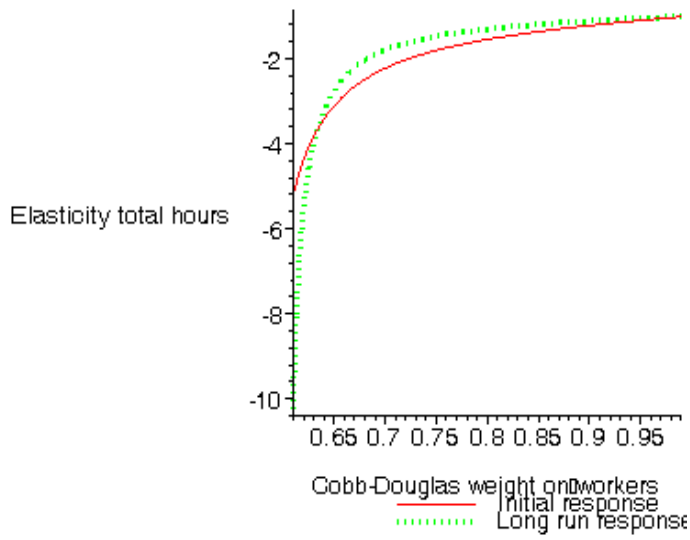
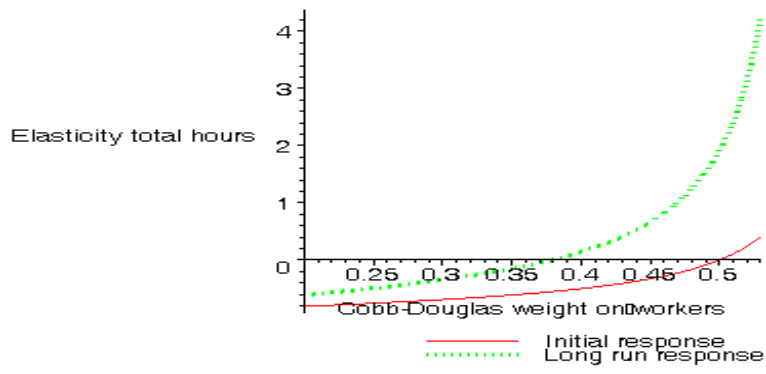
**Figure 3: Short and long run market equilibrium**



**Figure 4: Minimum wage and short run market demand**



**Figure 5: Minimum wage and total hours**



**Table A1: Equilibrium values for the Cobb Douglas example**

Output ( $y = h^{0.45} n^{0.55}$ )	0.55
Utility $u = wh(t - h)$ ]	0.5
Profit ( $\pi$ )	0.96
Hours per worker (h)	0.60
Number of workers (n)	0.52
Wage	2.08

Simulated model from appendix 1. where  $M=t=k=1$  and  $S=0$ . The first column solves for values at the equilibrium market clearing wage. Since we assume a unit mass of workers are available we set the output price at 3.835 such that  $u=0.5$  and labour demand is less than labour supply.

## Appendix 1

### *Cobb-Douglas production and utility function*

In this section we illustrate our results using a Cobb-Douglas example. This example is used to generate the results in Table A1 above and Figures 1 through 5<sup>18</sup>. The technology over hours and workers is  $f(n, h) = h^a n^b$ . A commonly used approach is to model the aggregate labour labour input as  $H = n^n h^\delta$ .<sup>19</sup> It is also important to remember that the technology above is for producing labour inputs and that one might expect that there is a diminishing marginal product associated with the labour input. One example would be  $f(H) = H^\beta = n^{\eta\beta} h^{\delta\beta} = h^a n^b$ <sup>20</sup> where both  $a$  and  $b$  are less than unity. Relative to  $b$ , a small value for  $a$  means a worker intensive production function, while a large value for  $a$  implies an hours intensive production function. We note that in this case equation (1.3) becomes:

<sup>18</sup> A maple file generating these simulations and graphs is available from the authors on request.

<sup>19</sup> See Hamermesh (1993) for example.

<sup>20</sup> Another way of looking at it is that there are  $s$  other inputs  $x_1...x_s$  in addition to labour  $L$  and a Cobb-Douglas production function  $f(x_1, ..x_s, H) = x_1^{\alpha_1} ...x_s^{\alpha_s} H^b$ . The comparative static analysis would be much more complicated if we add more inputs, but the example shows that it would be plausible to have a production function where the weights on  $h$  and  $n$  would be less than unity.

$$\frac{dn}{dw} = \frac{[\frac{f_h}{n} - f_{nh}]}{f_{nn}} \frac{dh}{dw} = -\frac{a}{b} \frac{n}{h} \frac{dh}{dw} \quad \text{A(1.18)}$$

We see that  $\frac{dn}{dw}$  and  $\frac{dh}{dw}$  have the opposite sign. Equation (1.6) becomes:

$$\frac{d(nh)}{dw} = n \frac{dh}{dw} + h \frac{dn}{dw} = n \left[ \frac{b-a}{b} \right] \frac{dh}{dw} \quad \text{A(1.19)}$$

To solve for equilibrium hours we continue by assuming that workers have the Cobb-Douglas utility function:  $u = cl - v = wh(T - h) - v$ . Workers have a reservation utility  $S$  as described in Section III and  $v$  is a fixed cost associated with going to work. At the equilibrium level of utility the indifference curve is:

$$w = \frac{u}{h(t-h)} \quad \text{A(1.20)}$$

For the moment we take the equilibrium value of  $u$  as given. We solve for labour market equilibrium in the simple example described below. Given the utility function the slope of the hours wage locus is just the slope of the indifference curve of the marginal worker:

$$\frac{dh}{dw} = \frac{[h(t-h)]^2}{(-t+2h)u} \quad \text{A(1.21)}$$

It follows that if  $t > 2h$  then  $\frac{dh}{dw} < 0$  and if  $t < 2h$  then  $\frac{dh}{dw} > 0$ . The representative firm's profit function is:

$$\pi(n, h) = h^a n^b - \frac{nu}{(t-h)} - kn \quad \text{A(1.22)}$$

From the first order conditions on  $n$  and  $w$  we get the following quadratic form for  $h$ :

$$(ut + kt^2) - \left[ \left( \frac{a+b}{a} \right) u + 2kt \right] h + kh^2 = 0 \quad \text{A(1.23)}$$



Noting that since  $h < t$  the solution is:

$$h = t + \left(\frac{a+b}{a}\right) \frac{u}{2k} - \frac{\sqrt{(a+b)^2 u^2 + 4ktabu}}{2ka} \quad \text{A(1.24)}$$

Given these considerations the easiest way to proceed is to simulate the model. Table A1 above assumes parameter values and equilibrium outcomes.

### *Figures 1 and 2*

Using equation A(1.20) above Figure 1 traces out the worker's indifference curve for the assumed values for  $u$  and  $t$  given in Table A1. Taking the first order condition on  $n$  we can rewrite A(1.22) in terms of the hourly wage  $w$  and trace out the relationship between hours and the wage at the equilibrium level of profit. This is the isoprofit curve plotted in Figure 1.

Again from the profit maximising solution for  $n$  and  $h$  we can solve for the equilibrium cost of production  $c$ . Solving  $c = whn + nk$  for  $h$  and using A(1.20) for  $w$ . This gives the isocost curve in Figure 2. Using the solutions for  $h$  and  $n$  from profit maximisation we calculate the equilibrium value of output. Using this value we rearrange the production function in terms of  $h$  and this gives the isoquant in Figure 2.

### *Long run market equilibrium*

If we assume that  $v$  has a uniform distribution over the unit interval and that reservation utility is normalised to zero then we get the equilibrium condition for worker participation  $D(h, w) = wh(T - h) = v$  and also that labour supply is:  $N^s = vM$ .

We assume a positive linear relationship between firm entry and profits  $F = f\pi$  where we set  $f$  equal to unity. Labour demand is  $N^d = nF = nf\pi$ . Since profit depends on the output price level  $p$ , for convenience we set this at 3.835 (this value

ensures that the equilibrium value of  $v^*=0.5$  and the demand for workers equals the supply where half the population are employed). Once we solve for the equilibrium number of firms we fix it and let  $v$  change to get the short run inverse demand curve in Figure 3. Figure 4. Graphs the two terms in the denominator of equation A(1.40) below for the Cobb-Douglas example. These are the slope of the market inverse supply and demand curves after a minimum wage is imposed. Figure 5 is described in the text.

## Appendix 2

*Condition for scale effect on hours to be positive.*

Firms that maximise the profit function in equation (1.1) must be minimising the cost of producing output. The cost minimisation problem is described in the following Lagrangean function where  $\lambda$  is the Lagrange multiplier.

$$\iota = wh(w)n + nk + \lambda\{q - f[n, h(w)]\} \quad \text{A(1.25)}$$

The first order conditions on  $w$ ,  $n$  and  $\lambda$  respectively are:

$$\iota_w(n, w, \lambda) = wh_w(w)n + h(w)n - \lambda f_h[n, h(w)]h_w(w) = 0$$

$$\iota_n(n, w, \lambda) = wh(w) + k - \lambda f_n[n, h(w)] = 0 \quad \text{A(1.26)}$$

$$\iota_\lambda(n, w, \lambda) = q - f[n, h(w)] = 0$$

Totally differentiating the first order conditions with respect to  $w, n, \lambda$  and  $q$  we get the following matrix system:

$$\begin{pmatrix} t_{ww} & t_{wn} & t_{w\lambda} \\ t_{nw} & t_{nn} & t_{n\lambda} \\ t_{\lambda w} & t_{\lambda n} & t_{\lambda\lambda} \end{pmatrix} \begin{pmatrix} \frac{dw}{dq} \\ \frac{dn}{dq} \\ \frac{d\lambda}{dq} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad \text{A(1.27)}$$

If we call the (3X3) matrix above  $A$ , we can use Cramer's rule and the assumption

that the scale effect on hours is greater than or equal to zero ( $\frac{\partial h}{\partial q} \geq 0$ ) and say that:

$$\frac{dh}{dq} = \frac{dh}{dw} \frac{dw}{dq} = \frac{dh}{dw} \frac{|A_1|}{|A|} = \frac{dh}{dw} \frac{(t_{w\lambda}t_{nn} - t_{wn}t_{n\lambda})}{|A|} \quad \text{A(1.28)}$$

$$= \frac{\lambda \frac{dh^2}{dw} \left[ \left( \frac{f_h}{n} - f_{hn} \right) f_n + f_h f_{nn} \right]}{|A|} \geq 0$$

Note that  $\lambda > 0$  (since  $\lambda$  is marginal cost). The second order conditions for cost minimisation imply that  $|A| < 0$ . Also of course we assume that:  $f_h > 0, f_n > 0, f_{nn} < 0$  while  $\frac{dh^2}{dw} > 0$ . Thus the term in squared brackets in A(1.28) must be negative implying:

$$\left( \frac{f_h}{n} - f_{hn} \right) f_n + f_h \geq 0 \quad \text{A(1.29)}$$

This also implies:

$$-\frac{f_h}{f_n} \leq \frac{(f_h - f_{hn})}{f_{nn}} \quad \text{A(1.30)}$$

The two terms in this inequality will be equal when scale effects are zero. We also note that we could reverse the signs of the inequalities given in A(1.28) to A(1.30) to

focus on the case where scale effects on hour are negative. In this case  $\frac{(f_h - f_{hn})}{f_{nn}} < 0$ .

It is instructive to analyse the impact of a minimum wage on the cost minimisation problem. We totally differentiate the first order conditions on  $n$  and  $\lambda$  from A(1.26) to get the following matrix system:

$$\begin{pmatrix} t_{nn} & t_{n\lambda} \\ t_{\lambda n} & t_{\lambda\lambda} \end{pmatrix} \begin{pmatrix} \frac{dn}{dw} \\ \frac{d\lambda}{dw} \end{pmatrix} = \begin{pmatrix} -t_{nw} \\ -t_{\lambda w} \end{pmatrix} \quad \text{A(1.31)}$$

Using Cramer's rule again we get the substitution effect of a change in the wage on the number of workers:

$$\frac{dn}{dw} = \frac{t_{n\lambda} t_{\lambda w}}{-t_{n\lambda}^2} = -\frac{f_h}{f_n} h_w \quad \text{A(1.32)}$$

### Appendix 3

#### (a) Proof of Proposition 2

Using (1.5) in (1.6) the elasticity of total hours with respect to the minimum wage can be written as:

$$\varepsilon_{nh,w} = \frac{d(nh)}{dw} \frac{w}{nh} = \frac{1 + \frac{h}{n} \left( \frac{f_h - f_{nh}}{f_{nn}} \right)}{\left( 1 + \frac{k}{wh} \right) \frac{\varepsilon_{qh}}{\varepsilon_{qn}} - 1} \quad \text{A(1.33)}$$

Next we assume that hours per worker is not an inferior input. That is  $\frac{dh}{dq} \geq 0$  for the production function  $q = f(n, h)$ . We show in Appendix 2 that this assumption holds when marginal costs are positive and equation A(1.29) is satisfied. Equation A(1.29)

also implies that there is some number  $x \geq 0$  such that  $\frac{f_h - f_{nh}}{f_{nn}} = -\frac{f_h}{f_n} + x$ . In the special case where the scale effect on hours equals zero  $x$  also equals zero. Using this definition of  $x$  we rewrite A(1.33):

$$\varepsilon_{nh,w} = \left[ \frac{1 + \frac{h}{n} x - \frac{\varepsilon_{qh}}{\varepsilon_{qn}}}{\left( 1 + \frac{k}{wh} \right) \frac{\varepsilon_{qh}}{\varepsilon_{qn}} - 1} \right] \quad \text{A(1.34)}$$

The condition for the numerator and denominator of A(1.34) to be positive, where it follows that the elasticity of total hours with respect to a minimum wage is positive gives the following inequality which is (1.7) in the main text:

$$\frac{1}{1 + \frac{k}{wh}} < \frac{\varepsilon_{q,h}}{\varepsilon_{q,n}} < 1 + x \frac{h}{n} \quad A(1.35)$$

From (1.5) if the hourly wage locus has a positive slope the denominator of A(1.34) is positive. But since  $x, h$  and  $n$  are positive if  $\varepsilon_{qn} > \varepsilon_{qh}$  the numerator of A(1.34) is positive implying  $\varepsilon_{nh,w} > 0$ .

(b) Showing that  $h_w$  and  $h_v$  will have the opposite signs

Using the fact that equation (1.8) will always hold in equilibrium, even after a minimum wage and remembering that the wage is held fixed we see that  $\frac{du[D(h, w), v^*]}{dv^*} \Big|_{w=w} = \frac{\partial u}{\partial D} \frac{\partial D}{\partial h} \frac{\partial h}{\partial v^*} + \frac{\partial u}{\partial v^*} = 0$ . Since  $\frac{\partial u}{\partial D} > 0$  and  $\frac{\partial u}{\partial v^*} < 0$  by assumption, then  $\frac{\partial D}{\partial h}$  and  $\frac{\partial h}{\partial v^*}$  must have the same sign to ensure that this equation equals zero. But  $\frac{\partial D}{\partial h}$  has the opposite sign to  $\frac{\partial h}{\partial w}$  when the wage is fixed. To illustrate this imagine that the indifference curve in Figure 1 is the equilibrium hours wage locus. An increase in  $h$  when the wage is held fixed at any point puts the workers on a higher indifference curve starting from any point where the indifference curve has a negative slope and on a lower indifference curve starting from a point where the indifference curve has a positive slope. Thus  $\frac{\partial h}{\partial v}$  and  $\frac{\partial h}{\partial w}$  have the opposite sign.

(c) Proof of Proposition 1(a)

Setting (1.9) equal to (1.10) we solve for the equilibrium change in  $v^*$ :

$$v_w = \frac{\frac{f_h - f_{nh}}{f_{nn}} h_w}{MG_v - F \left[ \frac{w - pf_{nh}}{pf_{nn}} \right] h_v} \quad A(1.36)$$

We can insert this in (1.12) or (1.13) to get the equilibrium change in employment:

$$\frac{dN^d}{dw} = F(n_w + n_v v_w) = F \left[ \frac{f_h - f_{nh}}{f_{nn}} \right] h_w \left\{ \frac{MG_v}{MG_v - F \left[ \frac{w - pf_{nh}}{pf_{nn}} \right] h_v} \right\} = F \left[ \frac{f_h - f_{nh}}{f_{nn}} \right] h_w Y$$

$$A(1.37)$$

The slope of the inverse supply curve is  $MG_v$  while from (1.11) the slope of the inverse demand curve is  $F \frac{[w - pf_{nh}]}{pf_{nn}} h_v$ . If  $MG_v - F \frac{[w - pf_{nh}]}{pf_{nn}} h_v > 0$  then  $Y > 0$  and Proposition one holds since A(1.37) is just (1.3) times a positive constant.

(d) Proof of Proposition 2 (a)

The first order conditions (1.2) will continue to hold. Noting from the first order conditions that  $w = \frac{f_h - f_{nh}}{f_{nn}} - \frac{h}{h_w pf_{nn}}$ , using (1.6) and A(1.37) we can write the change in total hours at a firm as:

$$\frac{d(Nh)}{dw} = N(h_w + h_v v_w) + h \frac{dN}{dw} \quad A(1.38)$$

Using A(1.37) and A(1.36) this can be written as:

$$\begin{aligned} \frac{d(Nh)}{dw} &= Nh_w \left[ 1 + \frac{\left( \frac{f_h - f_{nh}}{f_{nn}} \right) (Fh_v + \frac{F}{N} MG_v h)}{MG_v - F \frac{[w - pf_{nh}]}{pf_{nn}} h_v} \right] + \frac{Fhh_v}{h_w pf_{nn}} \\ &= \frac{Ah_w \left[ 1 + \left( \frac{f_h - f_{nh}}{f_{nn}} \right) \frac{h}{n} \right] + B}{C} \end{aligned} \quad A(1.39)$$

Where  $A = MG_v N > 0$  is the slope of the inverse labour supply Curve times the number of workers.  $B = \frac{Fhh_v}{h_w pf_{nn}} > 0$  since we are concerned with the case where  $h_w > 0$  in this proposition and we showed in Appendix 3 (b) that  $h_v < 0$  in this case.  $C = MG_v - F \frac{[w - pf_{nh}]}{pf_{nn}} h_v > 0$  since the slope of the inverse supply curve is greater than the slope of the inverse demand curve {the inverse demand curve is  $F \frac{[w - pf_{nh}]}{pf_{nn}} h_v$  from (1.11)}. Since  $h_w \left[ 1 + \left( \frac{f_h - f_{nh}}{f_{nn}} \right) \frac{h}{n} \right]$  is equation (1.6) the change in total hours at the level of the firm and A(1.39) is a positive transformation of this and Proposition 2 will continue to hold.

(e) Proof of Proposition 1 (b)

Supply and demand for workers will be equal in equilibrium so that one can equate (1.17) with the derivative of the supply of workers (1.13) to get:

$$v_w = \frac{F\left[\frac{f_h - f_{nh}}{f_{nn}}\right]h_w}{MG_v - \left\{F\left[\frac{w - pf_{nh}}{pf_{nn}}\right]h_v + nhn \frac{h_v}{h_w} \frac{dF}{d\Pi}\right\}} \quad A(1.40)$$

This in turn can be substituted back into (1.17) to get the change in equilibrium employment:

$$\begin{aligned} \frac{dN^d}{dw} &= F\left[\frac{f_h - f_{nh}}{f_{nn}}\right]h_w \left[ \frac{MG_v}{MG_v - \left\{F\left[\frac{w - pf_{nh}}{pf_{nn}}\right]h_v + nhn \frac{h_v}{h_w} \frac{dF}{d\Pi}\right\}} \right] \\ &= F\left[\frac{f_h - f_{nh}}{f_{nn}}\right]h_w Y1 \end{aligned} \quad A(1.41)$$

The slope of the inverse market supply curve is  $MG_v$  and the slope of the inverse market demand curve after a minimum wage is imposed is  $F\left[\frac{w - pf_{nh}}{pf_{nn}}\right]h_v + nhn \frac{h_v}{h_w} \frac{dF}{d\Pi}$ . If the slope of the inverse supply curve is larger  $Y1 > 0$  and comparing with equation (1.3) we see that Proposition 1 will continue to hold.

(f) Proof of Proposition 2 (b)

The change in total hours is:

$$\frac{d(Nh)}{dw} = N(h_w + h_v v_w) + h \frac{dN}{dw} \quad A(1.42)$$

Using A(1.40) and A(1.41) this can be written as:

$$\begin{aligned}
\frac{d(Nh)}{dw} &= Nh_w \left[ 1 + \frac{\left( \frac{f_h - f_{nh}}{f_m} \right) (Fh_v + \frac{F}{N} MG_v h)}{MG_v - \left\{ F \frac{[w - pf_{nh}]}{pf_m} h_v + nhn \frac{h_v}{h_w} \frac{dF}{d\Pi} \right\}} \right] \\
&= \frac{MG_v Nh_w \left[ 1 + \left( \frac{f_h - f_{nh}}{f_m} \right) \frac{h}{n} \right] + \frac{Fhh_v}{h_w pf_m} - nhn \frac{h_v}{h_w} \frac{dF}{d\Pi}}{MG_v - \left\{ F \frac{[w - pf_{nh}]}{pf_m} h_v + nhn \frac{h_v}{h_w} \frac{dF}{d\Pi} \right\}} \tag{A(1.43)} \\
&= \frac{Ah_w \left[ 1 + \left( \frac{f_h - f_{nh}}{f_m} \right) \frac{h}{n} \right] + B1}{C1}
\end{aligned}$$

The denominator of A(1.43) is the slope of the inverse labour supply curve less the slope of the restricted inverse labour demand curve which is positive by assumption.

The numerator is the same as the numerator of A(1.39) in the proof of proposition 2(a) above except that it includes the term  $-nhn \frac{h_v}{h_w} \frac{dF}{d\Pi}$ . Since we are concerned

with the case where  $h_w > 0$  in this proposition,  $\frac{dF}{d\Pi} > 0$  by assumption and as shown

earlier  $h_w$  and  $h_v$  have the opposite signs under a minimum wage we can say that

$$-nhn \frac{h_v}{h_w} \frac{dF}{d\Pi} > 0$$

It follows that as we showed Proposition 2 holds for A(1.39) above, it will also hold for A(1.43).

### (g) Proof of Proposition 3

If Proposition 1(b) holds  $Y1 > 0$  and from A(1.40)  $v_w$  has the opposite sign to  $h_w$ . Equation (1.16) gives us the change in profits from a minimum

wage  $\frac{d\Pi}{dv} \frac{dv}{dw} = hn \frac{h_v}{h_w} \frac{dv}{dw}$ . Since  $h_v$  and  $h_w$  have opposite signs from Appendix 3 (b)

the change in profits will have the same sign as the slope of the hours wage locus.