Title	The Goursat problem for a generalized Helmholtz operator in the plane
Authors(s)	Ebenfelt, Peter, Render, Hermann
Publication date	2008-09
Publication information	Ebenfelt, Peter, and Hermann Render. “The Goursat Problem for a Generalized Helmholtz Operator in the Plane” 105, no. 1 (September, 2008).
Publisher	Springer
Item record/more information	http://hdl.handle.net/10197/5500
Publisher's statement	The final publication is available at www.springerlink.com
Publisher's version (DOI)	10.1007/s11854-008-0033-5

THE GOURSAT PROBLEM FOR A GENERALIZED HELMHOLZ OPERATOR IN $\mathbb{R}^2$

PETER EBENFELT AND HERMANN RENDER

1. Introduction

Let us consider in $\mathbb{R}^2$ the mixed Cauchy problem

$$(1) \qquad (\textbf{cauchy0}) \begin{cases} \Delta^p u + \sum_{|\alpha| \le k_0} a_\alpha \frac{\partial^{|\alpha|} u}{\partial x^\alpha} = f \\ P|(u-g), \end{cases}$$

where p is a positive integer, k_0 is an integer with $0 \le k_0 \le 2p-1$, Δ denotes the standard Laplace operator in $\mathbb{R}^2$

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

the coefficients $a_\alpha = a_\alpha(x,y)$ as well as the data functions $f = f(x,y)$ and $g = g(x,y)$ are real-analytic functions near 0, and $P = P(x,y)$ is a homogeneous polynomial of degree $2p$. Here, the notation $P|(u-g)$ means that P divides $u-g$ in the ring of germs of real-analytic functions at 0. For instance, if $P(x,y) = L(x,y)^{2p}$ for some linear function $L(x,y)$ (which is equivalent to saying that the zero set of $P(x,y)$ consists of a single line with multiplicity $2p$), then (1) with $k_0 = 2p-1$ is a standard Cauchy problem and the classical Cauchy-Kowalevsky Theorem guarantees that (1) has a unique real-analytic solution u near 0 for every choice of data functions f and g. In the recent paper [1], the authors show that if P is elliptic (i.e. the zero set of $P(x,y)$ consists of only the origin), then (1) with $k_0 = p$ has a unique solution u for every choice of data functions f and g. In this paper, we shall consider the case where the zero set of $P(x,y)$ is a union of $2p$ distinct lines (in which case (1) may be called a Goursat problem). This case is much more subtle and leads to a small divisor problem. We shall give a sufficient condition (which is also necessary in the case $p=1$; see Section 7) on the divisor P (see Theorem 1 below) for the homogeneous Goursat problem

$$(2) \qquad (\textbf{goursatp1}) \begin{cases} \Delta^p u = f \\ P|(u-g) \end{cases}$$

The first author is supported in part by DMS-0401215. The second author is supported in part by Grant MTM2006-13000-C03-03 of the D.G.I. of Spain.

to have a unique real-analytic solution u for every real-analytic data f and g. We shall also give a sufficient condition on P (Theorem 3 below) for the perturbed Goursat problem

$$(\textbf{goursatp2}) \begin{cases} \Delta^p u + cu = f \\ P|(u-g), \end{cases} \tag{3}$$

where $c = c(x, y)$ is a real-analytic function near 0, to have a unique real-analytic solution u for every data function f and g.

The conditions on P in Theorems 1 and 3 involve Diophantine properties of a determinant constructed from the geometry of the lines constituting the zero set of P. For instance, if $p = 1$, so that P has degree two and its zero set consists of two distinct lines, then the condition can be phrased in terms of the (acute) angle $\theta = 2\pi\alpha$ between the two lines. The necessary and sufficient condition for the homogeneous Goursat problem

$$(\textbf{goursat11}) \begin{cases} \Delta u = f \\ P|(u-g) \end{cases} \tag{4}$$

to be solvable (Corollary 4) is that

$$(\textbf{dioleray}) \liminf_{\mathbb{Z}\ni m\to\infty} \left(\inf_{n\in\mathbb{Z}} \left| \alpha - \frac{n}{m} \right| \right) > 0, \tag{5}$$

a condition that is satisfied by e.g. all non-Liouville numbers. Our condition for the perturbed Goursat problem

$$(\textbf{goursat12}) \begin{cases} \Delta u + cu = f \\ P|(u-g), \end{cases} \tag{6}$$

to be solvable (Corollary 5) is more restrictive, namely there exists a constant $C > 0$ such that

$$(\textbf{diophantine}-\textbf{1}) \left| \alpha - \frac{n}{m} \right| \geq \frac{C}{m^2}, \quad \forall n, m \in \mathbb{Z}, m \neq 0. \tag{7}$$

We note that every irrational number α that satisfies an integral quadratic equation (like $\sqrt{k/l}$ for any integers k and l) satisfies (7) (by Liouville's Theorem on Diophantine approximation). We also point out that every irrational, algebraic number satisfies

$$(\textbf{diophantine}-\textbf{2}) \left| \alpha - \frac{n}{m} \right| \geq \frac{C_\mu}{m^\mu}, \quad \forall n, m \in \mathbb{Z}, m \neq 0, \tag{8}$$

for some constant C_μ (that depends on μ) and *every* $\mu > 2$ by the Thue-Siegel-Roth Theorem [7]). However, there are algebraic numbers that do not satisfy (7).

We also mention that it follows from our proof that (6) has a unique formal power series solution for all f and g if and only if α is irrational. Thus, as a consequence of our results, we conclude that the family of Goursat problems (6), parametrized by the angle $2\pi\alpha$ between the two lines in the zero set of P, displays "chaotic" behavior in that the set of parameters for which (6) is solvable is dense as is the set of parameters for which there is not even a formal solution.

The Goursat problem (4) (i.e. (2) with $p = 1$) can be transformed, by a simple linear change of coordinates, into a Goursat problem considered by J. Leray in [5]. His main result is equivalent our Corollary 4. The relationship between the two Goursat problems and Leray's work is briefly explained in Section 3 below. Leray's work was extended to complex parameters and to higher dimensions by Yoshino in [10] and [11]. Other related work on mixed Cauchy and Goursat problems include that of Gårding [3] (see also Theorem 9.4.2 in Hörmander [4]), Shapiro [8], the first author and Shapiro [2], and the authors [1]. Our approach to studying the Goursat problem is inspired by ideas from [8] (see also [2]). The proofs are based on a new estimate for an associated Fischer operator in the real Fischer norm (Theorem 6). The real Fischer norm was introduced in [6] and was also used in [1].

This paper is organized as follows. We present our main results in Section 2. In Section 3, we discuss the relation between our results in the case $p = 1$ and $c \equiv 0$ and those of Leray in [5]. An associated Fischer operator, which is used in the proofs of the main results, is introduced in Section 4 and a crucial estimate is proved for that operator (Theorem 6). The proof of Theorem 1 is also given in that section. The proof of Theorem 3 is given in the subsequent section. In Section 6, we consider the case $p = 2$ and present an explicit family of examples to which Theorem 3 can be applied (see Theorem 8). Finally, in Section 7, we show that our condition in Corollary 4 is also necessary in this case ($p = 1$).

2. Main results

(mainresults)

We shall now formulate our results more precisely. We must first introduce some notation. Let $B_R := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < R^2\}$ be the open disk of radius R in $\mathbb{R}^2$ (where $0 < R \leq \infty$). We denote by $A(B_R)$ the algebra of all infinitely differentiable functions $f : B_R \to \mathbb{C}$ such that for any compact subset $K \subset B_R$ the homogeneous Taylor series $\sum_{m=0}^{\infty} f_m(x, y)$ converges absolutely and uniformly to f on K; here, f_m is the homogeneous polynomial of degree m defined by the Taylor series of f

$$f_m(x, y) = \sum_{k+l=m} \frac{1}{k!l!} \frac{\partial^m f}{\partial x^k \partial y^l}(0)\, x^k y^l.$$

Note that the functions in $A(B_R)$ are real-analytic. For a real number a, we shall define the unimodular complex number

$$\textbf{(A)}\ A = A(a) := \frac{a + i}{a - i}. \tag{9}$$

As a goes from $-\infty$ to ∞, A ranges over the unit circle (from 1 to 1 in the negative direction) and, hence, there is a unique $\beta \in (0,1)$ such that $A = e^{2\pi i\beta}$. Note that β is rational precisely when A is a root of unity. For future reference, we observe, using the fact that $2\arctan a = i\log(1-ia)/(1+ia)$, that for $a \in [0,\infty)$ the acute angle between the lines $y=0$ and $x-ay=0$ is $\pi\beta$. Now, let us fix a positive integer p, distinct real numbers $a_1, a_2, \ldots, a_{2p-1}$, and write a for the vector $a = (a_1, \ldots, a_{2p-1})$. We shall denote by $P_a(x,y)$ the divisor

$$\textbf{(Pa)}\ P_a(x,y) := y\,\Pi_{j=1}^{2p-1}(x-a_jy). \tag{10}$$

If the divisor P in (1) is a homogeneous polynomial of degree $2p$ with $2p$ distinct lines as its zero set, then there is no loss of generality in assuming that P is of the form (10), since the Laplace operator is rotationally invariant. We associate to the vector a a sequence of $2p \times 2p$ matrices $\{M_{m,p,a}\}_{m=0}^{\infty}$, where

$$\textbf{(Ma)}\ M_{m,p,a} := \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 \\ 1 & A_1 & A_1^2 & \ldots & A_1^{p-1} & A_1^{m+p+1} & \ldots & A_1^{m+2p} \\ 1 & A_2 & A_2^2 & \ldots & A_2^{p-1} & A_2^{m+p+1} & \ldots & A_2^{m+2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & A_{2p-1} & A_{2p-1}^2 & \ldots & A_{2p-1}^{p-1} & A_{2p-1}^{m+p+1} & \ldots & A_{2p-1}^{m+2p} \end{pmatrix}. \tag{11}$$

Here, $A_j := A(a_j)$ where $A(a_j)$ is given by (9). We shall consider the Goursat problem

$$\textbf{(goursatp)}\ \begin{cases} \Delta^p u + cu = f \\ P_a | (u-g), \end{cases} \tag{12}$$

where the coefficient $c = c(x,y)$ as well as the data functions $f = f(x,y)$, $g = g(x,y)$ belong to $A(B_R)$. Our first result concerns the homogenenous problem, i.e. $c \equiv 0$.

Theorem 1. (homodelp) *Let p be a positive integer and $a_1, \ldots, a_{2p-1}$ real, distinct, non-zero numbers. Let $A_j := A(a_j)$, for $j = 1, \ldots, 2p-1$, be the unimodular complex numbers given by* (9)*, $P_a(x,y)$ the homogeneous polynomial given by* (10)*, and $\{M_{m,p,a}\}_{m=0}^{\infty}$ given by* (11)*. If $\det M_{m,p,a} \neq 0$ for all integers $m \geq 0$, and*

$$\textbf{(leraycond1)}\ \tau := \liminf_{m\to\infty} \left(\det M_{m,p,a}\right)^{1/m} > 0, \tag{13}$$

then the homogeneous Goursat problem

$$\textbf{(goursatp0)}\ \begin{cases} \Delta^p u = f \\ P_a | (u-g) \end{cases} \tag{14}$$

has a unique solution $u \in A(B_{\tau R})$ for every $f, g \in A(B_R)$.

Remark 2. (rmkmatrix) For future reference, we note the following identity
(15)

$$(\mathbf{Ma3})\ \det M_{m,p,a} = \det \begin{pmatrix} A_1 - 1 & A_1^2 - 1 & \dots & A_1^{p-1} - 1 & A_1^{m+p+1} - 1 & \dots & A_1^{m+2p} - 1 \\ A_2 - 1 & A_2^2 - 1 & \dots & A_2^{p-1} - 1 & A_2^{m+p+1} - 1 & \dots & A_2^{m+2p} - 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{2p-1} - 1 & A_{2p-1}^2 - 1 & \dots & A_{2p-1}^{p-1} - 1 & A_{2p-1}^{m+p+1} - 1 & \dots & A_{2p-1}^{m+2p} - 1 \end{pmatrix},$$

for $k \geq p-1$. In particular, for $p = 1$, we have $\det M_{m,p,a} = A_1^{m+2} - 1$.

We mention that e.g. all numbers $a_1, \dots, a_{2p-1}$ such that $A_1, \dots, A_{2p-1}$ are algebraic and $\det M_{m,p,a} \neq 0$ for all m satisfy (13) (see [9], Lemma 2.1).

It will follow from our proof of Theorem 3 below that the Goursat problem (12), and hence in particular (14), has a unique formal solution u if and only if $\det M_{m,p,a} \neq 0$ for all integers $m \geq 0$. The Diophantine condition (13) is sufficient (and necessary for $p = 1$; see Section 7 below) for the formal solution to (14) to converge. For the formal solution to the general Goursat problem (3) to converge, we need a stronger condition. We have the following result.

Theorem 3. (helmdelp) *Let p be a positive integer and $a_1, \dots, a_{2p-1}$ real, distinct, non-zero numbers. Let $A_j := A(a_j)$, for $j = 1, \dots, 2p-1$, be the unimodular complex numbers given by (9), $P_a(x, y)$ the homogeneous polynomial given by (10), and $\{M_{m,p,a}\}_{m=0}^{\infty}$ given by (11). If there exists a constant $C > 0$ such that*

$$(16) \qquad (\mathbf{leraycond2})\ \det M_{m,p,a} \geq \frac{C}{m^p},$$

for all natural numbers $m \geq 1$ then there exists $0 < r \leq R$ such that the Goursat problem (12) has a unique solution $u \in A(B_r)$ for every $f, g \in A(B_R)$.

In Section 6 below, we give some explicit examples of a_1, a_2, a_3 such that (16) holds for the corresponding unimodular numbers A_1, A_2, A_3.

In the case $p = 1$, the zero set of P_a is the union of the two distinct lines given by $y = 0$ and $x = ay$. By the rotational symmetry of Δ, we may also assume that $a \geq 0$. If we denote the acute angle between the two lines by $2\pi\alpha$ and by $\beta \in (0, 1/2]$ the number such that $A := A(a) = e^{2\pi i\beta}$, then as mentioned in the beginning of this section we have $\beta = 2\alpha$. As noted in Remark 2 above, we have $\det M_{m,p,a} = A^{m+2} - 1$. The condition $\det M_{m,p,a} = A^{m+2} - 1 \neq 0$ is clearly equivalent to α being irrational. Since

$$|A^{m+2} - 1| \approx \inf_{n \in \mathbb{Z}} |2\pi(m+2)\beta - 2\pi n| = 2\pi(m+2) \inf_{n \in \mathbb{Z}} \left| \beta - \frac{n}{m+2} \right|,$$

where by $E_k \approx F_k$ we mean $CF_k \le E_k \le DF_k$ for nonzero constants C, D, it is not difficult to see that Theorems 1 and 3, specialized to the case $p = 1$, can be formulated as follows.

Corollary 4. (homodel1) *Let Γ_1, Γ_2 be two distinct lines through the origin in $\mathbb{R}^2$, and denote by $\theta = 2\pi\alpha$ the acute angle between them. Suppose that α is irrational and satisfies the condition*

$$(17) \qquad (\textbf{leraycond3}) \ \tau := \liminf_{m\to\infty} \left(\inf_{n\in\mathbb{Z}} \left| \alpha - \frac{n}{m} \right| \right)^{1/m} > 0.$$

Then, the homogeneous Goursat problem

$$(18) \qquad (\textbf{goursat10}) \ \begin{cases} \Delta u = f \\ u = g \quad \text{on } \Gamma_1 \cup \Gamma_2 \end{cases}$$

has a unique solution $u \in A(B_{\tau R})$ for every $f, g \in A(B_R)$.

The condition (17) is also necessary for the conclusion of Corollary 4 to hold. This fact is proved in Section 7 below. As mentioned in the introduction, Corollary 4 is equivalent to the result of Leray in [5]. A more detailed explanation of this equivalence is given in Section 3 below.

We conclude this section by reformulating Theorem 3 in the case $p = 1$.

Corollary 5. (helmdel1) *Let Γ_1, Γ_2 be two distinct lines through the origin in $\mathbb{R}^2$, and denote by $\theta = 2\pi\alpha$ the acute angle between them. Suppose that α satisfies the Diophantine condition*

$$(19) \qquad (\textbf{diophantine}) \ \left| \alpha - \frac{n}{m} \right| \ge \frac{C}{m^2}, \quad \forall n, m \in \mathbb{Z}, m \ne 0$$

for some constant $C > 0$. Then, for any $c \in A(B_R)$, there exists $0 < r \le R$ such that the Goursat problem

$$(20) \qquad (\textbf{goursatp10}) \ \begin{cases} \Delta u + cu = f \\ u = g \quad \text{on } \Gamma_1 \cup \Gamma_2, \end{cases}$$

has a unique solution $u \in A(B_r)$ for every $f, g \in A(B_R)$.

3. Leray's Goursat problem

(lerayequiv)

Consider the homogeneous Goursat problem

$$\textbf{(goursat0)}\quad \begin{cases} \lambda \dfrac{\partial^2 u}{\partial x \partial y} + \Delta u = f \\ xy|(u-g), \end{cases} \tag{21}$$

where λ is a real constant. It follows from the general theory of Goursat (or mixed Cauchy) problems that (21) has a unique real-analytic solution near 0, for all f and g, if $|\lambda| > 2$ (see Gårding [3]; see also Theorem 9.4.2 in Hörmander [4]). The case where $\lambda \in [-2, 2]$ is much more subtle, and was analyzed by Leray in [5] (see also the work of Yoshino [10], [11] for extensions to complex parameters and higher dimensions). For $\lambda \in [-2, 2]$, let $\beta \in [-1/4, 1/4]$ denote the angle such that $\lambda = 2\sin(2\pi\beta)$. Leray showed that the unique solvability of (21) depends on Diophantine properties of β. For instance, there is a unique formal power series solution u for every f and g if and only if β is irrational. Leray also gave a necessary and sufficient Diophantine condition on irrational β quaranteeing that this formal solution u converges for all convergent f and g,

$$\textbf{(dioleray2)}\quad \liminf_{\mathbb{Z} \ni m \to \infty} \left(\inf_{n \in \mathbb{Z}} \left| \beta - \frac{n}{m} \right|^{1/m} \right) > 0. \tag{22}$$

Let us show that this result, for $\lambda \in (-2, 2)$, is equivalent to our Corollary 4 above. Consider the linear change of variables

$$\textbf{(trans)}\ x \to -\sqrt{1 - \frac{\lambda^2}{4}}\, x + \frac{\lambda}{2} y. \tag{23}$$

As the reader can easily verify, this change of variables leads to the following transformation for the principal symbol of the operator

$$\lambda \frac{\partial^2}{\partial x \partial y} + \Delta \to \Delta. \tag{24}$$

Hence, the Goursat problem (21) is transformed into the following

$$\textbf{(goursat01)}\quad \begin{cases} \Delta u = f \\ y(x - ay)|(u-g), \end{cases} \tag{25}$$

where

$$\textbf{(b)}\ a := \frac{\lambda/2}{\sqrt{1 - (\lambda/2)^2}}. \tag{26}$$

If we let $\theta = 2\pi\alpha$ denote the acute angle between the two lines $L_1 := \{y = 0\}$ and $L_2 := \{x = by\}$ and β the angle such that $\lambda := 2\sin(2\pi\beta)$, then we have

$$\alpha = \frac{1 - 2\beta}{4}.$$

Clearly, we have

$$\liminf_{\mathbb{Z}\ni m\to\infty}\left(\inf_{n\in\mathbb{Z}}\left|\beta-\frac{n}{m}\right|\right)^{1/m}=\liminf_{\mathbb{Z}\ni m\to\infty}\left(\inf_{n\in\mathbb{Z}}\left|\alpha-\frac{n}{m}\right|\right)^{1/m}.$$

This shows, as mentioned in the introduction, that Leray's result, with $\lambda\in(-2,2)$, is equivalent to our Corollary 4, with $0<a<\infty$.

4. An estimate for an associated Fischer operator and the proof of Theorem 1

(s:est)
Let $\mathbb{C}[x,y]$ denote the space of polynomials in x,y with complex coefficients. For each integer $m\geq 0$, we shall let $\mathcal{P}_m$ denote the subspace of homogeneous polynomials of degree m. We endow $\mathbb{C}[x,y]$ with the real Fischer inner product

$$\langle f,g\rangle:=\int_{\mathbb{R}^2}f(x,y)\overline{g(x,y)}e^{-(x^2+y^2)}dxdy, \tag{27}$$

and denote by $\|\cdot\|$ the corresponding norm (see [6]). We shall fix a positive integer p and distinct real numbers $a_1,\ldots,a_{2p-1}$ and consider the Fischer operator $F_a(q):=\Delta^p(P_aq)$, where P_a is given by (10). Observe that F_a is a linear operator sending $\mathcal{P}_m$ into $\mathcal{P}_m$. Our main result in this section is the following, in which the notation introduced above is used.

Theorem 6. (estimate) *Let p be a positive integer and $a_1,\ldots,a_{2p-1}$ real, distinct, non-zero numbers. Let $A_j:=A(a_j)$, for $j=1,\ldots,2p-1$, be the unimodular complex numbers given by* (9) *and $P_a(x,y)$ the homogeneous polynomial given by* (10)*. Then the Fischer operator $F_a\colon\mathcal{P}_m\to\mathcal{P}_m$, for $m\geq 0$, is a bijection if and only if* $\det M_{m,p,a}\neq 0$, *where $M_{m,p,a}$ is given by* (11)*. Moreover, if* $\det M_{m,p,a}\neq 0$, *then we have the estimate*

$$\|P_aq\|\leq\frac{C}{|\det M_{m,p,a}|}\|\Delta^p(P_aq)\|,\quad \forall q\in\mathcal{P}_m, \tag{28}$$

for some $C\geq 0$ (independent of m).

For the proof of Theorem 6, we shall need the following lemma. To state the lemma, we observe the well known fact that any homogeneous polynomial $f(x,y)$ of degree m can be expressed in the following way

$$\textbf{(expaninz)}\ f(x,y)=\sum_{k+l=m}f_{kl}z^k\bar{z}^l, \tag{29}$$

where $z=x+iy$ and $\bar{z}=x-iy$.

Lemma 7. (Fischer) *Let $f(x, y)$ be a homogeneous polynomial of degree m given by* (29)*. Then, we have*

$$\text{(normid)}\ \|f\|^2 = \pi m! \sum_{k+l=m} |f_{kl}|^2. \tag{30}$$

Proof. As in [6] (see also [1]), we observe that for any homogeneous polynomial $f(x, y)$ of degree m, we have

$$\text{(norm1)}\ \|f\|^2 = I_{2m+1} \int_{\mathbb{T}} |f(\eta)|^2 ds_\eta \tag{31}$$

where $\mathbb{T}$ denotes the unit circle in $\mathbb{R}^2$, ds arclength, and I_k the integral

$$I_k := \int_0^\infty e^{-r^2} r^k dr.$$

A simple substitution argument gives

$$\text{(int1)}\ I_{2m+1} = \int_0^\infty e^{-r^2} r^{2m+1} dr = \frac{1}{2} \int_0^\infty e^{-x} x^m dx = \frac{1}{2} m!. \tag{32}$$

Substituting (29) in (31), using the parametrization $z = e^{i\theta}$ for $\mathbb{T}$ and the identity (32), yields

$$\text{(norm2)}\ \|f\|^2 = \frac{1}{2} m! \sum_{k+l=m} \sum_{i+j=m} f_{kl} \overline{f_{ij}} \int_0^{2\pi} e^{i(k+j-l-i)\theta} d\theta, \tag{33}$$

from which (30) readily follows. □

Proof of Theorem 6. We fix $f \in \mathcal{P}_m$ and consider the equation

$$\text{(Faisf)}\ F_a(q) := \Delta^p(P_a q) = f, \tag{34}$$

for $q \in \mathcal{P}_m$. Note that $q \in \mathcal{P}_m$ solves (34) if and only if $u = P_a q$ solves the Goursat problem

$$\text{(goursat2)} \begin{cases} \Delta^p u = f \\ u(x, 0) = u(a_1 y, y) \ldots u(a_{2p-1} y, y) = 0. \end{cases} \tag{35}$$

We shall look for u of the form $u = v + w$, where $w(x, y) = (x^2 + y^2)^p s(x, y)$ for some $s \in \mathcal{H}_m$ such that

$$\text{(classic)}\ \Delta^p w(x, y) = \Delta^p((x^2 + y^2)^p s(x, y)) = f(x, y) \tag{36}$$

and $v \in \mathcal{H}_{m+2p}$ satifies

$$\text{(goursat3)} \begin{cases} \Delta^p v = 0 \\ v(x, 0) = -w(x, 0) \\ v(a_j y, y) = -w(a_j y, y), \quad j = 1, \ldots 2p - 1. \end{cases} \tag{37}$$

It is well known that (36) has a unique solution $w(x,y) = (x^2+y^2)^p s(x,y)$ (see e.g. [8] and references therein). Moreover, in view of the results in [1], we have

$$\|w\| \le C_1 \|f\| \tag{38}$$

for some constant $C_1 > 0$. Thus, to complete the proof of the theorem it suffices to show that (37) has a solution $v \in \mathcal{P}_{m+2p}$ for every $f \in \mathcal{P}_m$ if and only if $\det M_{m,p,a} \neq 0$, and that, in this case,

$$(\textbf{goal1})\ \|v\| \le \frac{C}{|\det M_{m,p,a}|} \|f\| \tag{39}$$

for some constant $C > 0$. To this end, we shall actually need the exact form of the solution to (36). Using $z = x+iy$ and $\bar z = x - iy$, we may write

$$(\textbf{id1})\ w(x,y) = W(z,\bar z) = z^p \bar z^p \sum_{k+l=m} s_{kl} z^k \bar z^l = \sum_{k+l=m} s_{kl} z^{k+p} \bar z^{l+p}. \tag{40}$$

We observe that $\Delta = 4\partial^2/\partial z \partial \bar z$. Thus, if we write $f(x,y) = \sum_{k+l=m} f_{kl} z^k \bar z^l$, then (36) is equivalent to

$$(\textbf{id2})\ s_{kl} = \frac{f_{kl}}{4^p (k+1)\dots(k+p)(l+1)\dots(l+p)}, \quad \forall\ k+l=m. \tag{41}$$

Now, we note that every function $v(x,y)$ that satisfies $\Delta^p v = 0$ is of the form

$$(\textbf{vform})\ v(x,y) = \sum_{t=0}^{p-1} \left(\bar z^t \phi_t(z) + z^t \psi_t(\bar z) \right), \tag{42}$$

where $\phi_t(z)$ and $\psi_t(\bar z)$ are holomorphic functions of z and $\bar z$, respectively. The function v is a homogeneous polynomial of degree $m+2p$ if and only if $\phi_t(z) = b_{p-1-t} z^{m+2p-t}$ and $\psi_t(\bar z) = c_t \bar z^{m+2p-t}$, for constants b_{p-1-t} and c_t and $t = 0, \dots, p-1$. Using this notation, equation (37) is equivalent to finding monomials

$$(\textbf{id3})\ \phi_t(z) = b_{p-1-t} z^{m+2p-t}, \quad \psi_t(\bar z) = c_t \bar z^{m+2p-t}, \tag{43}$$

for $t = 0, 1, \dots, p-1$, such that

$$(\textbf{get1})\ \sum_{t=0}^{p-1} \left(x^t \phi_t(x) + x^t \psi_t(x) \right) = -W(x,x) \tag{44}$$

and

$$(\textbf{get2})\ \sum_{t=0}^{p-1} \left(((a_j - i)y)^t \phi_t((a_j+i)y) + ((a_j+i)y)^t \psi_t((a_j-i)y) \right) = \\ -W((a_j+i)y, (a_j-i)y), \quad j = 1, \dots, 2p-1. \tag{45}$$

In (45), we use the fact that ϕ_t is homogeneous of degree $m+2p-t$ and we divide the equation by $(a_j-i)^{m+2p}$. With $A_j := A(a_j)$ and $A(a)$ given by (9), the equation becomes

$$\textbf{(get3)}\ \sum_{t=0}^{p-1}\left({A_j}^{m+2p-t}\phi_t(y)+A_j^t\psi_t(y)\right)=-W(A_jy,y),\quad j=1,\dots,2p-1. \tag{46}$$

Substituting (41) and (43) in (44) and (46), we obtain the following system of linear equations for the coefficients $b_0,\dots,b_{p-1},c_0,\dots,c_{p-1}$

(47)

$$\textbf{(system1)}\quad \begin{aligned} \sum_{t=0}^{p-1}(b_{p-1-t}+c_t) &= -\sum_{k+l=m}\frac{f_{kl}}{4^p(k+1)\dots(k+p)(l+1)\dots(l+p)}\\ \sum_{t=0}^{p-1}\left(A_1^{m+2p-t}b_{p-1-t}+A_1^tc_j\right) &= -\sum_{k+l=m}\frac{f_{kl}A_1^{m+2p}}{4^p(k+1)\dots(k+p)(l+1)\dots(l+p)}\\ &\vdots\\ \sum_{t=0}^{p-1}\left(A_{2p-1}^{m+2p-t}b_{p-1-t}+A_{2p-1}^tc_j\right) &= -\sum_{k+l=m}\frac{f_{kl}A_{2p-1}^{m+2p}}{4^p(k+1)\dots(k+p)(l+1)\dots(l+p)}\end{aligned}$$

If we write d for the column vector of coefficients $d=(c_0,\dots,c_{p-1},b_0,\dots,b_{p-1})^t$ and e for the column vector whose $(j+1)$th component, $j=0,\dots,2p-1$, is given by

$$-\sum_{k+l=m}\frac{f_{kl}A_j^{m+2p}}{4^p(k+1)\dots(k+p)(l+1)\dots(l+p)},$$

where we let $A_0 := 1$, then (47) can be written

$$\textbf{(matrixeq)}\ M_{m,p,a}d=e, \tag{48}$$

where $M_{m,p,a}$ is given by (11). We conclude, as claimed above, that (37) has a unique solution $v\in\mathcal{P}_{m+2p}$ for every $f\in\mathcal{P}_m$ if and only if $\det M_{m,p,a}\neq 0$.

Let us now suppose that $\det M_{m,p,a}\neq 0$ and write d_i for the ith component of d, $i=1,\dots,2p$. Using Cramer's rule and the fact that $|A_j|=1$, we conclude from (48) that

$$\textbf{(coeffest)}\ |d_i|\le C_1|\det M_{m,p,a}|^{-1}\sum_{k+l=m}\frac{|f_{kl}|}{(k+1)\dots(k+p)(l+1)\dots(l+p)}. \tag{49}$$

By the Cauchy-Schwarz inequality, we conclude that

$$\textbf{(coeffest2)}\ |d_i|\le C_1|\det M_{m,p,a}|^{-1}\left(\sum_{k+l=m}|f_{kl}|^2\right)^{1/2}S_m, \tag{50}$$

where S_m denotes the sum

$$\text{(51)} \qquad (\mathbf{Sm})\ S_m := \left(\sum_{k+l=m} \frac{1}{(k+1)^2 \dots (k+p)^2 (l+1)^2 \dots (l+p)^2} \right)^{1/2}.$$

By setting $l = m - k$, we obtain

$$\begin{aligned} S_m^2 &= \sum_{k=0}^{m} \left(\prod_{j=1}^{p} (k+j)^2 (m-k+j)^2 \right)^{-1} \\ \text{(52)} \qquad (\mathbf{Smest}) \qquad &\leq 2 \sum_{k=0}^{[m/2]+1} \left(\prod_{j=1}^{p} (k+j)^2 (m-k+j)^2 \right)^{-1} \\ &= 2m^{-2p} \sum_{k=0}^{[m/2]+1} \left(\prod_{j=1}^{p} (k+j)^2 \left((1+(j-k)/m\right)^2 \right)^{-1} \end{aligned}$$

Now, note that, for $j = 1, \dots, p$ and $k = 0, \dots, [m/2] + 1$, we have $(j-k)/m \geq -3/4$ when $m \geq 2$ and, hence, $(1 + (j-k)/m)^{-2} \leq 16$. Consequently, we have

$$\text{(53)} \qquad (\mathbf{Smest2})\ S_m^2 \leq \frac{32}{m^{2p}} \sum_{k=0}^{[m/2]+1} \left(\prod_{j=1}^{p} (k+j)^2 \right)^{-1} \leq \frac{32}{m^{2p}} \sum_{k=0}^{\infty} \frac{1}{(k+1)^{2p}} \leq \frac{C_2}{m^{2p}},$$

for some $C_2 > 0$ independent of m. Thus, by Lemma 7, we obtain from (50) and (53) the following estimates for the functions $\tilde{\phi}_t(z, \bar{z}) := \bar{z}^t \phi_t(z)$, where ϕ_t is given by (43),

$$\begin{aligned} \|\tilde{\phi}_t\| &= \sqrt{(m+2p)!}\, |b_{p-1-t}| \\ \text{(54)} \qquad (\mathbf{estphipsi}) \qquad &\leq C_1 C_2 |\det M_{m,p,a}|^{-1} \sqrt{(m+1)\dots(m+2p)} \|f\| m^{-p} \\ &\leq C_3 |\det M_{m,p,a}|^{-1} \|f\|. \end{aligned}$$

We obtain a similar estimate for $\tilde{\psi}_t(z, \bar{z}) := z^t \psi_t(\bar{z})$. These estimates yields (39) since v is given by (42). This completes the proof of Theorem 6. ☐

The arguments in the proof above also yield a proof of Theorem 1. We conclude this section by giving this proof.

Proof of Theorem 1. It is well known that to prove Theorem 1 it suffices to show that the equation

$$\text{(55)} \qquad (\mathbf{PDE})\ \Delta^p(Pq) = f$$

has a unique solution $q \in A(B_{\tau R})$ for every $f \in A(B_R)$ (see e.g. [1]). As in the proof of Theorem 6, we shall look for the solution $u := P_a q$ in the form $u = v + w$, where $w(x, y) = (x^2+y^2)^p s(x, y)$ satisfies (36) and v solves (37). It is well known that $w \in A(B_R)$ (see [8]; see also [1]). Thus, to complete the proof, it suffices to show that $v \in A(B_{\tau R})$. We expand v as a series $v = \sum_m v_m$, where the v_m are the homogeneous Taylor polynomials of degree m of v. Similarly, we expand $w = \sum_m w_m$ and $f = \sum_m f_m$. By homogeneity, we observe that the homogeneous polynomials v_m, w_m, f_m satisfy (37) (with $v = v_m$, $w = w_m$, and $f = f_m$). The fact that $v \in A(B_{\tau R})$ now follows easily from the definition (13) of τ, the form (42) of v, and the estimate (54). The details are left to the reader. $\square$

5. Proof of Theorem 3

Proof of Theorem 3. We fix $a = (a_1, \dots, a_{2p-1})$ as in the theorem. For brevity, we denote P_a simply by P. To prove Theorem 3, it suffices to show that there is $0 < r \leq R$ such that the equation

$$\text{(56)} \qquad (\textbf{PDE1})\ (\Delta^p + c)(Pq) = f$$

has a unique solution $q \in A(B_r)$ for every $f \in A(B_R)$. We shall look for the solution $u = Pq$ as a series $u = \sum_m u_m = \sum_m Pq_{m-2p}$, where the u_m are the homogeneous Taylor polynomials of degree m of u. To this end, we expand, similarly, both f and c as Taylor series $f = \sum_m f_m$ and $c = \sum_m c_m$. The equation (55) then implies

$$\text{(57)} \qquad (\textbf{basic0})\ \Delta^p(Pq_j) = f_j, \quad j = 0, 1, \dots, 2p-1,$$

and, for each $m \geq 2p$,

$$\text{(58)} \qquad (\textbf{basic1})\ \Delta^p(Pq_m) = f_m - \sum_{k=0}^{m-2p} c_{m-k-2p} Pq_k.$$

Since the Fischer operator $F = F_a$, given by $F(q) = \Delta^p(Pq)$, is bijective : $\mathcal{P}_m \to \mathcal{P}_m$ for every m (by Theorem 6), we can solve, uniquely, (57) and (58) inductively for q_m. This gives us a unique formal power series solution $u = \sum_m u_m$ with $u_m = Pq_{m-2p}$. It remains to prove that there is $r > 0$ such that this series converges to a function in $A(B_r)$. For this, we observe that Theorem 6 and the assumption (16) implies the following estimate

$$\text{(59)} \qquad (\textbf{basic2})\ \|u_{m+2p}\| \leq Cm^p \|\Delta(Pq_m)\| \leq Cm^p \left(\|f_m\| + \sum_{k=0}^{m-2p} \|c_{m-k-2p} u_{k+2p}\| \right)$$

To prove that $u \in A(B_r)$, we must show (see Proposition 16 in [1]) that for every $0 < \rho < r$ there is a constant $B > 0$ such that

$$\text{(60)} \qquad (\textbf{ind})\ \|u_k\| \leq B\rho^{-k}\sqrt{k!}$$

for every $k \geq 0$. Let us pick $\rho < \sigma < R$. In view of Proposition 16 in [1], we may assume that there are constants D and E such that

$$\text{(61)} \qquad (\textbf{assump})\ \max_{\theta \in \mathbb{T}} |c_k(\theta)| \leq D\sigma^{-k}, \quad \|f_k\| \leq E\rho^{-k}\sqrt{k!},$$

for all $k \geq 0$. We shall prove (60) by induction. Thus, assume that (60) holds for all $k \leq m + 2p - 1$. We shall prove that (60) holds also for $k = m + 2p$, provided that m

is large enough. By using (61), the induction hypothesis, and Proposition 8 in [1] (see also Proposition 7 in that paper), we conclude from (59) the following estimate, for some constant $F > 0$,

(62)
$$\begin{aligned}\|u_{m+2p}\| \le & Cm^p \left(E\rho^{-m}\sqrt{m!} + \sum_{k=0}^{m-2p} F\sigma^{-(m-k-2p)}[(k+2p+1)\dots(m-1)m]^{1/2}\|u_{k+2p}\| \right) \\ \le & Cm^{\mu-1}\left(E\rho^{-m}\sqrt{m!} + \sum_{k=0}^{m-2p} BF\sigma^{-(m-k-2p)}\rho^{-(k+2p)}\sqrt{m!}\right) \\ = & B\rho^{-(m+2p)}\sqrt{(m+2p)!}\,T_m,\end{aligned}$$

where

$$\begin{aligned}T_m := & Cm^p \frac{\rho^{2p}}{\sqrt{(m+1)(m+2)}}\left(E/B + F\sum_{k=0}^{m-2p}\left(\frac{\rho}{\sigma}\right)^{m-k-2p}\right) \\ \le & Cm^p\frac{\rho^{2p}}{\sqrt{(m+1)\dots(m+2p)}}\left(E/B + F\frac{1}{1-\rho/\sigma}\right).\end{aligned} \tag{63}$$

Since $\rho < r$, we can make $T_m \le 1$ for all m by requiring $0 < r \le R$ small enough (and keeping $\sigma < R$ fixed). This proves Theorem 3. □

6. Examples of solvable Goursat problems for $\Delta^2 + c$

(ex)

In this section, we shall consider the following one-parameter family of Goursat problems

$$\textbf{(goursat22)}\quad \begin{cases}\Delta^2 u + cu = f \\ P_t|(u-g),\end{cases} \tag{64}$$

where $P_t(x, y)$, for $t > 0$, denotes the divisor

$$\textbf{(Pt)}\ P_t(x,y) := xy(x-ty)(x-y/t). \tag{65}$$

Recall that $A = A(t)$ denotes the unimodular number given by (9) (with $a = t$). Let us denote by $\beta = \beta(t)$ the number $\beta \in (0, 2\pi)$ such that $A = e^{2\pi i\beta}$. We shall prove the following result.

Theorem 8. (helmdel2) *Let $t > 0$ and $\beta := \beta(t)$ as defined above. Suppose that β satisfies the Diophantine condition*

$$\textbf{(diophantine2)}\ \left|\beta - \frac{n}{m}\right| \ge \frac{C}{m^2},\quad \forall n, m \in \mathbb{Z}, m \ne 0, \tag{66}$$

for some constant $C > 0$. Then, for any $c \in A(B_R)$, there exists $0 < r \le R$ such that the Goursat problem (64) has a unique solution $u \in A(B_r)$ for every $f, g \in A(B_R)$.

Theorem 8 is a direct consequence of Theorem 3, with $p = 2$, and the following proposition.

Proposition 9. (matrixcomp) *Let $t > 0$, $a = (a_1, a_2, a_3) := (0, t, 1/t)$, and let $M_{m,p,a}$ be the matrix defined by (11) with $p = 2$. If $\beta = \beta(t)$ satisfies*

$$\text{(67)} \qquad (\textbf{diophantine3}) \ \left|\beta - \frac{n}{m}\right| \ge \frac{C}{m^\mu}, \quad \forall n, m \in \mathbb{Z}, m \ne 0,$$

for some constant $C > 0$, then

$$\text{(68)} \qquad (\textbf{detAk}) \ |\det M_{m,p,a}| \ge \frac{D}{m^{2\mu-2}},$$

for some $D > 0$.

Proof. It is easy to check that the unimodular numbers (A_1, A_2, A_3) that correspond to the vector a is $(-1, A, B)$, where $AB = -1$ and, in view of the discussion preceding Corollary 4,

$$\text{(69)} \qquad (\textbf{Am}-\textbf{1}) \ |A^m - 1| \ge \frac{C'}{m^{\mu-1}}.$$

(Of course, A is given by (9), but only the above two facts will be needed in the proof.) To prove the proposition, it suffices, in view of Remark 2, to show that $|N_m| \ge C'/m^{2\mu-2}$, where

$$\text{(70)} \qquad (\textbf{newmatrix}) \ N_m := M_{m-4,2,a} = \det \begin{pmatrix} -2 & (-1)^{m-1} - 1 & (-1)^m - 1 \\ A - 1 & A^{m-1} - 1 & A^m - 1 \\ B - 1 & B^{m-1} - 1 & B^m - 1 \end{pmatrix}.$$

We obtain, since $AB = -1$,

$$A^m N_m = \det \begin{pmatrix} -2 & (-1)^{m-1} - 1 & (-1)^m - 1 \\ A - 1 & A^{m-1} - 1 & A^m - 1 \\ -A^{m-1} - A^m & A(-1)^{m-1} - A^m & (-1)^m - A^m \end{pmatrix}.$$

If m is even, then

$$A^m N_m = \det \begin{pmatrix} -2 & -2 & 0 \\ A - 1 & A^{m-1} - 1 & A^m - 1 \\ -A^{m-1} - A^m & -A - A^m & 1 - A^m \end{pmatrix}.$$

A straightforward calculation shows that

$$\text{(71)} \qquad (\textbf{even}) \ A^m N_M = 4A(A^m - 1)(A^{m-2} - 1).$$

If m is odd, then

$$A^m N_m = \det \begin{pmatrix} -2 & 0 & -2 \\ A-1 & A^{m-1}-1 & A^m-1 \\ -A^{m-1}-A^m & A-A^m & -1-A^m \end{pmatrix}.$$

This time we get

$$\textbf{(odd)}\ A^m N_M = -2(A^{m-1}-1)^2(A^2+1). \tag{72}$$

The conclusion $|N_m| \geq C'/m^{2\mu-2}$ follows easily from (71) and (72). This completes the proof of the proposition. □

7. Divergence of formal solutions when $p = 1$ and $\tau = 0$.

(nec)

We now show that, for $p = 1$ and irrational angles α between the two lines Γ_1 and Γ_2, the formal solution u to (18), with f convergent and $g \equiv 0$, need not converge when τ, given by (13), is zero. Using the notation and setup in the proof of Theorem 6, let us choose f such that for each m we have, for $k + l = m$,

$$f_{kl} = \begin{cases} R^{-m}, & k = 0 \\ 0, & k > 0. \end{cases} \tag{73}$$

Note that $f \in A(B_R)$. Let us consider the Goursat problem (18) with $g = 0$. By following the argument in the proof of Theorem 6 above, we conclude that the formal solution is of the form $u = v+w$, where w is the formal solution to (36) and $v(x, y)$ is the formal solution to (37). Hence, v is of the form $v(x, y) = \phi(z) + \psi(\bar{z})$. It is well known that the solution w to (36) converges to a function in $A(B_R)$ (see [8]; see also [1]). Thus, the solution u to the Goursat problem converges if and only if the two power series $\phi(z) = \sum_m b_m z^m$ and $\psi(\bar{z}) = \sum_m c_m \bar{z}^m$ converge. With $p = 1$, it is easy to solve the system of equations (47) for b_m and c_m explicitly and we obtain

$$b_m = \frac{1}{(1 - A^m)} \frac{A - 1}{2R^{m-2}(m-1)}, \tag{74}$$

(A similar identity holds, of course, for c_m.) The radius of convergence of the series $\phi(z) = \sum_m b_m z^m$ is

$$R \liminf_{m\to\infty} |1 - A^m|^{1/m} = 0, \tag{75}$$

proving the assertion above that the solution u does not converge. We conclude this paper by giving an example of a number β in $A = e^{2\pi i\beta}$ such that $\tau = 0$.

Example 10. Let us define

$$\beta := \sum_{k=1}^{\infty} 10^{-p_k}, \tag{76}$$

where p_k is defined recursively by $p_1 = 1$ and $p_{k+1} = p_k + k\,10^{p_k}$. Note that, for every N, the rational number

$$r_N := \sum_{k=1}^{N} 10^{-p_k} = \frac{q_N}{10^{p_N}}$$

satisfies

$$|\beta - r_N| \leq \frac{2}{10^{p_{N+1}}}.$$

Consider the subsequence $m_N := 10^{p_N}$ and note that

$$|A^{m_N} - 1| \leq C \inf_{p,q\in\mathbb{Z}_+} q \left|\beta - \frac{p}{q}\right| \leq 2\frac{10^{p_N}}{10^{p_{N+1}}} = \frac{2}{10^{p_{N+1}-p_N}}$$

Thus, we have

$$|A^{m_N} - 1|^{1/m_N} \leq \frac{C}{10^{(p_{N+1}-p_N)/10^{p_N}}} = \frac{C}{10^N} \to 0,$$

which shows that $\tau = \liminf_{k\to\infty} |A^k - 1|^{1/k} = 0$.

References

[1] P. Ebenfelt, H. Render, *On the mixed Cauchy problem with data on singular conics,* Preprint, (2006).

[2] P. Ebenfelt, H.S. Shapiro, *The mixed Cauchy problem for holomorphic partial differential equations,* J. D'Analyse Math. 65 (1996) 237–295.

[3] L. Gårding, *Une variante de la mthode de majoration de Cauchy. (French)*, Acta Math. 114, (1965) 143–158.

[4] L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis.* Second edition. Grundlehren der Mathematischen Wissenschaften, **256**. Springer-Verlag, Berlin, 1990.

[5] J. Leray, *Caractre non fredholmien du problème de Goursat. (French),* J. Math. Pures Appl. 53 (1974), 133–136.

[6] H. Render, *Real Bargmann spaces, Fischer decompositions and Sets of Uniqueness for Polyharmonic Functions,* submitted.

[7] K. F. Roth, *Rational approximations to algebraic numbers and Corrigendum*, Mathematika, 2, (1955), 1–20 and 168.

[8] H.S. Shapiro, *An algebraic theorem of E. Fischer and the Holomorphic Goursat Problem,* Bull. London Math. Soc. 21 (1989), 513–537.

[9] M. Waldschmidt, *Diophantine approximation on linear algebraic groups. Transcendence properties of the exponential function in several variables.* Grundlehren der Mathematischen Wissenschaften, **326**. Springer-Verlag, Berlin, 2000.

[10] M. Yoshino, *Spectral property of Goursat problems.* Tokyo J. Math. 4 (1981), 55–71.

[11] M. Yoshino, *On the solvability of Goursat problems and a function of number theory.* Duke Math. J. 48 (1981), 685–696.

P. Ebenfelt: Department of Mathematics, University of California, San Diego, La Jolla, CA 92093–0112, USA.
E-mail address: pebenfelt@math.ucsd.edu

H. Render: Departamento de Matemáticas y Computación, Universidad de La Rioja, Edificio Vives, Luis de Ulloa s/n., 26004 Logroño, España.
E-mail address: render@gmx.de