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Ostrowski-type theorems for harmonic functions

Myrto Manolaki

Abstract

Ostrowski showed that there are intimate connections between the gap structure of a Taylor series and the behaviour of its partial sums outside the disk of convergence. This paper investigates the corresponding problem for the homogeneous polynomial expansion of a harmonic function. The results for harmonic functions display new features in the case of higher dimensions.

1 Introduction

Let $B(x_0, r)$ denote the open ball with centre $x_0$ and radius $r$ in Euclidean space $\mathbb{R}^N$ $(N \geq 2)$. If $h$ is a harmonic function on $B(x_0, r)$, then its multiple Taylor series does not necessarily converge on the whole of $B(x_0, r)$. However, if we group the terms of the series according to their degree, we obtain an expansion of $h$ which does converge on all of $B(x_0, r)$. We call this grouped Taylor series the homogeneous polynomial expansion of $h$ about $x_0$ and denote by $S_m(h, x_0)$ the $m$th partial sum. Thus

$$S_m(h, x_0)(x) = \sum_{j=0}^{m} H_j(x - x_0),$$

where $H_j$ is a homogeneous harmonic polynomial of degree $j$. (See Chapter 2 of [1].) The radius of the largest ball centred at $x_0$ inside which the above series converges locally uniformly is called the radius of convergence of the expansion.

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In the case of holomorphic functions, celebrated work of Ostrowski (see [8], for example) shows a deep connection between the gap structure of the Taylor series expansion and the phenomenon of overconvergence of a subsequence of partial sums outside the disk of convergence. Ostrowski’s insights have found new applications in recent years to the study of universal Taylor series (see [3], [5], [6], [7]). There is a corresponding notion of universal polynomial expansions for harmonic functions, but the theory is less well developed. As Tamptse has noted in [11], one of the barriers to progress is the absence of an Ostrowski-type theory for such expansions.

The purpose of this paper is to develop such a theory. It turns out that, in the case of harmonic functions, some of the results have a significantly different form, and this difference is essential in higher dimensions.

In order to state our results we need the following definition:

**Definition:** Let \( \sum_{j=0}^{\infty} H_j(x - x_0) \) be the homogeneous polynomial expansion of a harmonic function on an open neighbourhood of \( x_0 \) and let \((p_n)\) and \((q_n)\) be two sequences of natural numbers such that

\[
1 \leq p_1 < q_1 \leq p_2 < q_2 \leq ... \]

We say that the expansion possesses **Hadamard-Ostrowski gaps** \((p_n, q_n)\) if

1. There exists \( \theta > 0 \) such that \( q_n \geq (1 + \theta)p_n \) for all \( n \in \mathbb{N} \),
2. \( H_j \equiv 0 \) for \( j \in \bigcup_{n=1}^{\infty} \{p_n + 1, ..., q_n\} \).

If we replace (i) with the stronger condition

\[
(i') \quad \frac{q_n}{p_n} \to \infty \quad \text{as} \quad n \to \infty, \]

then we say that the expansion possesses **Ostrowski gaps** \((p_n, q_n)\).

Throughout this paper \( \mathcal{H}(\Omega) \) denotes the set of all harmonic functions on an open set \( \Omega \subset \mathbb{R}^N \). For simplicity we write \( S_m \) instead of \( S_m(h, 0) \).

Our first result is an analogue of Theorem I of Ostrowski [8].

**Theorem 1** Let \( h \in \mathcal{H}(B(0, 1)) \) and suppose that \( h \) has a harmonic extension to a neighbourhood of some point \( y \in \partial B(0, 1) \). If the homogeneous polynomial expansion of \( h \) about 0 has radius of convergence 1 and possesses Hadamard-Ostrowski gaps \((p_n, q_n)\), then the subsequence \((S_{p_n})\) of partial sums of \( h \) converges uniformly on a neighbourhood of \( y \).

The conclusion of Theorem 1 remains valid if to our initial function we
add a harmonic function on $B(0, 1 + \varepsilon)$ for some $\varepsilon > 0$. The following example, which was suggested by Stephen Gardiner, shows that the converse is not true for harmonic functions in higher dimensions. For the purposes of this example $B'(0, r)$ denotes the ball in $\mathbb{R}^{N-1}$ centred at 0 with radius $r$.

**Example** Let $N \geq 3$. Also, let $K(\cdot, y)$ be the Poisson kernel of $B'(0, 1)$ with pole at some fixed point $y \in \partial B'(0, 1)$. We consider the function $h : B'(0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$h(x_1, \ldots, x_{N-1}, x_N) = K((x_1, \ldots, x_{N-1}), y).$$

Then the homogeneous expansion of $h$ has radius of convergence 1 and its partial sums $(S_n)$ converge locally uniformly on $B'(0, 1)$ with pole at some fixed point $y \in \partial B'(0, 1)$. However, $h$ cannot be written in the form $h = g + v$ on $B(0, 1)$, where

(i) $v \in \mathcal{H}(B(0, 1 + \varepsilon))$ for some $\varepsilon > 0$,

(ii) $g \in \mathcal{H}(B(0, 1))$ and the homogeneous expansion of $g$ possesses Hadamard-Ostrowski gaps $(p_n, q_n)$.

Thus, in contrast to Theorem II of Ostrowski in [8], a harmonic function on $B(0, 1)$ which has a subsequence of partial sums converging uniformly on a neighbourhood of some $y \in \partial B(0, 1)$, need not be the sum of a harmonic function on a larger ball and one with Hadamard-Ostrowski gaps. However, as the following theorem shows, there is still a significant relationship between overconvergence and occurrence of Hadamard-Ostrowski gaps. We use the following notation:

If $\delta > 0$, $y \in \partial B(0, 1)$ and $a, b \in \mathbb{R}$ with $a < b$, then we write

$$P(y, \delta, a, b) = \{tu : u \in \partial B(0, 1) \cap \overline{B(y, \delta)}, t \in [a, b]\}.$$ 

**Theorem 2** Let $h \in \mathcal{H}(B(0, 1))$ with homogeneous polynomial expansion about 0 which has radius of convergence 1, and assume that there exists a subsequence $(S_{\lambda_n})$ of partial sums of $h$ which is uniformly bounded on some ball $B(w, \rho)$, disjoint from $B(0, 1)$. Then $h$ can be written in the form $h = g + v$, where $g, v \in \mathcal{H}(B(0, 1))$ and

(i) the homogeneous polynomial expansion of $g$ possesses Hadamard-Ostrowski gaps,

(ii) the homogeneous polynomial expansion of $v$ converges Hadamard-Ostrowski gaps locally uniformly on $B(0, 1) \cup P\left(\frac{w}{\|w\|}, \Delta, -r, r\right)$ for some $\Delta > 0$, $r > 1$.

**Corollary 1** Let $h$ be harmonic on the unit disk $D(0, 1)$ in the complex plane $\mathbb{C}$ and suppose that it has a homogeneous polynomial expansion with radius of convergence 1. If there exist $\rho > 0$ and $z_0 \in \partial D(0, 1)$ such that a subsequence $(S_{\lambda_n})$ of partial sums of $h$ converges uniformly
on the disk $D(z_0, \rho)$, then there exist $g \in \mathcal{H}(D(0, 1))$ with homogeneous polynomial expansion which possesses Hadamard-Ostrowski gaps and $v \in \mathcal{H}(D(0, 1 + \varepsilon))$, such that $h = g + v$ on $D(0, 1)$.

Finally, we prove an analogue of the third main theorem of Ostrowski concerning overconvergence (Theorem III of [8]) for expansions which fulfil a stronger gap condition.

**Theorem 3** Let $h \in \mathcal{H}(B(0, 1))$ and suppose that $h$ has a harmonic extension to a domain $G$, strictly containing $B(0, 1)$. If the homogeneous polynomial expansion of $h$ about $0$ has radius of convergence $1$ and possesses Ostrowski gaps $(p_n, q_n)$, then the subsequence $(S_{p_n})$ of partial sums of $h$ converges locally uniformly on $G$.

**Remark** For such a function $h$, there exists a largest domain $D$, containing $B(0, 1)$, to which $h$ can be extended harmonically. The maximum principle implies that $(\mathbb{R}^N \cup \{\infty\}) \setminus D$ is connected.

We will prove Theorems 1-3 and give details of the example in Section 3 following some preliminary material below.

## 2 Preliminaries

For the proofs of our results we will combine methods from the holomorphic case with tools from potential theory and some new arguments. We first prove a formula for the radius of convergence of a homogeneous polynomial expansion. If $y \in \partial B(0, \rho)$ and $j \in \mathbb{N}$, then $J_{y,j}$ denotes the $y$-axial homogeneous harmonic polynomial of degree $j$ (for details we refer to Theorem 2.3.2 of [1]). Finally, $\lambda$ denotes Lebesgue measure on $\mathbb{R}^N$ and $\sigma$ denotes surface area measure on a sphere.

**Lemma 1** Let $h$ be harmonic on an open set containing $B(0, \rho)$ and let $\sum_{j=0}^{\infty} H_j(x)$ be the homogeneous polynomial expansion of $h$ about 0.

(i) For each $j \in \mathbb{N}$ and $x \in B(0, \rho)$ we have

$$H_j(x) = \frac{1}{\sigma(\partial B(0, \rho))} \int_{\partial B(0, \rho)} J_{x,j} \left( \frac{y}{\rho} \right) h(y) d\sigma(y).$$

(ii) There exists a constant $C > 0$, depending only on the dimension $N$, such that for each $j \in \mathbb{N}$

$$L_j \leq \frac{C(j+1)^{N-2}}{\rho^2} \max_{\|y\|=\rho} |h(y)|.$$
where \( L_j = \max_{\|y\|=1} |H_j(y)| \).

(iii) The radius of convergence \( r \) of the expansion is given by

\[
r = R := \left( \limsup_{j \to \infty} L_j^{1/j} \right)^{-1},
\]

where we interpret \( R \) as \(+\infty\) when \( \limsup_{j \to \infty} L_j^{1/j} = 0 \).

Proof. (i) The formula can be derived by a suitable change of variable in formula (2.4.6) in [1].

(ii) By the \( j \)-homogeneity of \( H_j \) and the maximum principle,

\[
L_j = \max_{\|x\|=\rho} \left\{ \frac{1}{\|x\|^j} |H_j(x)| \right\} = \frac{1}{\rho^j} \sup_{\|x\|<\rho} |H_j(x)|.
\]

By Theorem 2.4.3 of [1], there is a constant \( C \), depending only on the dimension \( N \), such that

\[
\left| J_{y,\rho}^j \left( \frac{x}{\rho} \right) \right| \leq C(j+1)^{N-2} \quad (x \in B(0,\rho), \ y \in \partial B(0,\rho), \ j \in \mathbb{N}).
\]

Combining the above with part (i), we obtain the desired inequality.

(iii) We first observe that the radius of convergence coincides with the radius of the largest ball centred at 0 inside which \( h \) has a harmonic extension. By the \( j \)-homogeneity of \( H_j \) we see that \( |H_j(x)| \leq L_j \|x\|^j \) for all \( x \in \mathbb{R}^N \). Since the radius of convergence of the series \( \sum_{j=0}^\infty L_j \|x\|^j \) is \( R \), the series \( \sum_{j=0}^\infty H_j(x) \) converges locally uniformly on \( B(0,R) \). Hence \( r \geq R \). (If \( R = +\infty \) then \( r = +\infty \) as well.) Let \( \rho \in (0,r) \). Then \( h \) has a harmonic extension to an open set containing \( \overline{B(0,\rho)} \). If \( \max_{\|y\|=\rho} |h(y)| = 0 \), then \( h \) is identically 0 and \( L_j = 0 \), so \( r = R = +\infty \). If \( \max_{\|y\|=\rho} |h(y)| \neq 0 \), then

\[
\lim_{j \to \infty} \left( \frac{C(j+1)^{N-2}}{\rho^j} \max_{\|y\|=\rho} |h(y)| \right)^{1/j} = \frac{1}{\rho},
\]

and so \( \limsup_{j \to \infty} L_j^{1/j} \leq \frac{1}{\rho} \) by part (ii). Now, by letting \( \rho \to r^- \), we get \( \limsup_{j \to \infty} L_j^{1/j} \leq \frac{1}{r} \) and so \( r \leq R \), which gives the desired formula. \( \square \)

As we will see, the gap structure of the homogeneous polynomial expansion of a harmonic function \( h \) forces certain subsequences of
its partial sums to converge (to $h$) at a faster rate inside the ball of convergence. The following theorems, due to Korevaar and Meyers [4], allow us to transfer this good property to certain sets lying outside the ball of convergence.

If $u \in L^2(\partial B(w, r))$ we write

$$\|u\|_{w,r,2} = \sqrt{\frac{1}{\sigma(\partial B(w, r))} \int_{\partial B(w, r)} u^2 d\sigma}.$$ 

Also, if $u$ is bounded on a set $K$ we write

$$\|u\|_K = \sup\{|u(x)| : x \in K\}.$$

**Theorem A** Let $\Omega$ be a domain in $\mathbb{R}^N$, let $\Omega_0 \subset \Omega$ be a subdomain and $E \subset \Omega$ a compact subset. Then there is a constant $a = a(E, \Omega_0, \Omega) \in (0, 1]$ such that, for all harmonic functions $u$ on $\Omega$,

$$\|u\|_E \leq \|u\|_{\Omega_0}^a \|u\|_{\Omega}^{1-a}.$$ 

**Theorem B** Let $0 < \rho < t < R$ and $w \in \mathbb{R}^N$. Then, for all bounded harmonic functions $u$ on $B(w, R)$,

$$\|u\|_{w,t,2} \leq \|u\|_{w,\rho,2}^{\beta} \|u\|_{w,R,2}^{1-\beta},$$

where $\beta$ is the Hadamard exponent:

$$\beta = \frac{\log(t/R)}{\log(\rho/R)}.$$ 

Also we will make use of the following lemma which is a consequence of the subharmonic mean value inequality.

**Lemma 2** Let $(u_n)$ be a sequence of non-negative subharmonic functions on a ball $B(z,t)$. If

$$\int_{B(z,t)} u_n d\lambda \to 0 \text{ as } n \to \infty,$$

then $(u_n)$ converges to 0 locally uniformly on $B(z,t)$.

Finally we will use the next lemma in the proof of Theorem 2. This result is a “uniform” version of Theorem 3 of Gehlen [2].

**Lemma 3** Let $K$ be a compact set in $\mathbb{C}$ with positive logarithmic
capacity \( c(K) \), let \( M > 0 \) and let \( (\lambda_n) \) be a subsequence of the positive integers. Then, for each \( \varepsilon > 0 \), there exists \( \nu = \nu(\varepsilon) \in (\frac{1}{2}, 1) \) and \( n_0 \in \mathbb{N} \) such that for all power series \( \sum_{j=0}^\infty a_j z^j \) satisfying

\[
\sup_{n \in \mathbb{N}} \| s_{\lambda_n} \|_K \leq M \tag{2.1}
\]

(where \( s_m(z) = \sum_{j=0}^m a_j z^j \)) we have

\[
\max_{\nu \lambda_n \leq j \leq \lambda_n} |a_j|^{1/j} \leq \frac{1 + \varepsilon}{c(K)} \quad (n \geq n_0).
\]

**Proof.** We adapt the argument of Gehlen. Let \( \varepsilon > 0 \). We choose \( \delta > 0 \) such that \( e^\delta < 1 + \varepsilon \). From the definition of the Green function \( g_K \) of \( \mathbb{C} \setminus K \) with pole at \( \infty \), we can find \( R_\delta > 1 \) such that if \( |z| \geq R_\delta \), then

\[
g_K(z) \leq \log |z| - \log c(K) + \delta.
\]

Let \( T_j \) denote the set of the \( j \)th coefficients \( a_j \) of all power series \( \sum_{j=0}^\infty a_j z^j \) satisfying (2.1). By applying Bernstein’s lemma (see [9]) to the partial sums \( s_{\lambda_n} \), we obtain

\[
|s_{\lambda_n}(z)| \leq \| s_{\lambda_n} \|_K e^{\lambda_n g_K(z)} \leq M \left( \frac{|z|}{c(K)} e^\delta \right)^{\lambda_n} \quad (|z| \geq R_\delta, n \in \mathbb{N}).
\]

Further, Cauchy’s formula implies that, for all \( j = 1, 2, ..., \lambda_n \) and \( a_j \in T_j \),

\[
|a_j|^{1/j} = \left| \frac{1}{2\pi i} \int_{|z|=R_\delta} \frac{s_{\lambda_n}(z)}{z^j} \, dz \right|^{1/j} \leq M^{1/j} \left( \frac{e^\delta}{c(K)} \right)^{\lambda_n/j} R_\delta^{\lambda_n/j-1}.
\]

In particular, if \( \nu \in (0, 1) \) is sufficiently close to 1, then

\[
\limsup_{n \to \infty} \max_{\nu \lambda_n \leq j \leq \lambda_n} \sup\{|a_j|^{1/j} : a_j \in T_j\} \leq \frac{e^{\delta/\nu}}{\min\{c(K), c(K)^{1/\nu}\}} R_\delta^{1/\nu - 1}
\]

\[
< \frac{1 + \varepsilon}{c(K)}.
\]

\[ \square \]

**3 \ Proofs**

**Proof of Theorem 1.** Let \( \sum_{j=0}^\infty H_j(x) \) be the homogeneous polynomial expansion of \( h \) about 0. Without loss of generality, we may assume that \( y = (1, 0, ..., 0) \in \mathbb{R}^N \). Then, for sufficiently small \( \delta \in (0, \frac{1}{2}) \),
the function $h$ has a harmonic continuation to a neighbourhood of $B(z, \frac{1}{2} + \delta)$, where $z = (\frac{1}{2}, 0, ..., 0) \in \mathbb{R}^N$. On $B(z, \frac{1}{2} + \delta)$ we consider the functions $h_n$ with $h_n(x) = h(x) - S_{p_n}(x)$. We will show that $h_n$ converges locally uniformly to 0 on $B(z, \frac{1}{2} + \varepsilon)$ for sufficiently small $\varepsilon > 0$.

Since the homogeneous polynomial expansion of $h$ possesses Hadamard-Ostrowski gaps $(p_n, q_n)$, there is some $\theta > 0$ such that $q_n \geq (1 + \theta)p_n$ for all $n \in \mathbb{N}$ and $H_j \equiv 0$ for $j \in \bigcup_{n=1}^{\infty} \{p_n + 1, ..., q_n\}$. Let $\eta := \mu\delta$, where $\mu \in (0, \frac{1}{2})$ is chosen small enough that

$$1 + \frac{\theta}{\theta}(1 - \mu) - \frac{1}{\theta}(1 + \mu) > 0.$$ 

Let $L_j = \max_{\|x\|=1} |H_j(x)|$. Lemma 1(iii) shows that $\limsup_{j \to \infty} L_j^{1/j} = 1$. Hence, there exists $c > 1$ such that $L_j \leq c(1 - \eta)^{-j}$ for all $j \in \mathbb{N}$. Additionally, by the $j$-homogeneity of $H_j$, we have $|H_j(x)| \leq L_j \|x\|^j$, for all $x \in \mathbb{R}^N$.

From all the above we see that, for each $x \in B(z, \frac{1}{2} - \delta)$ and for each $n \in \mathbb{N}$

$$|h_n(x)| \leq \sum_{j=p_n+1}^{\infty} |H_j(x)| = \sum_{j=q_n}^{\infty} |H_j(x)|$$

$$\leq \sum_{j=q_n}^{\infty} c \frac{c}{(1 - \eta)^j} \|x\|^j \leq c \sum_{j=q_n}^{\infty} \left( \frac{1 - \delta}{1 - \eta} \right)^j$$

$$= c \left( 1 - \frac{1 - \delta}{1 - \eta} \right)^{-1} \left( \frac{1 - \delta}{1 - \eta} \right)^{q_n} \leq K \left( \frac{1 - \delta}{1 - \mu\delta} \right)^{(1 + \theta)p_n},$$

where $K = c \frac{1 - \mu\delta}{(1 - \mu)^\delta}$.

Moreover, since $h$ has a harmonic continuation to a neighbourhood of $B(z, \frac{1}{2} + \delta)$, the function $h$ is bounded there by a positive constant $M$. Hence for each $x \in B(z, \frac{1}{2} + \delta)$ and for each $n \in \mathbb{N}$,
\[ |h_n(x)| \leq |h(x)| + \sum_{j=0}^{p_n} |H_j(x)| \]
\[ \leq M + \sum_{j=0}^{p_n} L_j \|x\|^j \]
\[ \leq M + \sum_{j=0}^{p_n} \frac{c}{(1-\eta)^j} (1+\delta)^j \]
\[ = M + c \left( \frac{1+\delta}{1-\eta} \right)^{p_n} \sum_{j=0}^{p_n} \left( \frac{1-\eta}{1+\delta} \right)^j \]
\[ \leq L \left( \frac{1+\delta}{1-\mu\delta} \right)^{p_n}, \]

where \( L = M + c \frac{1+\delta}{(1+\mu)\delta} \).

Let \( \varepsilon \in (0, \delta) \). We apply Theorem B for the three spheres centred at \( z \) with radii \( \rho = \frac{1}{2} - \delta, \ t = \frac{1}{2} + \varepsilon, \ R = \frac{1}{2} + \delta \) and the harmonic functions \( h_n \). This tells us that, for each \( n \in \mathbb{N} \),
\[ \|h_n\|_{z,t,2} \leq \|h_n\|_{z,\rho,2}^{\beta} \|h_n\|_{z,R,2}^{1-\beta}, \text{ where } \beta = \frac{\log \left( \frac{\rho}{R} \right)}{\log \left( \frac{\rho}{t} \right)} = \frac{\log \left( \frac{1+2\delta}{1+2\varepsilon} \right)}{\log \left( \frac{1+2\varepsilon}{1+2\delta} \right)}. \]

By using the above estimates for the functions \( h_n \) on the balls \( B(z, \frac{1}{2} - \delta) \) and \( B(z, \frac{1}{2} + \delta) \) we deduce that
\[ \|h_n\|_{z,t,2} \leq K^\beta \left( \frac{1-\delta}{1-\mu\delta} \right)^{(1-\beta)p_n} \left( \frac{1+\delta}{1-\mu\delta} \right)^{(1-\beta)p_n} \leq c' (A_\delta(\varepsilon))^{p_n}, \]

where \( c' = \max \{ K, L \} \) and
\[ A_\delta(\varepsilon) = \left( \frac{1-\delta}{1-\mu\delta} \right)^{(1+\theta) \log \left( \frac{1+2\delta}{1+2\varepsilon} \right)} \left( \frac{1+\delta}{1-\mu\delta} \right)^{\log \left( \frac{1+2\varepsilon}{1+2\delta} \right)} \left( \frac{1+\delta}{1-\mu\delta} \right)^{1/\log \left( \frac{1+2\delta}{1+2\varepsilon} \right)} \]

We claim that \( A_\delta(\varepsilon) < 1 \) for sufficiently small \( \varepsilon \) and \( \delta \). Indeed,
\[ A_\delta(\varepsilon) \to A_\delta^{1/\log \left( \frac{1+2\varepsilon}{1+2\delta} \right)} \text{ as } \varepsilon \to 0^+, \]

where
\[ A_\delta = \left( 1 - \frac{(1-\mu)\delta}{1-\mu\delta} \right)^{(1+\theta) \log \left( 1+2\delta \right)} \left( 1 + \frac{(1+\mu)\delta}{1-\mu\delta} \right)^{-\log \left( 1-2\delta \right)}. \]
However, since 

\[ \frac{\log A_{\delta}}{-20\delta^2} \to \frac{1 + \theta}{\theta} (1 - \mu) - \frac{1}{\theta} (1 + \mu) > 0 \text{ as } \delta \to 0^+, \]

we can find a sufficiently small \( \delta > 0 \) such that \( \log A_{\delta} < 0 \), or equivalently, \( A_{\delta} < 1 \). Thus, for a suitable choice of \( \varepsilon \in (0, \delta) \), the quantity \( A_{\delta}(\varepsilon) \) is strictly less than 1, and so \( c'(A_{\delta}(\varepsilon))^{p_n} \to 0 \) as \( n \to \infty \). Consequently \( \| h_n \|_{z,t,2} \to 0 \) as \( n \to \infty \).

Since \( h_n \) is harmonic on a neighbourhood of \( \overline{B(z,t)} \), the function \( h_n^2 \) is subharmonic on the same neighbourhood. Therefore,

\[ \frac{1}{\lambda(B(z,t))} \int_{B(z,t)} h_n^2 d\lambda \leq \frac{1}{\sigma(\partial B(z,t))} \int_{\partial B(z,t)} h_n^2 d\sigma = \| h_n \|_{z,t,2}^2. \]

Hence \( \int_{B(z,t)} h_n^2 d\lambda \to 0 \), as \( n \to \infty \) and the result follows by applying Lemma 2 to the non-negative subharmonic functions (\( h_n^2 \)). \( \square \)

**Details of Example.** Since \( K(\cdot, y) \in \mathcal{H}(B'(0, 1)) \), the function \( h \) is harmonic on the cylinder \( B'(0, 1) \times \mathbb{R} \). Let \( \sum_{j=0}^{\infty} H_j(x_1, \ldots, x_N) \) be the homogeneous polynomial expansion of \( h \) about the origin. Then the radius of convergence of this expansion is 1 because \( h(x_0, 0) \to +\infty \) as \( x \to y \), where \( x \in B'(0, 1) \). Using Theorem 2.4.3 of [1] we obtain

\[ h(x_1, \ldots, x_{N-1}, x_N) = K((x_1, \ldots, x_{N-1}), y) = \sum_{j=0}^{\infty} \frac{1}{\sigma_{N-1}} J_{y,j}(x_1, \ldots, x_{N-1}), \]

where \( J_{y,j} \) denotes the \( y \)-axial homogeneous polynomial of degree \( j \) in \( \mathbb{R}^{N-1} \) and \( \sigma_{N-1} = \sigma(\partial B'(0, 1)) \). The uniqueness of the homogeneous polynomial expansion of \( h \) in \( B(0, 1) \) implies \( H_j(x_1, \ldots, x_{N-1}, x_N) = \frac{1}{\sigma_{N-1}} J_{y,j}(x_1, \ldots, x_{N-1}) \) for each \( j \in \mathbb{N} \) and for each \( (x_1, \ldots, x_{N-1}, x_N) \in \mathbb{R}^N \). Since the series \( \frac{1}{\sigma_{N-1}} \sum_{j=0}^{\infty} J_{y,j} \) converges locally uniformly on \( B'(0, 1) \) to \( K(\cdot, y) \), the sequence \( (S_n) \) of partial sums of \( h \) converges locally uniformly on \( B'(0, 1) \times \mathbb{R} \) (to \( h \)). We will show that \( h \) cannot be written in the form \( h = g + v \) on \( B(0, 1) \), where

(i) \( v \in \mathcal{H}(B(0, 1 + \varepsilon)) \) for some \( \varepsilon > 0 \),

(ii) \( g \in \mathcal{H}(B(0, 1)) \) and it has a homogeneous polynomial expansion with Hadamard-Ostrowski gaps \( (p_n, q_n) \).

For the sake of contradiction we assume that \( h \) can be written in the above form for some functions \( v \) and \( g \). Let \( \sum_{j=0}^{\infty} v_j \) and \( \sum_{j=0}^{\infty} g_j \) be the homogeneous polynomial expansions of \( v \) and \( g \) respectively. Then, using again the uniqueness of the homogeneous polynomial expansion of \( h \), we deduce that \( H_j = v_j + g_j \) in \( \mathbb{R}^N \). Therefore, for
each \((x_1, \ldots, x_{N-1}, x_N) \in \mathbb{R}^N\) and for each \(j \in \mathbb{N}\),

\[
\frac{1}{\sigma_{N-1}} J_{y,j}(x_1, \ldots, x_{N-1}) = g_j(x_1, \ldots, x_{N-1}, x_N) + v_j(x_1, \ldots, x_{N-1}, x_N).
\]

In particular, condition (ii) shows that, for each \((x_1, \ldots, x_{N-1}, x_N) \in \mathbb{R}^N\) and each \(j \in I = \bigcup_{n=1}^{\infty} \{p_n + 1, \ldots, q_n\}\),

\[
\frac{1}{\sigma_{N-1}} J_{y,j}(x_1, \ldots, x_{N-1}) = v_j(x_1, \ldots, x_{N-1}, x_N).
\]

Let

\[
V_j = \max\{|v_j(x_1, \ldots, x_{N-1}, x_N)| : (x_1, \ldots, x_{N-1}, x_N) \in \partial B(0, 1)\}.
\]

Then, for each \(j \in I\),

\[
V_j = \max\left\{\frac{1}{\sigma_{N-1}} |J_{y,j}(x_1, \ldots, x_{N-1})| : (x_1, \ldots, x_{N-1}, x_N) \in \partial B(0, 1)\right\}
= \max\left\{\frac{1}{\sigma_{N-1}} |J_{y,j}(x_1, \ldots, x_{N-1})| : (x_1, \ldots, x_{N-1}) \in \partial B'(0, 1)\right\}
\]

Consequently, since \(y \in \partial B'(0, 1)\),

\[
\limsup_{j \to \infty, j \in I} \left| \frac{1}{\sigma_{N-1}} J_{y,j}(y) \right|^{1/j} \leq \limsup_{j \to \infty} V_j^{1/j} \leq \limsup_{j \to \infty} V_j^{1/j}.
\]

Additionally, condition (i) and Lemma 1(iii) imply that \(\limsup_{j \to \infty} V_j^{1/j} < 1\), and so

\[
\limsup_{j \to \infty, j \in I} \left| \frac{1}{\sigma_{N-1}} J_{y,j}(y) \right|^{1/j} < 1.
\]

As a final step we will show that \(\lim_{j \to \infty} \left| \frac{1}{\sigma_{N-1}} J_{y,j}(y) \right|^{1/j} = 1\), which contradicts the above estimate. Indeed, from Corollary 2.3.7 of [1], we obtain \(J_{y,j}(y) = d_{j,N-1}\), where \(d_{j,N-1}\) is the dimension of the space of harmonic homogeneous polynomials of degree \(j\) in \(N - 1\) variables.

Further, Corollary 2.1.4 of [1] gives

\[
d_{j,N-1} = \binom{j + N - 2}{j} - \binom{j + N - 4}{j - 2}
= \frac{1}{(N-2)!} \{(j + N - 2) \cdot \ldots \cdot (j + 1) - (j + N - 4) \cdot \ldots \cdot (j - 1)\}.
\]
Thus \( d_{j,N-1} \) is a polynomial in \( j \), and so
\[
\lim_{j \to \infty} \left| \frac{1}{\sigma_{N-1}} J_{y,j}(y) \right|^{1/j} = \lim_{j \to \infty} d_{j,N-1}^{1/j} = 1.
\]

\[\Box\]

**Proof of Theorem 2.** Let \( h, (\lambda_n), w \) and \( \rho \) be as in the statement of the theorem and let \( \sum_{j=0}^{\infty} H_j(x) \) be the homogeneous polynomial expansion of \( h \) about 0. For each \( u \in \partial B(0,1) \), \( m \in \mathbb{N} \) we define the directional complexified partial sums
\[
S_m^{(u)}(z) = \sum_{j=0}^{m} H_j(u)z^j \quad (z \in \mathbb{C}).
\]
Clearly \( S_m^{(u)}(t) = S_m(tu) \) for every \( t \in \mathbb{R}, u \in \partial B(0,1) \). We observe that \( \mathcal{P}(\varphi_{|w|}, \Delta, \|w\|, \|w\| + \frac{\rho}{2}) \subset B(w, \rho) \) for sufficiently small \( \Delta > 0 \).

By considering the compact sets \( K_m = D(0,1 - \frac{1}{m}) \cup [\|w\|, \|w\| + \frac{\rho}{2}] \) we see that \( K_m \nrightarrow D(0,1) \cup [\|w\|, \|w\| + \frac{\rho}{2}] \) as \( m \to \infty \). Thus
\[
c(K_m) \to c(D(0,1) \cup [\|w\|, \|w\| + \frac{\rho}{2}])
= c(D(0,1) \cup [\|w\|, \|w\| + \frac{\rho}{2}])
> c(D(0,1)) = 1,
\]
where \( c(\cdot) \) denotes logarithmic capacity. Thus we can find \( m_0 \in \mathbb{N} \) such that \( c(K_{m_0}) > 1 \).

**Claim:** There exists \( M > 0 \) such that \( \|S_{\lambda_m}^{(u)}(z)\| \leq M \) for all \( z \in K_{m_0}, n \in \mathbb{N} \) and \( u \in T := \partial B(0,1) \cap B(\varphi_{|w|}, \Delta) \).

**Proof of the claim:** If \( t \in [\|w\|, \|w\| + \frac{\rho}{2}] \), then \( tu \in B(w, \rho) \) for every \( u \in \partial B(0,1) \cap B(\varphi_{|w|}, \Delta) \), from the choice of \( \Delta \). Hence, by hypothesis, there exists \( M_0 > 0 \) such that, for all \( t \in [\|w\|, \|w\| + \frac{\rho}{2}], n \in \mathbb{N} \) and \( u \in \partial B(0,1) \cap B(\varphi_{|w|}, \Delta) \),
\[
|S_{\lambda_m}^{(u)}(t)| = |S_{\lambda_m}(tu)| \leq M_0.
\]
If \( z \in D(0,1 - \frac{1}{m_0}) \), then \( \|z\|u \in B(0,1 - \frac{1}{m_0}) \) for every \( u \in \partial B(0,1) \). The local Weierstrass convergence of the homogeneous polynomial expansion of \( h \) (see Theorem 2.4.4 of [1]) implies that
\[
M_1 := \sum_{j=0}^{\infty} \sup \{H_j(x) : x \in B(0,1 - \frac{1}{m_0})\} < +\infty.
\]

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Hence, for all \( z \in \overline{D(0, 1 - \frac{1}{m_0})} \), \( n \in \mathbb{N} \) and \( u \in \partial B(0, 1) \),
\[
\left| S_{\lambda_n}^{(u)}(z) \right| \leq \sum_{j=0}^{\lambda_n} \left| H_j(u) z^j \right| = \sum_{j=0}^{\lambda_n} \left| H_j(|z| u) \right| \leq M_1.
\]

We finish the proof of the claim by setting \( M = \max\{M_0, M_1\} \).

Since \( c(K_m) > 1 \), we can choose \( \varepsilon > 0 \) such that \( 1 + \varepsilon c(K_m) < 1 \).

In view of the above claim, we can apply Lemma 3 to the Taylor polynomials \( (S_m^{(u)})_m \) for all \( u \in T \). Hence we find \( \nu \in \left( \frac{1}{2}, 1 \right) \), \( \mu < 1 \) and \( n_0 \in \mathbb{N} \) such that
\[
|H_j(u)|^{1/j} \leq \mu \quad (3.1)
\]
for all \( u \in T \) and \( j \in S = \bigcup_{n=n_0}^{\infty} \{[\nu \lambda_n] + 1, \ldots, \lambda_n\} \).

Without loss of generality we may assume that \( \lambda_{n+1} \geq 2\lambda_n \) (for otherwise we can choose a suitable subsequence of \( (\lambda_n) \)). Hence, if we set \( p_n = [\nu \lambda_{n_0+n-1}] \) and \( q_n = \lambda_{n_0+n-1} \), we have
\[
1 \leq p_1 < q_1 \leq p_2 < q_2 \leq \ldots \quad \text{and} \quad \frac{q_n}{p_n} \geq \frac{1}{\nu} > 1 \quad \text{for all} \quad n \in \mathbb{N}.
\]

We define
\[
G_j = \begin{cases} 
0 & \text{if} \quad j \in S \\
H_j & \text{if} \quad j \in \mathbb{N} \setminus S
\end{cases}
\]
and
\[
V_j = \begin{cases} 
H_j & \text{if} \quad j \in S \\
0 & \text{if} \quad j \in \mathbb{N} \setminus S
\end{cases}.
\]

The local Weierstrass convergence of the homogeneous polynomial expansion implies that the series
\[
g(x) = \sum_{j=0}^{\infty} G_j(x) \quad , \quad v(x) = \sum_{j=0}^{\infty} V_j(x)
\]
have radius of convergence at least 1, and so they define harmonic functions on \( B(0, 1) \). Clearly \( g \) possesses Hadamard-Ostrowski gaps \( (p_n, q_n) \) and \( h = g + v \) on \( B(0, 1) \). We choose \( r \in (1, \frac{1}{\mu}) \). From (3.1) we deduce that, for all \( j \in \mathbb{N}, t \in [-r, r] \) and \( u \in T = \partial B(0, 1) \cap \overline{B(\frac{1}{\mu} \|w\|, \Delta)} \),
\[
|V_j(tu)| = |V_j(u)t^j| \leq \mu^j r^j.
\]
Consequently, the choice of \( r \) gives

\[
\sum_{j=0}^{\infty} \sup \{|V_j(x)| : x \in P\left(\frac{w}{\|w\|}, \Delta, -r, r\right)\} \leq \sum_{j=0}^{\infty} (\mu r)^j < +\infty.
\]

Hence, by using the Weierstrass M-test, we conclude that the expansion of \( v \) converges uniformly on \( P\left(\frac{w}{\|w\|}, \Delta, -r, r\right) \), which completes the proof of the theorem.

**Proof of Corollary 1.** From Theorem 2 we can write \( h \) in the form \( h = g + v \), where \( g, v \in \mathcal{H}(D(0, 1)) \) and

(i) the homogeneous polynomial expansion of \( g \) possesses Hadamard-Ostrowski gaps,

(ii) the homogeneous polynomial expansion of \( v \) converges locally uniformly on \( D(0, 1) \cup P(z_0, \Delta, -r, r) \) for some \( \Delta > 0 \), \( r > 1 \).

Applying Proposition 1.4 (i) of Siciak and Kolodziej [10] to \( v \), we see that its expansion converges locally uniformly on \( D(0, r) \) and therefore \( h \) has the desired form.

**Proof of Theorem 3.** Let \( \sum_{j=0}^{\infty} H_j(x) \) be the homogeneous polynomial expansion of \( h \) about 0. Also let \( E \) be a compact subset of \( G \). We consider the functions \( h_n \) with \( h_n(x) = h(x) - S_{p_n}(x) \) and will show that \( h_n \to 0 \) uniformly on \( E \). Lemma 1(iii) and the \( j \)-homogeneity of \( H_j \) imply that there is a constant \( K > 1 \) such that

\[
|H_j(x)| \leq K \left(\frac{2}{3}\right)^j \|x\|^j
\]

for all \( x \in \mathbb{R}^N \) and \( j \in \mathbb{N} \). Since the homogeneous polynomial expansion of \( h \) possesses Ostrowski gaps \((p_n, q_n)\), for each \( x \in B(0, \frac{1}{2}) \) and \( n \in \mathbb{N} \), we have

\[
|h_n(x)| \leq \sum_{j=p_n+1}^{\infty} |H_j(x)| = \sum_{j=p_n+1}^{\infty} |H_j(x)| \leq 3K \left(\frac{2}{3}\right)^{q_n}
\]

Now we choose a bounded domain \( \Omega \) such that \( E \cup B(0, \frac{1}{2}) \subset \Omega \subset \overline{\Omega} \subset G \). Since \( h \) is continuous on the compact set \( \overline{\Omega} \), we know that \( \|h\|_{\overline{\Omega}} < +\infty \). Also \( \Omega \subset B(0, R) \) for some \( R > 1 \). Hence, for each \( x \in \Omega \) and \( n \in \mathbb{N} \),

\[
|h_n(x)| \leq |h(x)| + \sum_{j=0}^{p_n} |H_j(x)| \leq \|h\|_{\overline{\Omega}} + K \sum_{j=0}^{p_n} \left(\frac{4}{3}R\right)^j \leq L \left(\frac{4}{3}R\right)^{p_n},
\]

where \( L = K \left(\sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j + \|h\|_{\overline{\Omega}}\right) < +\infty \).

By applying Theorem A to the harmonic functions \( h_n \) and the sets \( E, \Omega \) and \( \Omega_0 = B(0, \frac{1}{2}) \), we find a constant \( a = a(E, \Omega_0, \Omega) \in (0, 1] \) such that, for every \( n \in \mathbb{N} \),

\[
\|h_n\|_{E} \leq \|h_n\|^a_{\Omega_0} \|h_n\|_{\overline{\Omega}}^{1-a}.
\]
If we set \( c = \max\{3K, L\} \) and \( M_n = \frac{q_n}{p_n} \), the above estimates give

\[
\|h_n\|_E \leq (3K)^a \left( \frac{2}{3} \right)^{M_n p_n a} L^{1-a} \left( \frac{4}{3} R \right)^{p_n (1-a)} \leq c \left( \left( \frac{2}{3} \right)^{a M_n} \left( \frac{4}{3} R \right)^{1-a} \right)^{p_n}
\]

for all \( n \in \mathbb{N} \). From the definition of Ostrowski gaps, \( M_n \to \infty \) as \( n \to \infty \), and since \( \left( \frac{2}{3} \right)^a < 1 \) we deduce that \( \|h_n\|_E \to 0 \) as \( n \to \infty \). Equivalently, \((S_{p_n})\) converges to \( h \) uniformly on \( E \) and the result follows from the arbitrary nature of \( E \).

\[\square\]

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References


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