DENSITIES OF 4-RANKS OF $K_2(\mathcal{O})$

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ABSTRACT. In [1], the authors established a method of determining the structure of the 2-Sylow subgroup of the tame kernel $K_2(\mathcal{O})$ for certain quadratic number fields. Specifically, the 4-rank for these fields was characterized in terms of positive definite binary quadratic forms. Numerical calculations led to questions concerning possible density results of the 4-rank of tame kernels. In this paper, we succeed in giving affirmative answers to these questions.

1. Introduction

Since the 1960's, relationships between algebraic K-theory and number theory have been intensely studied. For number fields F and their rings of integers $\mathcal{O}_F$, the K-groups $K_0(\mathcal{O}_F)$, $K_1(\mathcal{O}_F)$, $K_2(\mathcal{O}_F)$, ... were a main focus of attention. From [8] we have

$$K_0(\mathcal{O}_F) \cong \mathbb{Z} \times C(F)$$

where $C(F)$ is the ideal class group of F, and

$$K_1(\mathcal{O}_F) \cong \mathcal{O}_F^*,$$

the group of units of $\mathcal{O}_F$.

What can we say in general about $K_2(\mathcal{O}_F)$? Garland and Quillen in [3] and [11] showed that $K_2(\mathcal{O}_F)$ is finite. A conjecture of Birch and Tate connects the order of $K_2(\mathcal{O}_F)$ and the value of the zeta function of F at -1 when F is a totally real field. For abelian number fields, this conjecture has been confirmed up to powers of 2 [7]. In [12] a 2-rank formula for $K_2(\mathcal{O}_F)$ was given by Tate. Some results on the 4-rank of $K_2(\mathcal{O}_F)$ were given in [9], [10], and [13]. To gain further insight on the 4-rank of $K_2(\mathcal{O}_F)$, we consider the following specific families of fields.

In this paper we deal with the 4-rank of the Milnor K-group $K_2(\mathcal{O})$ for the quadratic number fields $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{2pl})$, $\mathbb{Q}(\sqrt{-pl})$, $\mathbb{Q}(\sqrt{-2pl})$ for primes $p \equiv 7 \bmod 8$, $l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$. In [1], the authors show that for the fields $E = \mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{2pl})$ and $F = \mathbb{Q}(\sqrt{-pl})$, $\mathbb{Q}(\sqrt{-2pl})$,

$$\text{4-rank } K_2(\mathcal{O}_E) = 1 \text{ or } 2,$$

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$$\text{4-rank } K_2(\mathcal{O}_F) = 0 \text{ or } 1.$$

Each of the possible values of 4-ranks is then characterized by checking which ones of the quadratic forms X^2+32Y^2, X^2+2pY^2, $2X^2+pY^2$ represent a certain power of l over $\mathbb{Z}$. This approach makes numerical computations accessible. We should note that this approach involves quadratic symbols and determining the matrix rank over $\mathbb{F}_2$ of 3×3 matrices with Hilbert symbols as entries, see [4]. Fix a prime $p \equiv 7 \bmod 8$ and consider the set

$$\Omega = \{l \text{ rational prime} : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = \left(\frac{p}{l}\right) = 1\}.$$

Let

$$\begin{aligned} \upsilon &= \text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) \\ \mu &= \text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{2pl})}) \\ \sigma &= \text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) \\ \tau &= \text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-2pl})}) \end{aligned}$$

and also consider the sets

$$\begin{aligned} \Omega_1 &= \{l \in \Omega : \upsilon = 1\} \\ \Omega_2 &= \{l \in \Omega : \upsilon = 2\} \\ \Omega_3 &= \{l \in \Omega : \mu = 1\} \\ \Omega_4 &= \{l \in \Omega : \mu = 2\} \\ \Lambda_1 &= \{l \in \Omega : \sigma = 0\} \\ \Lambda_2 &= \{l \in \Omega : \sigma = 1\} \\ \Lambda_3 &= \{l \in \Omega : \tau = 0\} \\ \Lambda_4 &= \{l \in \Omega : \tau = 1\}. \end{aligned}$$

We have computed the following (see Table 1 in Appendix): For $p = 7$, there are 9730 primes l in Ω with $l \leq 10^6$. Among them, there are 4866 primes (50.01%) in Ω_1 and Ω_3 and 4864 primes (49.99%) in Ω_2 and Ω_4. Also, there are 4878 primes (50.13%) in Λ_1 and Λ_3 and 4852 primes in Λ_2 and Λ_4. The goal of this paper is to prove the following theorem.

Theorem 1.1. *For the fields $\mathbb{Q}(\sqrt{pl})$ and $\mathbb{Q}(\sqrt{2pl})$, 4-rank 1 and 2 each appear with natural density $\frac{1}{2}$ in Ω. For the fields $\mathbb{Q}(\sqrt{-pl})$ and $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density $\frac{1}{2}$ in Ω.*

Now consider the tuple of 4-ranks $(\upsilon, \mu, \sigma, \tau)$. By Corollary (5.6) in [1], there are eight possible tuples of 4-ranks. For $p = 7$, among the 9730 primes $l \in \Omega$ with $l \leq 10^6$, the eight possible tuples are realized by 1215, 1213, 1228, 1210, 1210, 1228, 1225, 1201 primes l respectively (see Table 2 in Appendix). And, in fact:

Theorem 1.2. *Each of the eight possible tuples of 4-ranks appear with natural density $\frac{1}{8}$ in Ω.*

2. Preliminaries

Let $\mathcal{D}$ be a Galois extension of $\mathbb{Q}$, and $G = Gal(\mathcal{D}/\mathbb{Q})$. Let $Z(G)$ denote the center of G and $\mathcal{D}^{Z(G)}$ denote the fixed field of $Z(G)$. Let p be a rational prime which is unramified in $\mathcal{D}$ and β be a prime of $\mathcal{D}$ containing p. Let $\left(\frac{\mathcal{D}/\mathbb{Q}}{p}\right)$ denote the Artin symbol of p and $\{g\}$ the conjugacy class containing one element $g \in G$.

Lemma 2.1. $\left(\frac{\mathcal{D}/\mathbb{Q}}{p}\right) = \{g\}$ *for some* $g \in Z(G)$ *if and only if* p *splits completely in* $\mathcal{D}^{Z(G)}$.

Proof. $\left(\frac{\mathcal{D}/\mathbb{Q}}{p}\right) = \{g\}$ for some $g \in Z(G)$ if and only if $\left(\frac{\mathcal{D}/\mathbb{Q}}{\beta}\right) = g$ if and only if $\left(\frac{\mathcal{D}^{Z(G)}/\mathbb{Q}}{\beta}\right) = \left(\frac{\mathcal{D}/\mathbb{Q}}{\beta}\right)\Big|_{\mathcal{D}^{Z(G)}} = g|_{\mathcal{D}^{Z(G)}} = Id_{Gal(\mathcal{D}^{Z(G)}/\mathbb{Q})}$ if and only if p splits completely in $\mathcal{D}^{Z(G)}$. □

Thus if we can show that rational primes split completely in the fixed field of the center of a certain Galois group G, then we know the associated Artin symbol is a conjugacy class containing one element. Hence we may identify the Artin symbol with this one element and consider the symbol to be an automorphism which lies in Z(G). Thus determining the order of Z(G) gives us the number of possible choices for the Artin symbol.

Let G_1 and G_2 be finite groups and A a finite abelian group. Suppose $r_1 : G_1 \to A$ and $r_2 : G_2 \to A$ are two epimorphisms and $\mathcal{G} \subset G_1 \times G_2$ is the set $\{(g_1, g_2) \in G_1 \times G_2 : r_1(g_1) = r_2(g_2)\}$. Since A is abelian, there is an epimorphism $r : G_1 \times G_2 \to A$ given by $r(g_1, g_2) = r_1(g_1)r_2(g_2)^{-1}$. Thus $\mathcal{G} = ker(r) \subset G_1 \times G_2$. One can check that $Z(\mathcal{G}) = \mathcal{G} \cap Z(G_1 \times G_2)$.

Lemma 2.2. *(i) If* $r_2\Big|_{Z(G_2)}$ *is trivial, then* $Z(\mathcal{G}) = ker(r_1\Big|_{Z(G_1)}) \times Z(G_2)$.
(ii) $Z(\mathcal{G}) = Z(G_1) \times Z(G_2) \iff r_1\Big|_{Z(G_1)}$ *and* $r_2\Big|_{Z(G_2)}$ *are both trivial.*

Proof. (i) Suppose $(g_1, g_2) \in Z(\mathcal{G}) \subset ker(r)$ where $g_1 \in Z(G_1)$, $g_2 \in Z(G_2)$. Thus $1 = r(g_1, g_2) = r_1(g_1)r_2(g_2)^{-1}$ and so $r_1(g_1) = r_2(g_2)$. But $r_2(g_2) = 1$ which yields $r_1(g_1) = 1$. Thus $g_1 \in ker(r_1\Big|_{Z(G_1)})$. The other inclusion is clear.
(ii) Take $(g_1, 1), (1, g_2) \in Z(G_1) \times Z(G_2) = Z(\mathcal{G}) \subset ker(r)$, respectively to obtain that $r_1\Big|_{Z(G_1)}$ and $r_2\Big|_{Z(G_2)}$ are both trivial. The converse follows from part (i). □

We will use the following definition throughout this paper.

Definition 2.3. For primes $p \equiv 7 \bmod 8$, $l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = \left(\frac{p}{l}\right) = 1$, $\mathcal{K} = \mathbb{Q}(\sqrt{-2p})$, and $h(\mathcal{K})$ the class number of $\mathcal{K}$, we say:
l satisfies $\langle 1, 32\rangle$ if and only if $l = x^2 + 32y^2$ for some $x, y \in \mathbb{Z}$
l satisfies $\langle 2, p\rangle$ if and only if $l^{\frac{h(\mathcal{K})}{4}} = 2n^2 + pm^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \bmod l$
l satisfies $\langle 1, 2p\rangle$ if and only if $l^{\frac{h(\mathcal{K})}{4}} = n^2 + 2pm^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \bmod l$.

3. Three Extensions

In this section, we consider three degree eight field extensions of $\mathbb{Q}$. The idea will be to study composites of these fields and relate Artin symbols to 4-ranks. Rational primes which split completely in a degree 64 extension of $\mathbb{Q}$ will relate to Artin symbols and thus 4-ranks. Therefore calculating the density of these primes will answer density questions involving 4-ranks.

3.1. First Extension.

Consider $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$. Let $\epsilon = 1+\sqrt{2} \in (\mathbb{Z}[\sqrt{2}])^*$. Then ϵ is a fundamental unit of $\mathbb{Q}(\sqrt{2})$ which has norm -1. The degree 4 extension $\mathbb{Q}(\sqrt{2}, \sqrt{\epsilon})$) over $\mathbb{Q}$ has normal closure $\mathbb{Q}(\sqrt{2}, \sqrt{\epsilon}, \sqrt{-1})$. Set

$$N_1 = \mathbb{Q}(\sqrt{2}, \sqrt{\epsilon}, \sqrt{-1}).$$

Note that N_1 is the splitting field of the polynomial $x^4 - 2x^2 - 1$ and so has degree 8 over $\mathbb{Q}$. Therefore $Gal(N_1/\mathbb{Q})$ is the dihedral group of order 8. Note that the automorphism induced by sending $\sqrt{\epsilon}$ to $-\sqrt{\epsilon}$ commutes with every element of $Gal(N_1/\mathbb{Q})$. Thus $Z(Gal(N_1/\mathbb{Q})) = Gal(N_1/\mathbb{Q}(\sqrt{2}, \sqrt{-1}))$.

Observe that only the prime 2 ramifies in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{\epsilon})$, and so only the prime 2 ramifies in the compositum N_1 over $\mathbb{Q}$. Now as $l \in \Omega$ is unramified in N_1 over $\mathbb{Q}$, the Artin symbol $\left(\frac{N_1/\mathbb{Q}}{\beta}\right)$ is defined for primes β of $\mathcal{O}_{N_1}$ containing l. Let $\left(\frac{N_1/\mathbb{Q}}{l}\right)$ denote the conjugacy class of $\left(\frac{N_1/\mathbb{Q}}{\beta}\right)$ in $Gal(N_1/\mathbb{Q})$. The primes $l \in \Omega$ split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{-1})$ and ${N_1}^{Z(Gal(N_1/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$. Thus by Lemma 2.1, we have that $\left(\frac{N_1/\mathbb{Q}}{l}\right) = \{g\} \subset Z(Gal(N_1/\mathbb{Q}))$. As $Z(Gal(N_1/\mathbb{Q}))$ has order 2, there are two possible choices for $\left(\frac{N_1/\mathbb{Q}}{l}\right)$. Combining this statement with Addendum (3.4) from [1], we have

Remark 3.1.

$$\begin{aligned}\left(\frac{N_1/\mathbb{Q}}{l}\right) = \{id\} &\iff l \text{ splits completely in } N_1 \\ &\iff l \text{ satisfies } \langle 1, 32\rangle.\end{aligned}$$

3.2. **Second and Third Extension.** Consider the fixed prime $p \equiv 7 \bmod 8$. Note p splits completely in $\mathcal{L} = \mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$ and so

$$p\mathcal{O}_{\mathcal{L}} = \mathfrak{B}\mathfrak{B}^{'}$$

for some primes $\mathfrak{B} \neq \mathfrak{B}^{'}$ in $\mathcal{L}$. The field $\mathcal{L}$ has narrow class number $h^{+}(\mathcal{L}) = 1$ as $h(\mathcal{L}) = 1$ and $N_{\mathcal{L}/\mathbb{Q}}(\epsilon) = -1$ where $\epsilon = 1 + \sqrt{2}$ is a fundamental unit of $\mathbb{Q}(\sqrt{2})$, see [5]. From [1],

Lemma 3.2. *The prime $\mathfrak{B}$ which occurs in the decomposition of $p\mathcal{O}_{\mathcal{L}}$ has a generator $\pi = a + b\sqrt{2} \in \mathcal{O}_{\mathcal{L}}$, unique up to a sign and to multiplication by the square of a unit in $\mathcal{O}_{\mathcal{L}}^{*}$ for which $N_{\mathcal{L}/\mathbb{Q}}(\pi) = a^2 - 2b^2 = -p$.*

The degree 4 extension $\mathbb{Q}(\sqrt{2}, \sqrt{\pi})$ over $\mathbb{Q}$ has normal closure $\mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{-p})$ as $N_{\mathcal{L}/\mathbb{Q}}(\pi) = -p$. Set

$$N_2 = \mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{-p}).$$

Then N_2 is Galois over $\mathbb{Q}$ and $[N_2 : \mathbb{Q}] = 8$. Such an extension N_2 exists since the 2-Sylow subgroup of the ideal class group of $\mathbb{Q}(\sqrt{-2p})$ is cyclic of order divisible by 4 [2]. Thus the Hilbert class field of $\mathbb{Q}(\sqrt{-2p})$ contains a unique unramified cyclic degree 4 extension over $\mathbb{Q}(\sqrt{-2p})$. By Lemma 2.3 in [1], N_2 is the unique unramified cyclic degree 4 extension over $\mathbb{Q}(\sqrt{-2p})$. Also compare [6]. Similar to arguments in Section 2.1, $Gal(N_2/\mathbb{Q})$ is the dihedral group of order 8. Note that the automorphism induced by sending $\sqrt{\pi}$ to $-\sqrt{\pi}$ commutes with every element of $\mathrm{Gal}(N_2/\mathbb{Q}))$. Thus $Z(Gal(N_2/\mathbb{Q})) = Gal(N_2/\mathbb{Q}(\sqrt{2}, \sqrt{-p}))$.

Proposition 3.3. *If $l \in \Omega$, then l is unramified in N_2 over $\mathbb{Q}$.*

Proof. Since $p \equiv 7 \bmod 8$, the discriminant of $\mathbb{Q}(\sqrt{-2p})$ is $-8p$. For $l \in \Omega$, we have $\left(\frac{-2p}{l}\right) = 1$ and so l is unramified in $\mathbb{Q}(\sqrt{-2p})$. By Lemma 2.3 in [1], we have l is unramified in N_2 over $\mathbb{Q}$. □

As $l \in \Omega$ is unramified in N_2 over $\mathbb{Q}$, the Artin symbol $\left(\frac{N_2/\mathbb{Q}}{\beta}\right)$ is defined for primes β of $\mathcal{O}_{N_2}$ containing l. Let $\left(\frac{N_2/\mathbb{Q}}{l}\right)$ denote the conjugacy class of $\left(\frac{N_2/\mathbb{Q}}{\beta}\right)$ in $Gal(N_2/\mathbb{Q})$. The primes $l \in \Omega$ split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{-p})$ and ${N_2}^{Z(Gal(N_2/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-p})$. By Lemma 2.1, we have that $\left(\frac{N_2/\mathbb{Q}}{l}\right) = \{h\} \subset Z(Gal(N_2/\mathbb{Q}))$ for some $h \in Z(Gal(N_2/\mathbb{Q}))$. As $\mathrm{Z(Gal}(N_2/\mathbb{Q}))$ has order 2, there are two possible choices for $\left(\frac{N_2/\mathbb{Q}}{l}\right)$. Combining this statement and Lemmas (3.3) and (3.4) from [1], we have

Remark 3.4.

$$\left(\frac{N_2/\mathbb{Q}}{l}\right) = \{id\} \iff l \text{ splits completely in } N_2$$
$$\iff l \text{ satisfies } \langle 1, 2p\rangle.$$

$$\left(\frac{N_2/\mathbb{Q}}{l}\right) \neq \{id\} \iff l \text{ does not split completely in } N_2$$
$$\iff l \text{ satisfies } \langle 2, p\rangle.$$

Finally, for $l \in \Omega$, l splits completely in $\mathbb{Q}(\zeta_{16}) \iff l \equiv 1 \bmod 16$ This yields

Remark 3.5.

$$\left(\frac{\mathbb{Q}(\zeta_{16})/\mathbb{Q}}{l}\right) = \{id\} \iff l \text{ splits completely in } \mathbb{Q}(\zeta_{16})$$
$$\iff l \equiv 1 \bmod 16.$$

4. The Composite and Two Theorems

In this section we consider the composite field $N_1N_2\mathbb{Q}(\zeta_{16})$. Set

$$L = N_1N_2\mathbb{Q}(\zeta_{16}).$$

Note that $[L : \mathbb{Q}] = 64$. As N_1, N_2, and $\mathbb{Q}(\zeta_{16})$ are normal extensions of $\mathbb{Q}$, L is a normal extension of $\mathbb{Q}$.

For $l \in \Omega$, l is unramified in L as it is unramified in N_1, N_2, and $\mathbb{Q}(\zeta_{16})$. The Artin symbol $\left(\frac{L/\mathbb{Q}}{\beta}\right)$ is now defined for some prime β of $\mathcal{O}_L$ containing l. Let $\left(\frac{L/\mathbb{Q}}{l}\right)$ denote the conjugacy class of $\left(\frac{L/\mathbb{Q}}{\beta}\right)$ in $Gal(L/\mathbb{Q})$. Letting $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{-p}) \subset L$, we prove

Lemma 4.1. *$Z(Gal(L/\mathbb{Q})) = Gal(L/M)$ is elementary abelian of order 8.*

Proof. For $\sigma \in Gal(L/M)$, σ can only change the sign of $\sqrt{\epsilon}$, $\sqrt{\pi}$, and $\sqrt{\zeta_8}$ as $\epsilon \in M$. Since $L = M(\sqrt{\epsilon}, \sqrt{\pi}, \sqrt{\zeta_8})$, $Gal(L/M)$ is elementary abelian of order 8. Now consider the restrictions $r_1 : G_1 \to Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ and $r_2 : G_2 \to Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ where $G_1 = Gal(N_1/\mathbb{Q})$ and $G_2 = Gal(N_2/\mathbb{Q})$. Clearly $r_1\Big|_{Z(G_1)}$ and

$r_1\Big|_{Z(G_2)}$ are both trivial. Then by Lemma 2.2 part (ii), $Z(\mathcal{G})$ is elementary abelian of order 4 where $\mathcal{G} = Gal(N_1N_2/\mathbb{Q})$. Now consider the restrictions $R_1 : Gal(\mathbb{Q}(\zeta_{16})/\mathbb{Q}) \to Gal(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ and $R_2 : \mathcal{G} \to Gal(\mathbb{Q}(\zeta_8)/\mathbb{Q})$. Note that $ker(R_1)$ is cyclic of order 2 and $Z(\mathcal{G}) = Gal(M/\mathbb{Q})$. Thus $R_2\Big|_{Z(\mathcal{G})}$

is trivial and so by the above and Lemma 2.2 part (i), $Z(Gal(L/\mathbb{Q})) \cong \mathbb{Z}/2\mathbb{Z} \times Z(\mathcal{G}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus $Z(Gal(L/\mathbb{Q})) = Gal(L/M)$. □

Now for $l \in \Omega$, l splits completely in $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{2}, \sqrt{-p})$ and so splits completely in the composite field $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{-p})$. From Lemma 4.1, $L^{Z(Gal(L/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{-p})$. So by Lemma 2.1, we have $\left(\frac{L/\mathbb{Q}}{l}\right) = \{k\} \subset Z(Gal(L/\mathbb{Q}))$ for some $k \in Gal(L/\mathbb{Q})$. As $Z(Gal(L/\mathbb{Q}))$ has order 8, there are eight possible choices for $\left(\frac{L/\mathbb{Q}}{l}\right)$. Using Remarks 3.1, 3.4, and 3.5, we now make the following one to one correspondences.

Remark 4.2. (i) $\left(\frac{L/\mathbb{Q}}{l}\right) = \{id\} \iff l$ splits completely in L $\iff$

$$\left\{\begin{array}{l} l \text{ splits completely in } N_1, \\ N_2, \text{ and } \mathbb{Q}(\zeta_{16}) \end{array}\right\} \iff \left\{\begin{array}{l} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, 2p\rangle \\ l \equiv 1 \bmod 16 \end{array}\right\}.$$

(ii) $\left(\frac{L/\mathbb{Q}}{l}\right) \neq \{id\} \iff l$ does not split completely in L. Now there are seven cases.

1. $$\left\{\begin{array}{l} l \text{ splits completely in } N_1, \\ \text{but does not in } N_2 \text{ or } \mathbb{Q}(\zeta_{16}) \end{array}\right\} \iff \left\{\begin{array}{l} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 2, p\rangle \\ l \equiv 9 \bmod 16 \end{array}\right\}$$
2. $$\left\{\begin{array}{l} l \text{ splits completely in } N_1 \\ \text{and } N_2, \text{ but does not in } \mathbb{Q}(\zeta_{16}) \end{array}\right\} \iff \left\{\begin{array}{l} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, 2p\rangle \\ l \equiv 9 \bmod 16 \end{array}\right\}$$
3. $$\left\{\begin{array}{l} l \text{ splits completely in} \\ N_2, \text{ but does not in } N_1 \\ \text{or } \mathbb{Q}(\zeta_{16}) \end{array}\right\} \iff \left\{\begin{array}{l} l \text{ does not satisfy } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, 2p\rangle \\ l \equiv 9 \bmod 16 \end{array}\right\}$$
4. $$\left\{\begin{array}{l} l \text{ splits completely in} \\ N_2 \text{ and } \mathbb{Q}(\zeta_{16}), \\ \text{but does not in } N_1 \end{array}\right\} \iff \left\{\begin{array}{l} l \text{ does not satisfy } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, 2p\rangle \\ l \equiv 1 \bmod 16 \end{array}\right\}$$
5. $$\left\{\begin{array}{l} l \text{ splits completely in } N_1 \\ \text{and } \mathbb{Q}(\zeta_{16}), \text{ but does not in } N_2 \end{array}\right\} \iff \left\{\begin{array}{l} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 2, p\rangle \\ l \equiv 1 \bmod 16 \end{array}\right\}$$
6. $$\left\{\begin{array}{l} l \text{ splits completely in} \\ \mathbb{Q}(\zeta_{16}), \text{ but does not in } N_1 \\ \text{or } N_2 \end{array}\right\} \iff \left\{\begin{array}{l} l \text{ does not satisfy } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 2, p\rangle \\ l \equiv 1 \bmod 16 \end{array}\right\}$$
7. $$\left\{\begin{array}{l} l \text{ does not split completely} \\ \text{in } N_1, N_2, \text{ or } \mathbb{Q}(\zeta_{16}) \end{array}\right\} \iff \left\{\begin{array}{l} l \text{ does not satisfy } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 2, p\rangle \\ l \equiv 9 \bmod 16 \end{array}\right\}.$$

Now using Theorems (5.2), (5.3), (5.4), and (5.5) from [1], we relate each Artin symbol $\left(\frac{L/\mathbb{Q}}{l}\right)$ to each of the eight possible tuples of 4-ranks.

Remark 4.3. *From Remark 4.2, case (i) occurs if and only if we have* $(2, 2, 1, 1)$. *For case (ii),*
(1) occurs if and only if we have $(1, 2, 0, 1)$
(2) occurs if and only if we have $(2, 1, 1, 0)$
(3) occurs if and only if we have $(2, 1, 0, 1)$
(4) occurs if and only if we have $(2, 2, 0, 0)$
(5) occurs if and only if we have $(1, 1, 0, 0)$
(6) occurs if and only if we have $(1, 1, 1, 1)$
(7) occurs if and only if we have $(1, 2, 1, 0)$.

We can now prove Theorem 1.2.

Proof. Consider the set $X = \{l \text{ prime} : l \text{ is unramified in } L \text{ and} \left(\frac{L/\mathbb{Q}}{l}\right) = \{k\} \subset Z(Gal(L/\mathbb{Q}))\}$ for some $k \in \mathrm{Gal}(L/\mathbb{Q})$. By the Čebotarev Density Theorem, the set X has natural density $\frac{1}{64}$ in the set of all primes. Recall

$$\Omega = \{l \text{ rational prime} : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = \left(\frac{p}{l}\right) = 1\}$$

for some fixed prime $p \equiv 7 \bmod 8$. By Dirichlet's Theorem on primes in arithmetic progressions, Ω has natural density $\frac{1}{8}$ in the set of all primes. Thus X has natural density $\frac{1}{8}$ in Ω. By Remark 4.2 and 4.3, each of the eight choices for $\left(\frac{L/\mathbb{Q}}{l}\right)$ is in one to one correspondence with each of the possible tuples of 4-ranks. Thus each of the eight possible tuples of 4-ranks appear with natural density $\frac{1}{8}$ in Ω.

□

Now we can prove Theorem 1.1

Proof. We see from Remark 4.3, 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1$ in cases (ii), parts (1), (5), (6), and (7), 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{2pl})}) = 2$ in case (i) and case (ii) parts (1), (4), and (7), 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 0$ in case (ii) parts (1), (3), (4), and (5), 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-2pl})}) = 1$ in case (i) and case (ii) parts (1), (3), and (6). As each of the 4-rank tuples occur with natural density $\frac{1}{8}$, we have for the fields $\mathbb{Q}(\sqrt{pl})$ and $\mathbb{Q}(\sqrt{2pl})$, 4-rank 1 and 2 each appear with natural density $4 \cdot \frac{1}{8} = \frac{1}{2}$ in Ω. For the fields $\mathbb{Q}(\sqrt{-pl})$ and $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density $4 \cdot \frac{1}{8} = \frac{1}{2}$ in Ω.

□

Appendix

The following tables motivated possible density results of 4-ranks of tame kernels. We consider primes $l \in \Omega$ with $l \leq N$ for a fixed prime $p \equiv 7 \bmod 8$ and positive integer N. For Table 1, we consider the sets $\Omega_1, \ldots, \Omega_4$ and $\Lambda_1, \ldots, \Lambda_4$ as in the Introduction. For Table 2, we consider the sets

$$\begin{aligned}
I_1 &= \{l \in \Omega : \text{4-rank tuple is } (1,1,0,0)\} \\
I_2 &= \{l \in \Omega : \text{4-rank tuple is } (1,1,1,1)\} \\
I_3 &= \{l \in \Omega : \text{4-rank tuple is } (2,1,1,0)\} \\
I_4 &= \{l \in \Omega : \text{4-rank tuple is } (2,1,0,1)\} \\
I_5 &= \{l \in \Omega : \text{4-rank tuple is } (1,2,1,0)\} \\
I_6 &= \{l \in \Omega : \text{4-rank tuple is } (1,2,0,1)\} \\
I_7 &= \{l \in \Omega : \text{4-rank tuple is } (2,2,0,0)\} \\
I_8 &= \{l \in \Omega : \text{4-rank tuple is } (2,2,1,1)\}.
\end{aligned}$$

Table 1

Primes	$p = 7$		$p = 23$		$p = 31$	
Cardinality	$N = 1000000$	%	$N = 1000000$	%	$N = 1000000$	%
$\mid \Omega \mid$	9730		9742		9754	
$\mid \Omega_1 \mid$	4866	50.01	4905	50.35	4916	50.40
$\mid \Omega_2 \mid$	4864	49.99	4837	49.65	4838	49.60
$\mid \Omega_3 \mid$	4866	50.01	4911	50.41	4851	49.73
$\mid \Omega_4 \mid$	4864	49.99	4831	49.59	4903	50.27
$\mid \Lambda_1 \mid$	4878	50.13	4912	50.42	4930	50.54
$\mid \Lambda_2 \mid$	4852	49.87	4830	49.58	4824	49.46
$\mid \Lambda_3 \mid$	4878	50.13	4876	50.05	4943	50.68
$\mid \Lambda_4 \mid$	4852	49.87	4866	49.95	4811	49.32

TABLE 2

Primes	$p = 7$		$p = 23$		$p = 31$	
Cardinality	$N = 1000000$	%	$N = 1000000$	%	$N = 1000000$	%
$\mid \Omega \mid$	9730		9742		9754	
$\mid I_1 \mid$	1215	12.49	1246	12.79	1246	12.77
$\mid I_2 \mid$	1213	12.46	1229	12.62	1203	12.33
$\mid I_3 \mid$	1228	12.62	1211	12.43	1214	12.45
$\mid I_4 \mid$	1210	12.44	1225	12.57	1188	12.18
$\mid I_5 \mid$	1210	12.44	1204	12.36	1227	12.58
$\mid I_6 \mid$	1228	12.62	1226	12.58	1240	12.71
$\mid I_7 \mid$	1225	12.59	1215	12.47	1256	12.88
$\mid I_8 \mid$	1201	12.34	1186	12.17	1180	12.10

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References

[1] P. E. Conner and J. Hurrelbrink, *On the 4-rank of the tame kernel $K_2(\mathcal{O})$ in positive definite terms*, J. Number Th. **88** (2001), 263–282.
[2] P. E. Conner and J. Hurrelbrink, *Class Number Parity*, Ser. Pure Math. **8**, World Sci. Publ. Singapore, 1988.
[3] H. Garland, *A finiteness theorem for K_2 of a number field*, Ann. of Math. **94** (1971), 534–548.
[4] J. Hurrelbrink and M. Kolster, *Tame kernels under relative quadratic extensions and Hilbert symbols*, J. reine angew. Math. **499** (1998), 145–188.
[5] G. Janusz, *Algebraic Number Fields*, Academic Press, New York, 1973.
[6] C. U. Jensen and N. Yui, *Quaternion Extensions*, Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata, 1987, 155–182.
[7] B. Mazur and A. Wiles, *Class fields of abelian extensions of* $\mathbb{Q}$, Invent. Math. **76** (1984), 179–330.
[8] J. Milnor, *An Introduction to Algebraic K-Theory*, Ann. Math. Studies. Vol. **72**, Princeton Univ. Press, Princeton, 1971.
[9] H. Qin, *The 2-Sylow subgroups of the tame kernel of imaginary quadratic fields*, Acta Arith., **69** (1995), 153–169.
[10] H. Qin, *The 4-rank of $K_2(\mathcal{O}_F)$ for real quadratic fields F*, Acta Arith., **72** (1995), 323–333.
[11] D. Quillen, *Higher algebraic K-theory*, Algebraic K-Theory I, Lecture Notes in Mathematics, Vol. **341**, Springer-Verlag, New York, 1973, 85–147.
[12] J. Tate, *Relations between K_2 and Galois cohomology*, Inventiones Math., **36**, (1976), 257–274.
[13] A. Vazzana, *4-ranks of K_2 of rings of integers in quadratic number fields*, Ph.D. Thesis, University of Michigan, 1998.

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