<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Sets of determination for the Nevanlinna class</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Authors(s)</strong></td>
<td>Gardiner, Stephen J.</td>
</tr>
<tr>
<td><strong>Publication date</strong></td>
<td>2010-11-06</td>
</tr>
<tr>
<td><strong>Publication information</strong></td>
<td>Gardiner, Stephen J. “Sets of Determination for the Nevanlinna Class” 42, no. 6 (November 6, 2010).</td>
</tr>
<tr>
<td><strong>Publisher</strong></td>
<td>London Mathematical Society</td>
</tr>
<tr>
<td><strong>Item record/more information</strong></td>
<td><a href="http://hdl.handle.net/10197/2728">http://hdl.handle.net/10197/2728</a></td>
</tr>
<tr>
<td><strong>Publisher's version (DOI)</strong></td>
<td>10.1112/blms/bdq073</td>
</tr>
</tbody>
</table>
Sets of determination for the Nevanlinna class

Stephen J. Gardiner

Abstract

This paper characterizes the subsets $E$ of the unit disc $\mathbb{D}$ with the property that $\sup_{E} |f| = \sup_{\mathbb{D}} |f|$ for all functions $f$ in the Nevanlinna class.

1 Introduction

Let $\mathcal{A}$ be a collection of holomorphic functions on the unit disc $\mathbb{D}$, and let $\mathbb{T}$ denote the unit circle. A set $E \subset \mathbb{D}$ is called a set of determination for $\mathcal{A}$ if $\sup_{E} |f| = \sup_{\mathbb{D}} |f|$ for all $f \in \mathcal{A}$. Brown, Shields and Zeller [3] have shown that $E$ is a set of determination for $H^\infty$, the space of bounded holomorphic functions on $\mathbb{D}$, if and only if almost every point of $\mathbb{T}$ can be approached nontangentially by a sequence of points in $E$. Massaneda and Thomas [6] have observed that the same characterization remains valid when $\mathcal{A}$ is the Smirnov class $\mathcal{N}^+$. However, the situation is more complicated for the Nevanlinna class $\mathcal{N}$, which consists of all holomorphic functions $f$ on $\mathbb{D}$ that satisfy

$$\sup_{0 < r < 1} \int_{0}^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta < \infty.$$ 

This is the main focus of [6], where a variety of conditions are shown to be either necessary or sufficient for $E$ to be a set of determination for $\mathcal{N}$, and some illustrative special cases are examined. (See also Stray [7], p.256.) The purpose of this paper is to give a complete characterization of such sets.

First we recall a related result of Hayman and Lyons [5] for the harmonic Hardy space $h^1$, which consists of those functions on $\mathbb{D}$ that can be expressed as the difference of two positive harmonic functions. For $n \in \mathbb{N}$ and $0 \leq m < 2^n+4$ let

$$z_{m,n} = (1 - 2^{-n}) \exp(2\pi im/2^{n+4})$$

and

$$S_{m,n} = \left\{ re^{i\theta} : 2^{-n-1} \leq 1 - r \leq 2^{-n} \text{ and } \frac{2\pi m}{2^{n+4}} \leq \theta \leq \frac{2\pi (m+1)}{2^{n+4}} \right\},$$

$^0$2000 Mathematics Subject Classification 30D50, 30C80, 31A15.

This research was supported by Science Foundation Ireland under Grant 09/RFP/MTH2149 and is also part of the programme of the ESF Network “Harmonic and Complex Analysis and Applications” (HCAA).
and let \( E_{m,n} = E \cap S_{m,n} \). The Poisson kernel for \( D \) is given by
\[
P(z, w) = \frac{1 - |z|^2}{|z - w|^2} \quad (z \in D, w \in T).
\]

**Theorem A** [5] Let \( E \subset D \). The following conditions are equivalent:
(a) \( \sup_\overline{E} h = \sup_D h \) for all \( h \in h^1 \);
(b) \( \sum_{E_{m,n} \neq \emptyset} 2^{-n} P(z_{m,n}, w) = \infty \) for every \( w \in T \).

For any set \( A \) which is contained in a disc of radius less than 1, and any \( t \geq 0 \), we define a capacity-related quantity \( Q(A, t) \) as follows. We put
\[
Q(A, t) = 0 \quad \text{if either} \quad t = 0 \quad \text{or} \quad A = \emptyset; \quad \text{otherwise,}
\]
\[
Q(A, t) = \min\{k \in \mathbb{N} : \exists \xi_1, \ldots, \xi_k \in \mathbb{C} \text{ such that } \sum_{j=1}^{k} \log \frac{1}{|z - \xi_j|} \geq t \quad (z \in A)\}.
\]
Clearly \( Q(\cdot, t) \) is translation-invariant and \( Q(\{\zeta\}, \cdot) = \chi_{(0, \infty)} \) for any \( \zeta \in \mathbb{C} \).
Also,
\[
Q(\{\zeta_1, \zeta_2\}, t) = \begin{cases} 
0 & \text{if } t = 0 \\
1 & \text{if } |\zeta_1 - \zeta_2| \leq 2e^{-t} \text{ and } t > 0 \\
2 & \text{otherwise}
\end{cases}
\]
and, if \( A \) is a disc of radius of \( r < 1 \), then \( Q(A, t) \) is the least integer \( k \) satisfying \( k \geq t/\log(1/r) \). We use \( [t] \) to denote the integer part of a non-negative number \( t \), and \( tA \) to denote the set \( \{t z : z \in A\} \). Our characterization of sets of determination for the Nevanlinna class is as follows.

**Theorem 1** Let \( E \subset D \). The following conditions are equivalent:
(a) \( \sup_\overline{E} |f| = \sup_D |f| \) for all \( f \in N \);
(b) \( \sum_{m,n} 2^{-n} Q(2^n E_{m,n}, [P(z_{m,n}, w)]) = \infty \) for every \( w \in T \).

Since
\[
\log \frac{2^{-n}}{|z - z_{m,n}|} \geq -\frac{1}{2} \log \left( \left( \frac{\pi}{8} \right)^2 + \left( \frac{1}{2} \right)^2 \right) > \frac{1}{3} \quad (z \in S_{m,n}),
\]
we have
\[
3P(z_{m,n}, w) \log \frac{2^{-n}}{|z - z_{m,n}|} \geq P(z_{m,n}, w) \quad (z \in S_{m,n}, w \in T).
\]
By separate consideration of the cases \( P(z_{m,n}, w) \geq 1 \) and \( P(z_{m,n}, w) < 1 \), we see that
\[
Q(2^n E_{m,n}, [P(z_{m,n}, w)]) \leq 4P(z_{m,n}, w). \quad (1)
\]
Applying this inequality to terms where \( E_{m,n} \neq \emptyset \), it is now clear that condition (b) of Theorem 1 implies the corresponding condition of Theorem A. It is not difficult to check that condition (a) of Theorem 1 is equivalent to the assertion that, if \( \log |f| \leq h \) on \( E \), where \( f \in \mathcal{N} \) and \( h \in h^1 \), then \( \log |f| \leq h \) on all of \( \mathbb{D} \) (cf. [6]).

**Examples** Let \( U = \{ z : |z - \frac{1}{2}| < \frac{1}{2} \} \) and \( F = U \cap \{ z_{m,n} \} \).

(i) The set \( E = \mathbb{D} \setminus U \) is not a set of determination (for \( \mathcal{N} \)) because the series in condition (b) of Theorem A then converges when \( w = 1 \) (cf. Example 6.2 in [5]).

(ii) Further, even \( E \cup F \) is not a set of determination because each of the sets \( F_{m,n} \) contains at most 5 points and so

\[
\sum_{m,n} 2^{-n} Q(2^n F_{m,n}, [P(z_{m,n}, 1)]) \leq 5 \sum_{z_{m,n} \in F} 2^{-n} < \infty
\]

(cf. Example 1 in [6]).

(iii) On the other hand, \( E \cup [\frac{1}{2}, 1) \) is a set of determination since

\[
Q(2^n [1 - 2^{-n}, 1 - 2^{-n-1}], [P(z_{0,n}, 1)]) = Q\left([0, \frac{1}{2}], 2^n\right)
\]

and \( \inf_n 2^{-n} Q\left([0, \frac{1}{2}], 2^n\right) > 0 \) because \([0, \frac{1}{2}]\) is non-polar.

## 2 Proof of Theorem 1

Let \( G_U(\cdot, \cdot) \) denote the Green function of an open set \( U \), let

\[
D_\rho(z) = \left\{ \zeta : |\zeta - z| < \rho(1 - |z|) \right\} \quad (z \in \mathbb{D}, 0 < \rho < 1),
\]

and let \( A(g, z) \) denote the mean value of a function \( g \) over the disc \( D_{1/8}(z) \). For potential theoretic background we refer to the book [2].

Suppose firstly that condition (b) of Theorem 1 holds and let \( f \in \mathcal{N} \). We will assume that \( \sup_E |f| < \infty \), for otherwise it is trivially true that \( \sup_E |f| = \sup_{\mathbb{D}} |f| \). Further, multiplication by a suitable constant enables us to arrange that \( \sup_E |f| \in [0, 1] \). Now let \( a \in (-\infty, 0] \) be such that \( a \geq \log \sup_E |f| \). We can write

\[
\log |f| = h_1 - h_2 - G_{\mathbb{D}} \mu,
\]

where \( h_1 \) and \( h_2 \) are positive harmonic functions and \( \mu \) is a sum of unit point masses on \( \mathbb{D} \) satisfying

\[
\int (1 - |z|) d\mu(z) < \infty.
\]
Further, by addition to both $h_1$ and $h_2$, we may assume that $h_1 \geq 1$. By the Riesz-Herglotz theorem there is a Borel measure $\nu_1$ on $\mathbb{T}$ such that

$$h_1(z) = \int P(z, w) d\nu_1(w) \quad (z \in \mathbb{D}).$$

We know that

$$h_1 - a \leq h_2 + G_{\mathbb{D}} \mu \quad \text{on } E.$$  \hspace{1cm} (2)

Also,

$$G_{\mathbb{D}}(z, \xi) - A(G_{\mathbb{D}}(\cdot, \xi), z) \leq G_{D_1/8}(z, \xi) = \log \frac{(1 - |z|)/8}{|z - \xi|} \quad (\xi \in D_1/8(z)) \quad (3)$$

and $G_{\mathbb{D}}(z, \xi) - A(G_{\mathbb{D}}(\cdot, \xi), z) = 0$ otherwise. Let $\varepsilon \in (0, 1)$ and

$$I_\varepsilon = \{(m, n) : G_{\mathbb{D}} \mu \geq A(G_{\mathbb{D}} \mu, \cdot) + \varepsilon h_1 \text{ on } E_{m,n}\},$$

and let $I'_\varepsilon$ denote the complementary set of pairs $(m, n)$. (We note that $(m, n) \in I_\varepsilon$ whenever $E_{m,n} = \emptyset$.) If $(m, n) \in I_\varepsilon$, then we see from (3) that

$$\varepsilon h_1(z) \leq G_{\mathbb{D}} \mu(z) - A(G_{\mathbb{D}} \mu, z)$$

$$= \int_{D_1/8(z)} (G_{\mathbb{D}}(z, \xi) - A(G_{\mathbb{D}}(\cdot, \xi), z)) d\mu(\xi)$$

$$\leq \int_{A_{m,n}} \log \frac{2^{-n}}{|z - \xi|} d\mu(\xi) \quad (z \in E_{m,n}),$$

where

$$A_{m,n} = \{\xi : \text{dist}(\xi, S_{m,n}) < 2^{-n-3}\}.$$  \hspace{1cm} (4)

(Here we have used the fact that the diameter of $2^n A_{m,n}$ is less than 1.) By Harnack’s inequalities there is an absolute constant $c_1 > 1$ such that $h(\zeta_1) \leq c_1 h(\zeta_2)$ for any positive harmonic function $h$ on $\mathbb{D}$, any points $\zeta_1, \zeta_2 \in S_{m,n}$, and any choice of $(m, n)$. For any $w \in \mathbb{T}$ we thus have

$$P(z_{m,n}, w) \leq \frac{c_1}{\varepsilon h_1(z_{m,n})} P(z_{m,n}, w) \int_{A_{m,n}} \log \frac{2^{-n}}{|z - \xi|} d\mu(\xi) \quad (z \in E_{m,n}),$$

and so

$$Q(2^n E_{m,n}, [P(z_{m,n}, w)]) \leq \left( \frac{c_1}{\varepsilon h_1(z_{m,n})} P(z_{m,n}, w) + 1 \right) \mu(A_{m,n}).$$

Integration of the above inequality with respect to $d\nu_1(w)$ yields

$$\int Q(2^n E_{m,n}, [P(z_{m,n}, w)]) d\nu_1(w) \leq \left( \frac{c_1}{\varepsilon} + h_1(0) \right) \mu(A_{m,n}).$$
Since no point of $\mathbb{D}$ can lie in more than 4 of the sets $A_{m,n}$, and $1 - |z| > 2^{-n-2}$ when $z \in A_{m,n}$, we see that
\[
\int \sum_{(m,n) \in I_\varepsilon} 2^{-n} \mathcal{Q}(2^n E_{m,n}, [P(z_{m,n}, w)]) \, d\nu_1(w) \leq 2^4 \left( \frac{c_1}{\varepsilon} + h_1(0) \right) \int (1 - |z|) \, d\mu(z) < \infty,
\]
so
\[
\sum_{(m,n) \in I_\varepsilon} 2^{-n} \mathcal{Q}(2^n E_{m,n}, [P(z_{m,n}, w)]) < \infty \text{ for } \nu_1\text{-almost every } w \in \mathbb{T},
\]
and hence, by hypothesis,
\[
\sum_{(m,n) \in I'_\varepsilon} 2^{-n} \mathcal{Q}(2^n E_{m,n}, [P(z_{m,n}, w)]) = \infty \text{ for } \nu_1\text{-almost every } w \in \mathbb{T}.
\]

In view of (1) we now see that
\[
\sum_{(m,n) \in I'_\varepsilon} 2^{-2n} |w - z_{m,n}|^{-2} = \infty \text{ for } \nu_1\text{-almost every } w \in \mathbb{T}. \tag{4}
\]

For each $(m, n) \in I'_\varepsilon$ we can find $\zeta_{m,n} \in E_{m,n}$ such that
\[
G_{\mathbb{D}}\mu(\zeta_{m,n}) < A(G_{\mathbb{D}}\mu, \zeta_{m,n}) + \varepsilon h_1(\zeta_{m,n}).
\]

Let $F = \{ \zeta_{m,n} : (m, n) \in I'_\varepsilon \}$. Then
\[
(1 - \varepsilon)h_1 - a \leq h_2 + A(G_{\mathbb{D}}\mu, \cdot) \text{ on } F, \tag{5}
\]
in view of (2). Also, by (4),
\[
\int_{F_\rho} |w - z|^{-2} \, d\lambda(z) = \infty \quad (0 < \rho < 1) \tag{6}
\]
for $\nu_1$-almost every $w \in \mathbb{T}$, where $F_\rho = \bigcup_{\zeta \in F} D_\rho(\zeta)$ and $\lambda$ denotes area measure. At this point we could invoke Theorem 2 of [4], but for the sake of completeness we will extract the relevant reasoning in the next paragraph.

Let $0 < \rho < 1/8$. If $z' \in D_\rho(z)$, then by the mean value inequality
\[
G_{\mathbb{D}}\mu(z') \geq \frac{1}{\pi(\rho + 1/8)^2(1 - |z|)^2} \int_{\{\zeta : |z - z'| < (\rho + 1/8)(1 - |z|)\}} G_{\mathbb{D}}\mu(\zeta) \, d\lambda(\zeta) \geq \frac{(1/8)^2}{(\rho + 1/8)^2} A(G_{\mathbb{D}}\mu, z),
\]
and by Harnack’s inequalities
\[
\frac{1 - \rho}{1 + \rho} h_j(z) \leq h_j(z') \leq \frac{1 + \rho}{1 - \rho} h_j(z) \quad (j = 1, 2),
\]

5
so (5) yields
\[(1 - \varepsilon) \frac{1 - \rho}{1 + \rho} h_1 - a \leq \frac{1 + \rho}{1 - \rho} h_2 + (8\rho + 1)^2 G_{\mathbb{D}} \mu \quad \text{on } F_{\rho}. \tag{7}\]

Condition (6) is known to ensure that the reduced function $R_{\rho}^{F_{\varphi}}$, where
\[R_{\rho}^{F_{\varphi}} = \inf \{ v : v \text{ is positive and superharmonic on } \mathbb{D} \text{ and } v \geq u \text{ on } F_{\rho} \},\]
coinsides with $P(\cdot, w)$ (see Corollary 7.4.6 in [1]). Since this condition holds $\nu_1$-almost everywhere on $\mathbb{T}$, we have
\[R_{h_1}^{F_{\rho}} = \int R_{F_{\varphi}(\cdot, w)}^{F_{\rho}} d\nu_1(w) = \int P(\cdot, w) d\nu_1(w) = h_1.\]

Also, $h_1 \geq 1$, so $\nu_1$ majorizes normalized arclength measure on $\mathbb{T}$, and we similarly have $R_{h_1}^{F_{\rho}} \equiv 1$. Hence, on taking reductions over $F_{\rho}$, we see that the inequality in (7) extends to all of $\mathbb{D}$. (Recall that $a \leq 0$.) We can now let $\rho \to 0+$ and $\varepsilon \to 0+$ to see that $\log |f| \leq a$ on $\mathbb{D}$. It is now clear that (b) implies (a).

Next suppose that condition (b) of Theorem 1 fails. Then there exists $w_0 \in \mathbb{T}$ such that
\[\sum_{m,n} 2^{-n} q_{m,n} < \infty, \quad \text{where } q_{m,n} = Q(2^n E_{m,n}, [P(z_{m,n}, w_0)]). \tag{8}\]

For each $m, n$ we can choose points $\xi_{k,m,n}$ ($k = 1, \ldots, q_{m,n}$) such that
\[\sum_{k=1}^{q_{m,n}} \log \frac{2^n}{|z - \xi_{k,m,n}|} \geq P(z_{m,n}, w_0) - 1 \quad (z \in E_{m,n}), \tag{9}\]
and without loss of generality we can assume that $\xi_{k,m,n}$ lies in the convex hull $\text{conv}(S_{m,n})$ of $S_{m,n}$. In view of (8), the Blaschke product
\[B(z) = \prod_{k,m,n} \frac{|\xi_{k,m,n}|}{\xi_{k,m,n}} \left( \frac{\xi_{k,m,n} - z}{1 - z \bar{\xi}_{k,m,n}} \right)\]
converges on $\mathbb{D}$. There is an absolute constant $c_2 > 0$ such that
\[G_{\mathbb{D}}(z, \xi) \geq c_2 \log \frac{2^{-n}}{|\xi - z|} \quad (z, \xi \in \text{conv}(S_{m,n}))\]
for any pair $(m, n)$. For a given pair $(m_0, n_0)$ we thus have
\[-\log |B(z)| = \sum_{k,m,n} G_{\mathbb{D}}(z, \xi_{k,m,n}) \geq \sum_{k=1}^{q_{m_0,n_0}} G_{\mathbb{D}}(z, \xi_{k,m_0,n_0}) \geq c_2 \sum_{k=1}^{q_{m_0,n_0}} \log \frac{2^{-n_0}}{|\xi_{k,m_0,n_0} - z|} \quad (z \in S_{m_0,n_0})\]
\[\text{(8)}\]
so, by (9),

\[ c_2 - \log |B(z)| \geq c_2 P(z_{m_0,n_0}, w_0) \geq \frac{c_2}{c_1} P(z, w_0) \quad (z \in E_{m_0,n_0}). \quad (10) \]

Let

\[ f(z) = B(z) \exp \left( \frac{c_2}{c_1} \left( \frac{w_0 + z}{w_0 - z} \right) \right) \quad (z \in \mathbb{D}). \]

Then \( \log |f(z)| \leq (c_2/c_1) P(z, w_0) \), so \( f \in \mathcal{N} \), and certainly \( f \) is unbounded on \( \mathbb{D} \). However, \( |f| \leq e^{c_2} \) on \( E \), by (10). Hence condition (a) of Theorem 1 also fails.

References


School of Mathematical Sciences
University College Dublin
Dublin 4, Ireland.

e-mail: stephen.gardiner@ucd.ie