The Khavinson-Shapiro conjecture and polynomial decompositions

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Abstract

The main result of the paper states the following: Let ψ be a polynomial in n variables of degree t. Suppose that there exists a constant $C > 0$ such that any polynomial f has a polynomial decomposition $f = \psi q_f + h_f$ with $\Delta^k h_f = 0$ and $\deg q_f \leq \deg f + C$. Then $\deg \psi \leq 2k$. Here Δ^k is the kth iterate of the Laplace operator Δ. As an application, new classes of domains in $\mathbb{R}^n$ are identified for which the Khavinson-Shapiro conjecture holds.

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1. Introduction

A real-valued function h defined on an open set U in $\mathbb{R}^n$ is called *k-harmonic* or *polyharmonic of order k* if h is differentiable up to the order $2k$ and satisfies the equation $\Delta^k h(x) = 0$ for all $x \in U$. Here Δ denotes the Laplacian $\frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}$ and Δ^k is the kth iterate of the Laplace operator Δ. Polyharmonic functions have been studied extensively in [6], and they are useful in many branches in mathematics, see [22]. For example, in elasticity theory and dynamics of slow, viscous fluids polyharmonic functions of order 2, or more briefly, *biharmonic functions*, are very important.

Before discussing our main results we still need some notation. By $\mathbb{R}[x_1, ..., x_n]$ we denote the space of all polynomials with real coefficients in the variables $x_1, ..., x_n$. Frequently we use the fact that any polynomial ψ of degree m can be expanded into a sum of homogeneous polynomials ψ_j of degree j for $j = 0, ..., m$, and we write shortly $\psi = \psi_0 + ... + \psi_m$; here $\psi_m \neq 0$ is called the *principal part* or *leading part* of the polynomial ψ. The degree of a polynomial ψ is denoted by $\deg \psi$.

In this article we will be concerned with a conjecture (see below) which arises naturally from of the following statement proven in [25, Theorem 3] (for $k = 1$ see also [8]):

Theorem 1.1. *Let $\psi \in \mathbb{R}[x_1, ..., x_n]$ be a polynomial of degree $2k$ such that the leading part ψ_{2k} is non-negative. Then for any polynomial $f \in \mathbb{R}[x_1, ..., x_n]$ there exist unique polynomials q_f and h_f in $\in \mathbb{R}[x_1, ..., x_n]$ such that*

$$f = \psi q_f + h_f \text{ and } \Delta^k (h_f) = 0. \tag{1.1}$$

Moreover, the decomposition is degree preserving, meaning that $\deg h_f \leq \deg f$ and, consequently, $\deg q_f \leq \deg f - 2k$.

Theorem 1.1 is related to the polynomial solvability of Dirichlet-type problems. For example, let us consider the polynomial

$$\psi_0(x) = \sum_{j=1}^{n} \frac{x_j^2}{a_j^2} - 1, \tag{1.2}$$

so $E_0 := \left\{ x \in \mathbb{R}^d : \psi_0(x) < 0 \right\}$ is an ellipsoid. Then the decomposition (1.1) (where $k = 1$) shows the well known and old fact that for any polynomial f, restricted to the boundary ∂E_0, there exists a harmonic *polynomial* h which coincides with the data function f on ∂E_0. In other words: the solutions for polynomial data functions of the Dirichlet problem for the ellipsoid are again polynomials, see [7], [9], [12], or [20].

In [20] D. Khavinson and H.S. Shapiro formulated the following two conjectures (i) and (ii) for bounded domains Ω for which the Dirichlet problem is solvable:

(KS): Ω is an ellipsoid if for every polynomial f the solution of the Dirichlet problem u_f is (i) a polynomial and, respectively, (ii) entire.

Conjectures (i) and (ii) are still open, but important contributions have been made by several authors. Most of the results are proven for the two-dimensional case, see e.g. [12], [13], [23] and [17]. M. Putinar and N. Stylianopoulos have shown recently in [24] that the conjecture (i) for a simply connected bounded domain Ω in the complex plane is true if and only if the Bergman orthogonal polynomials satisfy a finite recurrence relation. D. Khavinson and N. Stylianopoulos proved among other things that the Bergman orthogonal polynomials satisfy a recurrence relation of order $N+1$ if and only if conjecture (i) holds and a degree condition for the solution u_f is satisfied, for details and further discussion see [21]. In [25] the second author has given a solution for (i) and (ii) for arbitrary dimension and for a large but not exhaustive class of domains.

The authors believe that the validation of the following conjecture for the case $k=1$ would be an important step for proving the Khavinson-Shapiro conjecture (e.g. confer the proof of Theorem 27 in [25]):

Conjecture 1.2. *Suppose* $\psi \in \mathbb{R}[x_1, ..., x_n]$ *is a polynomial, such that every polynomial* $f \in \mathbb{R}[x_1, ..., x_n]$ *has a decomposition* $f = \psi q_f + h_f$*, where* h_f *is polyharmonic of order* k*. Then* $\deg \psi \leq 2k$*.*

We are able to prove the conjecture if we add a degree condition on the involved polynomials which is in the spirit of the above-mentioned work [21]. More precisely, the main result of the present paper is the following:

Theorem 1.3. *Let* $\psi \in \mathbb{R}[x_1, ..., x_n]$ *be a polynomial. Suppose that there exists a constant* $C > 0$ *such that for any polynomial* $f \in \mathbb{R}[x_1, ..., x_n]$ *there exists a decomposition* $f = \psi q_f + h_f$ *with* $\Delta^k h_f = 0$ *and*

$$\deg q_f \leq \deg f + C. \tag{1.3}$$

Then $\deg \psi \leq 2k$*.*

Theorem 1.3 will be a consequence of a somewhat stronger result proved in Section 3 after a short discussion of harmonic divisors in Section 2. In passing we note that the conjecture 1.2 does not hold for polynomials ψ with complex coefficients, see [18].

It is a natural question under which conditions at the given polynomial $\psi(x)$ the degree condition in Theorem 1.3 is automatically satisfied. In other words, can we conclude from the equation

$$f = \psi q_f + h_f \text{ with } \Delta^k h_f = 0$$

a restriction on the degree of q_f or h_f in terms of the degree of f? For the case $k = 1$ we shall prove in Section 4 that the degree condition (1.3) is satisfied if (i) the leading part ψ_t of ψ contains a non-negative non-constant factor or (ii) ψ has a homogeneous expansion of the form $\psi = \psi_t + \psi_s + ... + \psi_0$ where $\psi_s \neq 0$ contains a non-negative non-constant factor. An extension of these results for arbitrary k is also given. These results allow to identify new types of domains in $\mathbb{R}^n$ for which the Khavinson-Shapiro conjecture is true.

2. Fischer operators and harmonic divisors

For $Q \in \mathbb{R}[x_1, ..., x_n]$ let us define $Q(D)$ as the differential operator replacing a monomial x^α appearing in Q by the differential operator $\partial^\alpha / \partial x^\alpha$, where α is a multi-index. For two polynomials Q and ψ we call the operator $F_\psi^Q : \mathbb{R}[x_1, ..., x_n] \to \mathbb{R}[x_1, ..., x_n]$ defined by

$$F_\psi^Q(q) := Q(D)(\psi q) \qquad q \in \mathbb{R}[x_1, ..., x_n] \tag{2.1}$$

the *Fischer operator*; for the significance of this notion we refer to the excellent exposition [26], or [8], [25]. We shall need the following result due to E. Fischer [16] which is in a slightly modified form valid for polynomials with complex coefficients, see [26]:

Theorem 2.1. *Let $Q \in \mathbb{R}[x_1, ..., x_n]$ be a homogeneous polynomial. Then the operator $q \longmapsto Q(D)(Qq)$ is bijective.*

At first we observe that the conjecture 1.2 is equivalent to the surjectivity of the Fischer operator with $Q = (\sum_{i=1}^n x_i^2)^k$; this fact is well known but for convenience of the reader we include the short proof.

Proposition 2.2. *Suppose $k \in \mathbb{N}$ and ψ is a polynomial. The operator*

$$F_{\psi}^{k}(q) := \Delta^{k}(\psi q)$$

is surjective if and only if every polynomial f can be decomposed as $f = \psi q_f + h_f$, where h is polyharmonic of order k.

Proof. Taking Δ^k of both sides of $f = \psi q + h$ gives $\Delta^k f = F_{\psi}^{k}(q)$. Given g we can find f such that $g = \Delta^k f$, showing F_{ψ}^{k} is surjective. Conversely, if F_{ψ}^{k} is surjective, then given f there is a q such that $\Delta^k f = F_{\psi}^{k}(q)$, showing that $h = f - \psi q$ is polyharmonic of order k. □

A polynomial f_m is called *homogeneous* of degree m if $f_m(rx) = r^m f_m(x)$ for all $r > 0$ and for all $x \in \mathbb{R}^n$. We will use $\mathbb{P}^N$ to denote the space of polynomials of degree at most N, and $\mathbb{P}_{\text{hom}}^{N}$ the space of homogeneous polynomials of degree N. For a homogeneous polynomial ψ we define the space of all homogeneous k-harmonic divisors of degree m of ψ by

$$D_k^m(\psi) = \left\{ q \in \mathbb{P}_{\text{hom}}^{m} : \Delta^k(\psi q) = 0 \right\}.$$

For $k = 1$ we obtain the definition of a harmonic divisor (of degree m) which arises in the investigation of stationary sets for the wave and heat equation, see [2], [3], and the injectivity of the spherical Radon transform, see [4], [1]. It is an interesting but difficult problem to compute the dimension of the space $D_k^m(\psi)$ in dependence of the polynomial ψ. In the proof of our main result Theorem 1.3 we shall use the rough upper estimate provided in the next proposition and the remarks following:

Proposition 2.3. *Let $\psi \in \mathbb{R}[x_1, ..., x_n]$ be a homogeneous polynomial. Then*

$$\dim D_k^m(\psi) \leq \dim \left\{ f \in \mathbb{P}_{\text{hom}}^{m} : \Delta^k f = 0 \right\}.$$

Proof. Let $q \in D_k^m(\psi)$. Then $q \in \mathbb{P}_{\text{hom}}^{m}$ and $q\psi = h$ for some $h \in \mathbb{P}_{\text{hom}}^{m+t}$ with $\Delta^k h = 0$, where t is the degree of ψ. Clearly we have $\psi(D)(\psi q) = \psi(D) h$ and

$$0 = \psi(D)\left(\Delta^k h\right) = \Delta^k(\psi(D) h). \tag{2.2}$$

By Theorem 2.1 the operator F defined by $F(q) = \psi(D)(\psi q)$ is bijective, and from $\psi q = h$ we infer that $q = F^{-1}(w)$ with $w := \psi(D) h$. Equation (2.2) shows that $w \in \left\{ f \in \mathbb{P}_{\text{hom}}^{m} : \Delta^k f = 0 \right\}$. Thus

$$D_k^m(\psi) \subset F^{-1}\left(\left\{ f \in \mathbb{P}_{\text{hom}}^{m} : \Delta^k f = 0 \right\}\right).$$

Since F^{-1} is a bijective operator the claim is now obvious. □

Let us define $H_k^m := \left\{ f \in \mathbb{P}_{\text{hom}}^m : \Delta^k f = 0 \right\}$. By Theorem 2.1 for $Q(x) = |x|^{2k}$ it follows that any polynomial f has a Fischer decomposition $f = |x|^{2k} q + h$ where h is k-harmonic. Moreover, h and q are homogeneous iff f is. So we have

$$\mathbb{P}_{\text{hom}}^m = |x|^{2k} \mathbb{P}_{\text{hom}}^{m-2k} \oplus H_k^m.$$

Thus we obtain

$$\dim D_k^m(\psi) \leq \dim H_k^m = \dim \mathbb{P}_{\text{hom}}^m - \dim \mathbb{P}_{\text{hom}}^{m-2k}. \tag{2.3}$$

The following question was posed by M. Agranovsky for the case $k = 1$ in [1], where it was also answered in the case that ψ factors completely into linear factors.

Question 2.4 (Agranovksy)**.** *What is the asymptotic behavior of* $\dim D_k^m(\psi)$, *as* $m \to \infty$*?*

We expect that a full answer to this question would allow to relax the asumption on degree appearing in Theorem 1.3.

3. Proof of the main result

Assume that $2k \leq t$ and let ψ be a polynomial of degree $\leq t$ and let F_ψ^k be the Fischer operator defined in Proposition 2.2. The following technical notion is a crucial tool for proving our main result Theorem 1.3: For a natural number M define $S_i \subset \mathbb{P}^i$ as the subspace whose image under F_ψ^k is contained in $\mathbb{P}^{M+t-2k}$, i.e.,

$$S_i := \{ q \in \mathbb{P}^i : F_\psi^k(q) \in \mathbb{P}^{M+t-2k} \}$$

for $i \in \mathbb{N}_0$. Since ψ has degree $\leq t$ it follows that

$$\mathbb{P}^M = S_M \subset S_{M+1} \subset \ldots \subset S_{M+j}$$

for all $j \geq 1$.

Proposition 3.1. *Let* $\psi = \psi_t + \ldots + \psi_0$ *be a polynomial of degree* $\leq t$ *and let* M *be a natural number. Then for all* $j \in \mathbb{N}$

$$\dim S_{M+j} \leq \dim S_{M+j-1} + \dim D_k^{M+j}(\psi_t).$$

Proof. For given $j \in \mathbb{N}$ we will construct a space Q_j such that $S_{M+j} = S_{M+j-1} \oplus Q_j$, and $\dim Q_j \leq \dim D_k^{M+j}(\psi_t)$. First define $Q_{H,j} := \{q_{M+j} : q_{M+j}$ is the degree-$(M+j)$ homogeneous term of some $q \in S_{M+j}\}$. Choose (finitely many) polynomials in S_{M+j} whose leading terms form a basis for $Q_{H,j}$, and define Q_j to be the subspace of S_{M+j} spanned by these polynomials. Suppose $\hat{q} \in S_{M+j}$. The degree-$(M+j)$ homogeneous term $\hat{q}_{M+j}$ (possibly zero) can be matched by the leading homogeneous term of some $q \in Q_j$ so that $\hat{q} - q \in S_{M+j-1}$. This shows that $S_{M+j} = S_{M+j-1} \oplus Q_j$.

Now, we will establish $\dim Q_j \leq \dim D_k^{M+j}(\psi_t)$. It suffices to show that $Q_{H,j} \subset D_k^{M+j}(\psi_t)$, since $\dim Q_j = \dim Q_{H,j}$ by construction. Suppose $q_{M+j} \in Q_{H,j}$ is nonzero, i.e., there is a $q \in S_{M+j}$ and $\deg q = M+j$ such that q_{M+j} is the leading homogeneous term of q. Since $F_\psi^k(q) \in \mathbb{P}^{M+t-2k}$, we have $\deg(\Delta^k(\psi q)) \leq M+t-2k$. This implies that the leading term, $\Delta^k(\psi_t q_{M+j})$, of $\Delta^k(\psi q)$ is zero (since it has degree $M+j+t-2k$). i.e., $\psi_t q_{M+j}$ is k-harmonic. Therefore, $Q_{H,j} \subset D_k^{M+j}$. □

The main result of this paper, Theorem 1.3, follows now from the following more general result by taking $\alpha = 1$:

Theorem 3.2. *Let ψ be a polynomial of degree t. Suppose that there exist constants $\alpha \geq 1$, $C > 0$ such that for any polynomial f there exists a decomposition $f = \psi q_f + h_f$ with $\Delta^k h_f = 0$ and*

$$\deg q_f \leq \alpha \deg f + C.$$

Then $t \leq 2k \cdot \alpha^{n-1}$.

Proof. Suppose $t \geq 2k$. (If $t < 2k$, there is nothing to prove.) Let $f \in \mathbb{P}^{M+t-2k}$ and suppose that $M > 2k$. Choose a polynomial $g \in \mathbb{P}^{M+t}$ with $\Delta^k g = f$. By assumption there exists q_f and h_f with $g = \psi q_f + h_f$ and $\Delta^k h_f = 0$ and $\deg q_f \leq \alpha(M+t) + C$. Then $f = \Delta^k g = F_\psi^k(q_f)$ and we infer the inclusion

$$\mathbb{P}^{M+t-2k} \subset F_\psi^k\left(\mathbb{P}^{B_M}\right) \tag{3.1}$$

with $B_M := \alpha M + \alpha t + C \geq M$. Using the above notation $S_{B_M} = \{q \in \mathbb{P}^{B_M} : F_\psi^k(q) \in \mathbb{P}^{M+t-2k}\}$ we see that (3.1) implies that $\mathbb{P}^{M+t-2k} \subset F_\psi^k(S_{B_M})$. Since F_ψ^k is a linear operator, we have

$$\dim \mathbb{P}^{M+t-2k} \leq \dim F_\psi^k(S_{B_M}) \leq \dim S_{B_M}. \tag{3.2}$$

Applying Proposition 3.1 inductively we obtain

$$\dim S_{B_M} \leq \dim(\mathbb{P}^M) + \sum_{j=M+1}^{B_M} \dim D_k^j(\psi_t) \tag{3.3}$$

Since $\mathbb{P}^{M+t-2k} = \mathbb{P}^M \oplus \mathbb{P}_{\text{hom}}^{M+1} \oplus ... \oplus \mathbb{P}_{\text{hom}}^{M+t-2k}$ and $\dim \mathbb{P}_{\text{hom}}^{M+1} \leq \dim \mathbb{P}_{\text{hom}}^{M+j}$ for $j \geq 1$ we infer from (3.2) and (3.3) the interesting formula

$$(t-2k) \dim \mathbb{P}_{\text{hom}}^{M+1} \leq \sum_{j=M+1}^{B_M} \dim D_k^j(\psi_t)\,. \tag{3.4}$$

Further we know from (2.3) that $\dim D_k^j(\psi_t) \leq \dim \mathbb{P}_{\text{hom}}^j - \dim \mathbb{P}_{\text{hom}}^{j-2k}$. Thus the right hand side in (3.4) is a telescoping sum. Using that $\dim \mathbb{P}_{\text{hom}}^j \leq \dim \mathbb{P}_{\text{hom}}^{B_M}$ for $j = B_M - 2k + 1, ..., B_M$ and $\dim \mathbb{P}_{\text{hom}}^{M+1-2k} \leq \dim \mathbb{P}_{\text{hom}}^j$ for the lower indices we can estimate

$$\sum_{j=M+1}^{B_M} \dim D_k^j(\psi_t) \leq 2k \dim \mathbb{P}_{\text{hom}}^{B_M} - 2k \dim \mathbb{P}_{\text{hom}}^{M+1-2k}.$$

Thus we infer from (3.4) and the well known fact

$$\dim \mathbb{P}_{\text{hom}}^{M+1} = \binom{n+M}{n-1} = \binom{n+M}{M+1},$$

proven in [7] that

$$\begin{aligned} & (t-2k)\,\frac{(M+2)\,...\,(M+n)}{(n-1)!} \\ \leq \quad & 2k\frac{(B_M+1)\,...\,(B_M+n-1)-(M+2-2k)\,....\,(M+n-2k)}{(n-1)!} \end{aligned}$$

Clearly the term $(n-1)!$ can be canceled in the inequality. Divide the inequality by M^{n-1} on both sides and recall that $B_M = \alpha M + \alpha t + C$. Now take the limit $M \to \infty$ and we obtain

$$t - 2k \leq 2k\left(\alpha^{n-1} - 1\right).$$

This implies $t \leq 2k\alpha^{n-1}$ and the proof is complete. □

4. Criteria for degree-related decompositions

We are now turning to the question under which conditions the degree condition is automatically satisfied. The first criterion is simple to prove:

Proposition 4.1. *Suppose that ψ is a polynomial of degree $t > 2$ and $\psi = \psi_t + ... + \psi_0$ is the decomposition into a sum of homogeneous polynomials. Assume the polynomial ψ_t contains a non-negative non-constant factor. Let f be a polynomial and assume that there exists a decomposition*

$$f = \psi q + h$$

where h is harmonic and q is a polynomial. Then $\deg q \leq \deg f - t$ and $\deg h \leq \deg f$.

Proof. Write $q = q_M + ... + q_0$ with homogeneous polynomials q_j of degree $j = 0, ..., M$. Expand the product ψq into a sum of homogeneous polynomials, so $\psi q = \psi_t q_M + R(x)$ where $R(x)$ is a polynomial of degree $< M + t$. Suppose that $M + t > \deg f$. Since $\Delta f = \Delta(\psi q)$ we conclude that $\Delta(\psi_t q_M) = 0$, so $\psi_t q_M$ is harmonic. By the Brelot-Choquet theorem, a harmonic polynomial cannot have non-negative factors, see [11]. Thus $\psi_t q_M = 0$, and we obtain a contradiction. □

The next criterion is more difficult to prove and uses again ideas from the proof of the Brelot-Choquet theorem:

Theorem 4.2. *Suppose that ψ is a polynomial of degree $t > 2$ and $\psi = \psi_t + \psi_s + \psi_{s-1} + ... + \psi_0$ is the decomposition into a sum of homogeneous polynomials. Assume the polynomial ψ_s is non-zero and contains a non-negative non-constant factor. Let f be a polynomial and assume that there exists a decomposition*

$$f = \psi q + h$$

where h is harmonic and q is a polynomial. Then $\deg q \leq 2 - s + \deg f$ and $\deg h \leq t + 2 - s + \deg f$.

Before proving Theorem 4.2 we notice the following conclusion:

Corollary 4.3. *Suppose that ψ is a polynomial with a non-zero second-highest degree term that contains a non-negative factor. If every polynomial f has a Fischer decomposition $f = \psi q_f + h_f$ with h_f harmonic, then $\deg(\psi) \leq 2$.*

Proof. Suppose $\deg(\psi) > 2$. By Theorem 4.2, $\deg q_f - \deg f$ is bounded. Now we can apply Theorem 1.3, to obtain $\deg \psi \leq 2$. □

The following lemma is needed for the proof of Theorem 4.2:

Lemma 4.4. *Suppose that ψ is a polynomial of degree $t > 2$ and $\psi = \psi_t + \psi_s + \psi_{s-1} + ... + \psi_0$ is the decomposition into a sum of homogeneous polynomials. Assume that $g \in \mathbb{P}^m$ and q is a polynomial of degree M such that $F_\psi^k(q) := \Delta(\psi q) = g$ and $M + s > m$. Then for every $p \in \mathbb{P}^{s-1}$,*

$$\int_{\mathbb{S}^{n-1}} q_M^2 \cdot \psi_s \cdot p \; d\theta = 0,$$

where $q_M \neq 0$ is the senior term of q.

Proof (of lemma). Write $q = q_M + ... + q_0$ with homogeneous polynomials q_j of degree $j = 0, ..., M$. Expand the product ψq into a sum of homogeneous polynomials,

$$\psi q = \psi_t q_M + ... + \psi_t q_{M-t+s+1} + (\psi_t q_{M-t+s} + \psi_s q_M) + R(x) \tag{4.1}$$

where $R(x)$ is a polynomial of degree $< M+s$. Since $\Delta(\psi q) = g$ and $M+s > m$, we conclude that $\Delta(\psi_t q_M) = 0$ and $\Delta(\psi_t q_{M-t+s} + \psi_s q_M) = 0$. Thus, we can write

$$\psi_t q_M = h_{M+t} \tag{4.2}$$

$$\psi_t q_{M-t+s} + \psi_s q_M = h_{M+s}, \tag{4.3}$$

where h_{M+t} and h_{M+s} are homogeneous harmonic polynomials.

Take $p \in \mathbb{P}^{s-1}$, and multiply equation (4.3) by $q_M p$ and integrate over the unit sphere, $\mathbb{S}^{n-1}$. Then

$$\int_{\mathbb{S}^{n-1}} \psi_t q_{M-t+s} \cdot q_M p \; d\theta + \int_{\mathbb{S}^{n-1}} \psi_s q_M^2 \cdot p \; d\theta = \int_{\mathbb{S}^{n-1}} h_{M+s} \cdot q_M p \; d\theta.$$

Since $\deg(q_M p) < M+s$ and h_{M+s} is harmonic, the integral on the right-hand side is zero. Indeed, homogeneous harmonics of different degree are orthogonal in the space $L^2(\mathbb{S}^{n-1})$ (see [7]), and, moreover, $q_M p$ can be matched on $\mathbb{S}^{n-1}$ by a harmonic polynomial of not higher degree. Substituting equation 4.2 into the first integral on the left-hand side gives $\int_{\mathbb{S}^{n-1}} h_{M+t} \cdot p \cdot q_{M-t+s} d\theta$, which is also zero, since $\deg(p q_{M-t+s}) < M + t$. □

Proof of Theorem 4.2. By assumption we may write $\psi_s = \phi P$ where ϕ is non-negative and P has degree $< s$. Suppose that $M + s > \deg f + 2$. We have $\Delta(\psi q) = \Delta f$ and $M + s > deg(\Delta f)$. Then, q, ψ satisfy Lemma 4.4 with $g = \Delta f$, and thus $\int_{\mathbb{S}^{n-1}} q_M^2 \cdot \psi_s \cdot p \, d\theta = 0$, for all p of degree $< s$. In particular, this is true for $p = P$. Hence,

$$0 = \int_{\mathbb{S}^{n-1}} q_M^2 \cdot \psi_s \cdot P d\theta = \int_{\mathbb{S}^{n-1}} q_M^2 \cdot \phi \cdot P^2 d\theta.$$

Since $P \neq 0$, $\phi \neq 0$, and $\phi(\theta) \geq 0$ for all $\theta \in \mathbb{S}^{n-1}$, we have the contradiction $q_M = 0$. □

The following instructive example is due to L. Hansen and H.S. Shapiro [17]; it was also suggested in [19] as a simple example for which the Khavinson-Shapiro conjecture is unresolved (whenever φ is a cubic): Let $\varphi \in \mathbb{R}[x_1, ..., x_n]$ be a homogeneous harmonic polynomial of degree > 2, in particular φ does not contain a nonnegative non-constant factor, see [11]. We perturb the equation for the unit ball $|x|^2 - 1$ by $\varepsilon\varphi$, i.e. we consider

$$\psi_\varepsilon (x) := |x|^2 - 1 + \varepsilon\varphi (x) \ \text{ for } \varepsilon > 0. \tag{4.4}$$

If $\varepsilon > 0$ is small enough, then the component of $E_\varepsilon := \{\psi_\varepsilon < 0\}$ containing 0 is a bounded domain in $\mathbb{R}^d$. Then the Dirichlet problem for the data function $|x|^2 = x_1^2 + ... + x_n^2$ restricted to ∂E_ε has a harmonic polynomial solution $u_f (x) = 1 - \varepsilon\varphi (x)$ since

$$|x|^2 = \psi_\varepsilon (x) \cdot 1 + 1 - \varepsilon\varphi (x) .$$

Note that in this example the degree of the solution u_f for the Dirichlet problem is higher than the degree of the data function f.

The question arises whether any polynomial data function may have a polynomial solution. If this is the case, and ψ_ε is irreducible and changes the sign in a neighborhood of some point in ∂E_ε then the proof of Theorem 27 in [25] implies that for any polynomial f there exists a decomposition $f = \psi_\varepsilon q_f + h_f$ where h_f is harmonic. By Corollary 4.3 $\deg \psi_\varepsilon \leq 2$. Thus we have proved that for this class of examples the Khavinson-Shapiro conjecture is true.

In the rest of this section we extend Theorem 4.2 to the case $k \geq 1$. We consider the following inner product

$$\langle f, g\rangle := \int_{\mathbb{R}^n} f(x)g(x)e^{-|x|^2} dx \tag{4.5}$$

and the following orthogonality condition established in [25].

Theorem 4.5. *Suppose that f is a homogeneous polynomial, and let $k \in \mathbb{N}$ with $2(k-1) \leq$ deg f. Then $\Delta^k f = 0$ if and only if $\langle f, g\rangle = 0$ for all polynomials g with $2(k-1)+$ deg $g <$ deg f.*

Theorem 4.6. *Suppose that ψ is a polynomial of degree $t > s$ and $\psi = \psi_t + \psi_s + \psi_{s-1} + \ldots + \psi_0$ is the decomposition into a sum of homogeneous polynomials. Assume the polynomial $\psi_s \neq 0$ is non-negative. If the polynomial f has the decomposition*

$$f = \psi q + h$$

where h is k-harmonic, then $\deg(q) \leq 2k - s + \deg f$.

Proof. Suppose that $M + s > 2k + \deg f$, where $f = \psi q + h$ and $M = \deg q$. We will derive a contradiction. We proceed as in the proof of Lemma 4.4 writing $q = q_M + \ldots + q_0$ with homogeneous polynomials q_j of degree $j = 0, \ldots, M$. Expand the product ψq as in (4.1). Then we conclude that $\Delta^k(\psi_t q_M) = 0$ and $\Delta^k(\psi_t q_{M-t+s} + \psi_s q_M) = 0$. Thus, we can write

$$\psi_t q_M = H_{M+t} \tag{4.6}$$

$$\psi_t q_{M-t+s} + \psi_s q_M = H_{M+s}, \tag{4.7}$$

where H_{M+t} and H_{M+s} are homogeneous k-harmonic polynomials. Next take the inner product (4.5) of both sides of equation (4.7) with q_M. Then

$$\langle q_{M-t+s}, q_M \psi_t\rangle + \langle \psi_s, q_M^2\rangle = \langle H_{M+s}, q_M\rangle$$

Using equation 4.6, we arrive at $\langle q_{M-t+s}, H_{M+t}\rangle + \langle \psi_s, q_M^2\rangle = \langle H_{M+s}, q_M\rangle$. Now we use Theorem 4.5. Since $\deg H_{M+t} > \deg q_{M-t+s} + 2(k-1)$ and $\deg H_{M+s} > \deg q_M + 2(k-1)$, the first term on the left and the term on the right are both zero. Thus, $\langle \psi_s, q_M^2\rangle = 0$ implies $q_M = 0$ (since $\psi \neq 0$ is non-negative), a contradiction. □

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