Title	On a class of singular elliptic systems
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Publication date	2015-06
Publication information	Ghergu, Marius. "On a Class of Singular Elliptic Systems" 119 (June, 2015).
Publisher	Elsevier
Item record/more information	http://hdl.handle.net/10197/7191
Publisher's statement	This is the author's version of a work that was accepted for publication in Nonlinear Analysis: Theory Methods and Applications. Changes resulting from the publishing process, such as peer review, editing, corrections, structural formatting, and other quality control mechanisms may not be reflected in this document. Changes may have been made to this work since it was submitted for publication. A definitive version was subsequently published in Nonlinear Analysis: Theory Methods and Applications (VOL 119, (2015)) DOI: 10.1016/j.na.2014.08.013.
Publisher's version (DOI)	10.1016/j.na.2014.08.013

On a class of singular elliptic systems

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Abstract

We study the semilinear elliptic system

$$\begin{cases} -\Delta u = u^{-p} + v^{-q}\ ,\ u > 0 & \text{in } \Omega,\\ -\Delta v = u^{-r} + v^{-s}\ ,\ v > 0 & \text{in } \Omega,\\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a smooth and bounded domain, $p, q, r, s > 0$. Under suitable ranges of exponents we obtain various results regarding the well posedness of our system.

Key words: Singular elliptic system, negative exponents, boundary behavior

1991 MSC: 35J55, 35B40, 35B50

1 Introduction and the main results

We are concerned in this paper with qualitative properties of solutions to the system

$$\begin{cases} -\Delta u = u^{-p} + v^{-q}\ ,\ u > 0 & \text{in } \Omega,\\ -\Delta v = u^{-r} + v^{-s}\ ,\ v > 0 & \text{in } \Omega,\\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a smooth and bounded domain, $p, q, r, s > 0$.

Solutions (u, v) to (1) are understood in the classical sense, that is, $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$.

The system (1) appears as a natural extension of the single singular problem

$$\begin{cases} -\Delta u = u^{-p}\ ,\ u > 0 & \text{in } \Omega,\\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

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which was considered, among other works, in [3,9,15]. A particular feature of (2) in the case $p > 0$, and in contrast to the case $p < -1$ is that it has a unique solution. This fact will be used in dealing with (1) in order to study the existence of solutions.

Another singular elliptic system recently investigated in the literature is

$$\begin{cases} -\Delta u = u^{-p}v^{-q} \ , \ u > 0 & \text{in } \Omega, \\ -\Delta v = u^{-r}v^{-s} \ , \ v > 0 & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

The case $p, q, r, s > 0$ represents the singular counterpart of the standard Lane-Emden system and was discussed in [5,6,10,14,17].

The case $q, s > 0$ and $p, r < 0$ corresponds to the Gierer-Meinhardt system [11,12] with homogeneous Dirichlet boundary conditions (see [1,2,4,7–10]). Such a system describes the pattern formation of spatial tissue structures in morphogenesis, a biological phenomenon discovered by Trembley [16] in 1744.

Coming back to our system (1), we first state a nonexistence result.

Theorem 1.1 (Nonexistence) *Assume that one of the following two conditions hold.*

(i) $2r < 1 + \max\{1, s\}$ *and*

$$\min\left\{\frac{1}{1+\max\{1,s\}}, 1 - \max\left\{\frac{1}{2}, \frac{r}{1+\max\{1,p\}}\right\}\right\} > \frac{1}{q}, \tag{4}$$

or

(ii) $2q < 1 + \max\{1, p\}$ *and*

$$\min\left\{\frac{1}{1+\max\{1,p\}}, 1 - \max\left\{\frac{1}{2}, \frac{q}{1+\max\{1,s\}}\right\}\right\} > \frac{1}{r}. \tag{5}$$

Then, the systems (1) has no solutions.

Corollary 1.2 *Assume that one of the following conditions hold:*

$$q > 1 + \max\{1, s\} \quad \text{and} \quad 2r < 1 + \max\{1, p\},$$

or

$$r > 1 + \max\{1, p\} \quad \text{and} \quad 2q < 1 + \max\{1, s\}.$$

Then, the systems (1) has no solutions.

More clearly but perhaps less precise, Corollary 1.2 states that if one of the exponents q and r is too small and the other is too big, then the systems (1) has no solutions.

In particular, from Corollary 1.2 we deduce that the system (1) has no solutions if

$$q > 2 + s \quad \text{and} \quad 2r < 1 + p,$$

or

$$r > 2 + p \quad \text{and} \quad 2q < 1 + s.$$

We shall next be concerned with the existence of a solution to (1). Our main result in this case is the following.

Theorem 1.3 (Existence) *Assume $p, q, r, s > 0$ satisfy*

$$q < 1 + \max\{1, s\} \quad \textit{and} \quad r < 1 + \max\{1, p\}. \tag{6}$$

Then, the system (1) has at least one classical solution.

Corollary 1.4 *Assume $0 < q, r < 2$. Then, the system (1) has at least one classical solution.*

In other words, and in contrast to Corollary 1.2, if q and r are both small, then a classical solution to system (1) always exists, regardless to the size of p and s.

We should point out that there are regions for exponents $p, q, r, s > 0$ where we do not know whether the system (1) admits solutions. For instance, if

$$q > \max\{1, s\} \quad \text{and} \quad r > 1 + \max\{1, p\}$$

then, none of the conditions (4), (5) or (6) hold. In particular, for q and r large enough, we are not able to decide the (non)existence of a solution to (1).

Theorem 1.5 (C^1-regularity of solutions up to the boundary)

Let (u, v) be a classical solution of (1).

(i) If $p < 1$ and $2q < 1 + \max\{1, s\}$ then $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$;
(ii) If $s < 1$ and $2r < 1 + \max\{1, p\}$ then $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

We are next concerned with the uniqueness of a solution to system (1). For the singular systems considered in [4–6,8] the uniqueness of the solution was deduced for some ranges of exponents p, q, r and s which imply either $u \in C^1(\overline{\Omega})$ or $v \in C^1(\overline{\Omega})$.

Theorem 1.6 (Uniqueness) *Assume $p, q, r, s > 0$ satisfy (6) and that one of the following holds.*

(i) $\dfrac{q}{1 + \max\{1, s\}} < \dfrac{\max\{1, p\}}{1 + \max\{1, p\}}$ *and* $qr < 1$;
(ii) $\dfrac{r}{1 + \max\{1, p\}} < \dfrac{\max\{1, s\}}{1 + \max\{1, s\}}$ *and* $qr < 1$.

Then, the system (1) has a unique classical solution.

Corollary 1.7 *Assume $p, q, r, s > 0$ satisfy*

$$2q < 1 + \max\{1, s\}\,, \qquad 2r < 1 + \max\{1, p\} \quad \text{and} \quad qr < 1. \tag{7}$$

Then, the system (1) has a unique classical solution.

Corollary 1.8 *Assume $p, s > 0$ and $0 < q, r < 1$. Then, the system (1) has a unique classical solution.*

2 Some preliminary results

In this section we collect some basic results which will be useful in proving our main results. In the sequel Ω will be assumed to be a smooth and bounded domain of $\mathbb{R}^N$. We also denote by $\delta(x)$ the distance from $x \in \Omega$ to the boundary $\partial\Omega$. Given two positive functions f, g defined in Ω, we shall use $f \sim g$ to signify that $c^{-1}f \leq g \leq cf$ in Ω for some constant $c > 1$.

We start this section with the following comparison principle whose proof relies on the maximum principle.

Lemma 2.1 *Let $K \in C(\Omega)$ and assume $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy*

(i) $\Delta u + u^{-p} + K(x) \leq 0 \leq \Delta v + v^{-p} + K(x)$ *in* Ω,
(ii) $u, v > 0$ *in* Ω *and* $v = 0$ *on* $\partial\Omega$.

Then $u \geq v$ in Ω.

The following result stems from Crandall, Rabinowitz and Tartar [3].

Lemma 2.2 *For any $p > 0$ there exists a unique solution $w \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

$$\begin{cases} -\Delta w = w^{-p} & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

Furthermore, w has the following asymptotic behavior

$$w(x) \sim \begin{cases} \delta(x) & \text{if } 0 < p < 1, \\ \delta(x) \log^{1/2} \dfrac{1}{\delta(x)} & \text{if } p = 1, \\ \delta(x)^{\frac{2}{1+p}} & \text{if } p > 1. \end{cases} \tag{9}$$

In particular, there exists $C > 0$ such that

$$w \geq C\delta(x)^{\frac{2}{1+\max\{1,p\}}} \quad \text{in } \Omega, \tag{10}$$

and for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$w \le C_\varepsilon \delta(x)^{\frac{2-\varepsilon}{1+\max\{1,p\}}} \quad \text{in } \Omega. \tag{11}$$

Lemma 2.3 *Let $w \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy*

$$0 \le -\Delta w \le C\delta(x)^{-a} \quad \text{in } \Omega\,, \quad w = 0 \text{ in } \partial\Omega,$$

where $a \in (0,2)$ and $C > 0$. Then, $w \in C^{0,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$ and

(i) If $a \in (0,1)$ then $w \in C^2(\Omega) \cap C^{1-a}(\overline{\Omega})$ and $w(x) \le c\delta(x)$ in Ω, for some $c > 0$;
(ii) If $a = 1$ then for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that $w(x) \le c_\varepsilon \delta(x)^{1-\varepsilon}$ in Ω;
(iii) If $a \in (1,2)$ then $w(x) \le c\delta(x)^{2-a}$ in Ω, for some $c > 0$.

Proof. Let $\mathcal{G}$ be the Green's function for the negative Laplace operator. This yields

$$u(x) = -\int_\Omega \mathcal{G}(x,y)\Delta u(y)dy,$$

for all $x \in \Omega$. Hence

$$\begin{aligned} |u(x_1) - u(x_2)| &\le -\int_\Omega |\mathcal{G}(x_1,y) - \mathcal{G}(x_2,y)|\Delta u(y)dy \\ &\le c\int_\Omega |\mathcal{G}(x_1,y) - \mathcal{G}_x(x_2,y)|\delta(y)^{-a}dy, \end{aligned}$$

for all $x_1, x_2 \in \Omega$. Next, using the method in [13, Theorem 1.1] we have

$$|u(x_1) - u(x_2)| \le C|x_1 - x_2|^\gamma \quad \text{for some } 0 < \gamma < 1.$$

Hence $u \in C^{0,\gamma}(\overline{\Omega})$.

(i) Suppose $0 < a < 1$. Then,

$$\nabla u(x) = -\int_\Omega \mathcal{G}_x(x,y)\Delta u(y)dy \quad \text{for all } x \in \Omega,$$

and

$$\begin{aligned} |\nabla u(x_1) - \nabla u(x_2)| &\le -\int_\Omega |\mathcal{G}_x(x_1,y) - \mathcal{G}_x(x_2,y)|\Delta u(y)dy \\ &\le c\int_\Omega |\mathcal{G}_x(x_1,y) - \mathcal{G}_x(x_2,y)|\delta(y)^{-a}dy. \end{aligned}$$

The same technique as in [13, Theorem 1.1] yields

$$|\nabla u(x_1) - \nabla u(x_2)| \le C|x_1 - x_2|^{1-a} \quad \text{for all } x_1, x_2 \in \Omega.$$

Therefore $u \in C^{1,1-a}(\overline{\Omega})$. This also implies $w(x) \le c\delta(x)$ in Ω, for some $c > 0$.

(iii) Denote by λ_1 (resp. φ_1) the first eigenvalue (resp. eigenfunction) of $-\Delta$ in Ω. By normalization, we can assume $\varphi_1 > 0$ in Ω. Let now $\overline{w} := M\varphi_1^{2-a}$. A straightforward calculation yields

$$-\Delta\overline{w} = M(2-a)\varphi_1^{-a}\Big[(a-1)|\nabla\varphi_1|^2 + \lambda_1\varphi_1^2\Big] \quad \text{in } \Omega.$$

By Hopf boundary point lemma and the maximum principle, we have $(a-1)|\nabla\varphi_1|^2 + \lambda_1\varphi_1^2 > c > 0$ in Ω, for some positive constant $c > 0$. Thus, we can choose $M > 1$ suitably large such that

$$-\Delta\overline{w} \geq C\delta^{-a}(x) \geq -\Delta w \quad \text{in } \Omega.$$

This yields $w(x) \leq \overline{w}(x) \leq c\delta(x)^{2-a}$ in Ω.

(ii) This follows directly from part (iii) by noting that for $a = 1$ and $\varepsilon > 0$ we have $-\Delta w \leq C_\varepsilon \delta(x)^{-1-\varepsilon}$ in Ω.

Our last result in this section concerns the problem

$$\begin{cases} -\Delta w = w^{-p} + K(x) & \text{in } \Omega, \\ \quad w > 0 & \text{in } \Omega, \\ \quad w = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

where $p > 0$ is a constant and $K \in C(\Omega)$ is a positive function in Ω.

Lemma 2.4 (i) *If $K(x)$ satisfies*

$$\int_\Omega \delta(x)K(x)dx = \infty,$$

then (12) has no solutions.

(ii) *If $K(x) \geq c\delta(x)^{-a}$ in Ω, for some $a \geq 2$ then (12) has no solutions.*

(iii) *If $K(x) \leq C\delta(x)^{-a}$ in Ω, for some $a \in (0,2)$, then (12) has a unique solution $w \in C^2(\Omega) \cap C(\overline{\Omega})$. Furthermore, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$w(x) \leq C_\varepsilon \delta(x)^{\min\left\{\frac{2-\varepsilon}{1+\max\{1,p\}},\, 2-\varepsilon-\max\{1,a\}\right\}} \quad \text{in } \Omega. \tag{13}$$

Proof. Assume there exists a classical solution $w \in C^2(\Omega) \cap C(\overline{\Omega})$ of (12). For $\varepsilon > 0$ denote

$$\Omega_\varepsilon := \{x \in \Omega : \delta(x) > \varepsilon\}.$$

Thus, Ω_ε is a smooth domain provided $\varepsilon > 0$ is small enough. Denote by $\lambda_{1,\varepsilon}$ (resp. $\varphi_{1,\varepsilon}$) the first eigenvalue (resp. eigenfunction) of $-\Delta$ in Ω_ε. We can normalize φ_1 and $\varphi_{1,\varepsilon}$ and assume $\|\varphi_1\|_\infty = \|\varphi_{1,\varepsilon}\|_\infty = 1$. Consider the problem

$$\begin{cases} -\Delta w = (w+\varepsilon)^{-p} + K(x) & \text{in } \Omega_\varepsilon, \\ \quad w > 0 & \text{in } \Omega_\varepsilon, \\ \quad w = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{14}$$

Clearly, the solution w of (12) is a supersolution to (14) while $\underline{w} = m\varphi_{1,\varepsilon}$ is a subsolution of (14) provided $m > 0$ is small enough. By Lemma 2.1 we have $w \geq \underline{w}$ in Ω_ε.

Thus, (14) has a solution w_ε which, by elliptic regularity, satisfies $w_\varepsilon \in C^2(\overline{\Omega}_\varepsilon)$. Let us next multiply by $\varphi_{1,\varepsilon}$ in (14) and integrate over Ω_ε. We obtain

$$\lambda_{1,\varepsilon} \int_{\Omega_\varepsilon} w dx \geq \lambda_{1,\varepsilon} \int_{\Omega_\varepsilon} \varphi_{1,\varepsilon} w dx \geq \lambda_{1,\varepsilon} \int_{\Omega_\varepsilon} \varphi_{1,\varepsilon} w_\varepsilon dx \geq \int_{\Omega_\varepsilon} \varphi_{1,\varepsilon} K(x) dx.$$

A passage to the limit in the above estimate together with Fatou lemma yield

$$\begin{aligned} M := \lambda_1 \int_\Omega w dx &\geq \liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \varphi_{1,\varepsilon} K(x) dx \\ &\geq \int_\Omega \liminf_{\varepsilon \to 0} \varphi_{1,\varepsilon} \chi_{\Omega_\varepsilon} K(x) dx \\ &= \int_\Omega \varphi_1 K(x) dx \\ &\geq C \int_\Omega \delta(x) K(x) dx = \infty, \end{aligned}$$

contradiction. Hence, (12) has no classical solution.

(ii) This follows from part (i) since

$$\int_\Omega \delta(x)^{-\gamma} dx = \infty, \qquad \text{for all } \gamma \geq -1.$$

This can be seen by using local co-ordinates near the boundary of Ω as explained in [15].

(iii) Let w_1 be the solution of (8) and denote by $\tilde{w}$ be solution of

$$\begin{cases} -\Delta \tilde{w} = K(x) & \text{in } \Omega, \\ \tilde{w} > 0 & \text{in } \Omega, \\ \tilde{w} = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

Note that such a solution $\tilde{w}$ always exist due to the fact that $K(x) \leq C\delta(x)^{-a}$ in Ω, with $a \in (0,2)$. It is easy to check that $M\varphi_1^{2-a}$ is a supersolution of (15) while the zero function is a subsolution. This simple observation together with the maximum principle yield the existence and uniqueness of a solution $\tilde{w} \in C^2(\Omega) \cap C(\overline{\Omega})$ of (15). By Lemma 2.3 we have

$$\tilde{w}(x) \leq \begin{cases} C\delta(x) & \text{if } 0 < a < 1, \\ C_\varepsilon \delta(x)^{1-\varepsilon} & \text{if } a = 1, \\ C\delta(x)^{2-a} & \text{if } 1 < a < 2. \end{cases}$$

We can summarize the above estimates by noting that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\tilde{w}(x) \leq C_\varepsilon \delta(x)^{2-\varepsilon-\max\{1,a\}} \qquad \text{in } \Omega. \tag{16}$$

Let w_1 be the unique solution of (8). Thus, w_1 is a subsolution while $w_1 + \tilde{w}$ is a supersolution of (12). Thus, (12) admits a solution w such that

$$w_1 \leq w \leq w_1 + \tilde{w} \quad \text{in } \Omega. \tag{17}$$

The uniqueness of the solution to (12) follows from Lemma 2.1. The estimate (13) follows from (17), (11) and (16).

3 Proof of Theorem 1.1.

Assume that condition (i) in Theorem 1.1 holds and there exists a solution (u, v) of system (1). From (4) we can find $\varepsilon > 0$ small such that

$$\sigma := \min\left\{\frac{2-\varepsilon}{1+\max\{1,s\}}, 2-\varepsilon-\max\left\{1, \frac{2r}{1+\max\{1,p\}}\right\}\right\} > \frac{2}{q}.$$

Let w_1 be the unique solution of (8). Then, by the comparison principle and (10), there exists a constant $C > 0$ such that

$$u \geq w_1 \geq C\delta(x)^{\frac{2}{1+\max\{1,p\}}} \quad \text{in } \Omega.$$

Using this fact in the second equation of (1) we find

$$-\Delta v \leq v^{-s} + c\delta(x)^{-\frac{2r}{1+\max\{1,p\}}} \quad \text{in } \Omega,$$

for some constant $c > 0$. Since $r < 1+\max\{1, p\}$, we can use Lemma 2.4(iii) to deduce

$$v \leq C\delta(x)^{\sigma} \quad \text{in } \Omega,$$

where $C > 0$ is a constant. Using now this last estimate in the first equation of (1) we find

$$-\Delta u \geq u^{-p} + c\delta(x)^{-\sigma q} \quad \text{in } \Omega,$$

for some constant $c > 0$. But this is impossible according to Lemma 2.4(ii), since $\sigma q > 2$.

4 Proof of Theorems 1.3 and Theorem 1.5

Proof of Theorem 1.3. Let w_1, w_2 be solutions of (8) and

$$\begin{cases} -\Delta w_2 = w_2^{-s} & \text{in } \Omega, \\ w_2 > 0 & \text{in } \Omega, \\ w_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{18}$$

respectively. By Lemma 2.2 we have

$$w_1(x) \geq C\delta(x)^{\frac{2}{1+\max\{1,p\}}} \quad \text{and} \quad w_2(x) \geq C\delta(x)^{\frac{2}{1+\max\{1,s\}}} \quad \text{in } \Omega. \tag{19}$$

Hence, by (6) and Lemma 2.4, we may find $w_3, w_4 \in C^2(\Omega)\cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta w_3 = w_3^{-p} + w_2^{-q} & \text{in } \Omega, \\ \quad w_3 > 0 & \text{in } \Omega, \\ \quad w_3 = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta w_4 = w_4^{-s} + w_1^{-r} & \text{in } \Omega, \\ \quad w_4 > 0 & \text{in } \Omega, \\ \quad w_4 = 0 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

Set

$$\mathcal{A} = \left\{ (u,v) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : \begin{array}{ll} w_1 \le u \le w_3 & \text{in } \Omega \\ w_2 \le v \le w_4 & \text{in } \Omega \end{array} \right\}.$$

For any $(u,v)\in\mathcal{A}$, we consider (Tu,Tv) the unique solution of the decoupled system

$$\begin{cases} -\Delta(Tu) = (Tu)^{-p} + v^{-q}\ ,\ Tu > 0 & \text{in } \Omega, \\ -\Delta(Tv) = (Tv)^{-s} + u^{-r}\ ,\ Tv > 0 & \text{in } \Omega, \\ \qquad Tu = Tv = 0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

Since $u,v\in C(\overline{\Omega})$ and $u\ge w_1$, $v\ge w_2$ in Ω, by (19) we have

$$w_1^{-r}(x) \le c\delta(x)^{-\frac{2q}{1+\max\{1,s\}}}, \quad w_2^{-q}(x) \le c\delta(x)^{-\frac{2r}{1+\max\{1,p\}}} \quad \text{in } \Omega.$$

Thus, from (6) and Lemma 2.4, we deduce that the solution (Tu,Tv) of (21) is well posed. Define next

$$\mathscr{F}: \mathcal{A} \to C(\overline{\Omega}) \times C(\overline{\Omega}) \quad \text{by} \quad \mathscr{F}(u,v) = (Tu,Tv) \quad \text{for any } (u,v)\in\mathcal{A}. \tag{22}$$

Thus, the existence of a solution to system (1) follows once we prove that $\mathscr{F}$ has a fixed point in $\mathcal{A}$. To this aim, we shall prove that $\mathscr{F}$ satisfies the conditions:

$$\mathscr{F}(\mathcal{A}) \subseteq \mathcal{A},\ \mathscr{F} \text{ is compact and continuous.}$$

Then, by Schauder's fixed point theorem we deduce that $\mathscr{F}$ has a fixed point in $\mathcal{A}$, which, by standard elliptic estimates, is a classical solution to (1).

Let us show first that $\mathscr{F}(\mathcal{A})\subseteq\mathcal{A}$. Indeed, comparing (8), (21) and using Lemma 2.1 we easlily deduce $Tu\ge w_1$ in Ω. Further, since $v\ge w_2$ we have

$$\Delta w_3 + w_3^{-p} + w_2^{-q} \le 0 \le \Delta(Tu) + (Tu)^{-p} + w_2^{-q} \quad \text{in } \Omega$$

and $Tu, w_3 > 0$ in Ω, $Tu = w_3 = 0$ on $\partial\Omega$. By Lemma 2.1 it follows $Tu \le w_3$ in Ω, and thus $w_1\le Tu\le w_3$ in Ω. Similarly $w_2\le Tv\le w_4$ in Ω which shows that $\mathscr{F}(\mathcal{A})\subseteq\mathcal{A}$.

Out next aim is to show that $\mathscr{F}$ is compact and continuous. Let $(u,v)\in\mathcal{A}$. Since u and v are bounded, we deduce from (21) that $Tu,Tv\in C^{0,\gamma}(\overline{\Omega})$ for some $\gamma\in(0,1)$. Since the embedding $C^{0,\gamma}(\overline{\Omega})\hookrightarrow C(\overline{\Omega})$ is compact, it follows that $\mathscr{F}$ is also compact.

It remains to prove that $\mathscr{F}$ is continuous. To this aim, let $\{(u_n,v_n)\}\subset\mathcal{A}$ be such that $u_n\to u$ and $v_n\to v$ in $C(\overline{\Omega})$ as $n\to\infty$. Using the fact that $\mathscr{F}$ is compact, there exists

$(U,V) \in \mathcal{A}$ such that up to a subsequence we have

$$Tu_n \to U, \quad Tv_n \to V \quad \text{in } C(\overline{\Omega}) \quad \text{as } n \to \infty.$$

On the other hand, by standard elliptic estimates, the sequences $\{Tu_n\}$ and $\{Tv_n\}$ are bounded in $C^{2,\beta}(\overline{\omega})$ $(0 < \beta < 1)$ for any smooth open set $\omega \subset\subset \Omega$. Therefore, up to a diagonally subsequence, we have

$$Tu_n \to U, \quad Tv_n \to V \quad \text{in } C^2(\overline{\omega}) \quad \text{as } n \to \infty,$$

for any smooth open set $\omega \subset\subset \Omega$. Passing to the limit in the definition of Tu_n and Tv_n we find that (U,V) satisfies

$$\begin{cases} -\Delta U = U^{-p} + v^{-q}\ ,\ U > 0 & \text{in } \Omega, \\ -\Delta V = V^{-s} + u^{-r}\ ,\ V > 0 & \text{in } \Omega, \\ \quad U = V = 0 & \text{on } \partial\Omega. \end{cases}$$

By uniqueness of (21), we have that $Tu = U$ and $Tv = V$. Hence

$$Tu_n \to Tu, \quad Tv_n \to Tv \quad \text{in } C(\overline{\Omega}) \quad \text{as } n \to \infty.$$

This proves that $\mathscr{F}$ is continuous.

We are now in a position to apply the Schauder's fixed point theorem. Thus, there exists $(u,v) \in \mathcal{A}$ such that $\mathscr{F}(u,v) = (u,v)$, that is, $Tu = u$ and $Tv = v$. By standard elliptic estimates, it follows that (u,v) is a classical solution of system (1).

Proof of Theorem 1.5. (i) Assume $p < 1$, $2q < 1 + \max\{1,s\}$ and let (u,v) be a solution to (1). By Lemma 2.1 we have $u \ge w_1$, $v \ge w_2$ in Ω, where w_1, w_2 are solutions of (8) and (18) respectively. Using the asymptotic behavior described in (10) it follows that

$$u(x) \ge C\delta(x)\,, \quad v(x) \ge C\delta(x)^{\frac{2}{1+\max\{1,s\}}} \quad \text{in } \Omega,$$

where $C > 0$ is a constant. We next use these estimates for u and v in the first equation of our system (1). We find

$$-\Delta u \le C\Big[\delta(x)^{-p} + \delta(x)^{-\frac{2q}{1+\max\{1,s\}}}\Big] \quad \text{in } \Omega.$$

By our assumption on p, q, r, s and Lemma 2.3(i) it follows that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

The proof of (ii) is similar.

5 Proof of Theorem 1.6

We shall prove Theorem 1.6 under the assumption (i). The case where (ii) holds can be treated similarly. Our arguments are divided into two steps.

Step 1: For any solution (u, v) of (1) we have $u \sim w_1$, where w_1 satisfies (8).

With similar arguments to those used in the proof of Theorem 1.3 we have

$$w_1 \le u \le w_3 \quad \text{in } \Omega\,, \qquad w_2 \le v \le w_4 \quad \text{in } \Omega,$$

where w_2, w_3 and w_4 are solutions of (18) and (20). Using the above estimates, Lemma 2.2 and (19) we have

$$-\Delta u \le w_1^{-p} + \delta(x)^{-\frac{2q}{1+\max\{1,s\}}} \quad \text{in } \Omega. \tag{23}$$

If $0 < p \le 1$ then condition (i) reads $\frac{2q}{1+\max\{1,s\}} < 1$ so by (17) we have

$$u \sim w_1 \sim \begin{cases} \delta(x) & \text{if } 0 < p < 1, \\ \delta(x) \log^{1/2} \dfrac{1}{\delta(x)} & \text{if } p = 1. \end{cases}$$

If $p > 1$ then from (i) and (23) we deduce

$$-\Delta u \le c\delta(x)^{-\frac{2p}{1+p}} + \delta(x)^{-\frac{2q}{1+\max\{1,s\}}} \le C\delta(x)^{-\frac{2p}{1+p}} \quad \text{in } \Omega,$$

where $c, C > 0$ are constants. Thus, by Lemma ... we find

$$u(x) \le C\delta(x)^{\frac{2}{1+p}} \le C_0 w_1(x) \quad \text{in } \Omega.$$

Hence, in both the above cases $u \sim w_1$.

Step 2: System (1) has a unique classical solution.

Let (u_1, v_1) and (u_2, v_2) be two solutions of system (1). From Step 1 we have $u_1 \sim w_1 \sim u_2$. This means that we can find a constant $C > 1$ such that $Cu_1 \ge u_2$ and $Cu_2 \ge u_1$ in Ω.

We claim that $u_1 \le u_2$ in Ω. Supposing the contrary, let

$$M := \inf\{A > 1 : u_1 \le Au_2 \;\text{ in } \Omega\}.$$

By our assumption, we have $M > 1$. From $u_1 \le Mu_2$ in Ω, it follows that

$$-\Delta v_1 = u_1^{-r} + v_1^{-s} \ge M^{-r}u_2^{-r} + v_1^{-s} \quad \text{in } \Omega.$$

Therefore v_2 is a solution and $M^r v_1$ is a supersolution of

$$\begin{cases} -\Delta w = w^{-s} + u_2^{-r} \ , \ w > 0 & \text{in } \Omega, \\ \quad w = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.1 we obtain $M^r v_1 \geq v_2$ in Ω which reads

$$v_1 \geq M^{-r} v_2 \quad \text{in } \Omega.$$

The above estimate yields

$$-\Delta u_1 = u_1^{-p} + v_1^{-q} \leq u_1^{-p} + M^{rq} v_2^{-q} \quad \text{in } \Omega.$$

It follows that u_2 is a solution and $M^{-rq} u_1$ is a subsolution of

$$\begin{cases} -\Delta w = w^{-p} + v_2^{-q} \ , \ w > 0 & \text{in } \Omega, \\ \quad w = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.1 we now deduce

$$u_1 \leq M^{rq} u_2 \quad \text{in } \Omega.$$

Since $M > 1$ and $qr < 1$, the above inequality contradicts the minimality of M. Hence, $u_1 \geq u_2$ in Ω. Similarly we deduce $u_1 \leq u_2$ in Ω, so $u_1 \equiv u_2$ which also yields $v_1 \equiv v_2$. Therefore, the system (1) has a unique solution. This completes the proof of Theorem 1.6.

Acknowledgments. The author acknowledges the financial support of the IRC Ulysses Project between France and Ireland.

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