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A REMARK ON A CONJECTURE OF BORWEIN AND CHOI

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Abstract. We prove the remaining case of a conjecture of Borwein and Choi concerning an estimate on the square of the number of solutions to \( n = x^2 + Ny^2 \) for a squarefree integer \( N \).

1. Introduction

We consider the positive definite quadratic form \( Q(x, y) = x^2 + Ny^2 \) for a squarefree integer \( N \). Let \( r_{2,N}(n) \) denote the number of solutions to \( n = Q(x, y) \) (counting signs and order). In this note, we estimate

\[
\sum_{n \leq x} r_{2,N}(n)^2.
\]

A positive squarefree integer \( N \) is called solvable (or more classically “numerus idoneus”) if \( x^2 + Ny^2 \) has one form per genus. Note that this means the class number of the form class group of discriminant \(-4N\) equals the number of genera, \( 2^t \), where \( t \) is the number of distinct prime factors of \( N \). Concerning \( r_{2,N}(n) \), Borwein and Choi [1] proved the following:

**Theorem 1.1.** Let \( N \) be a solvable squarefree integer. Let \( x > 1 \) and \( \epsilon > 0 \). We have

\[
\sum_{n \leq x} r_{2,N}(n)^2 = \frac{3}{N} \left( \prod_{p \nmid 2N} \frac{2p}{p+1} \right) x \log x + \alpha(N)x + O(N^{4+\epsilon}x^{3+\epsilon})
\]

where the product is over all primes dividing \( 2N \) and

\[
\alpha(N) = -1 + 2\gamma + \sum_{p \nmid 2N} \frac{\log p}{p+1} + \frac{2L'(1, \chi_{-4N})}{L(1, \chi_{-4N})} - \frac{12}{\pi^2} \zeta'(2)
\]

where \( \gamma \) is the Euler-Mascheroni constant and \( L(1, \chi_{-4N}) \) is the L-function corresponding to the quadratic character mod \(-4N\).

Based on this result, Borwein and Choi posed the following:

**Conjecture 1.2.** For any squarefree \( N \),

\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p \nmid 2N} \frac{2p}{p+1} \right) x \log x
\]

The main result in [10] was the following.

**Theorem 1.3.** Let \( Q(x, y) = x^2 + Ny^2 \) for a squarefree integer \( N \) with \(-N \neq 1 \mod 4\). Let \( r_{2,N}(n) \) denote the number of solutions to \( n = Q(x, y) \) (counting signs and order). Then

\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p \nmid 2N} \frac{2p}{p+1} \right) x \log x.
\]

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In this note, we settle the conjecture in the remaining case, namely

**Theorem 1.4.** For \( -N \equiv 1 \mod 4 \), we have

\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p \mid 2N} \frac{2p}{p+1} \right) x \log x.
\]

2. Preliminaries

Let \( Q(x, y) = ax^2 + bxy + cy^2 \) denote a positive definite integral quadratic form with discriminant \( D = b^2 - 4ac \) and \( \gcd(a, b, c) = 1 \). Given \( Q \), let \( \kappa \) be the largest positive integer with \( D/\kappa^2 \) an integer congruent to 0 or 1 modulo 4. We call \( \kappa \) the **conductor** of \( Q \) and set \( d = D/\kappa^2 \). Let \( r(Q, n) \) be the number of representations of the integer \( n \) by the form \( Q \). We now relate \( r(Q, n) \) to counting the number of integral ideals of norm \( n \) in a given class in a generalized ideal class group.

Given \( D = \kappa^2 d \) we consider ideals in \( \mathcal{O}_K \) where \( K = \mathbb{Q} (\sqrt{d}) \). Let \( I_\kappa \) be the group of fractional ideals of \( \mathcal{O}_K \) which are quotients of ideals coprime to \( \kappa \) and \( P_\kappa \) be the subgroup of fractional ideals which are quotients of principal ideals \( (\alpha) \in I_{\kappa} \) where \( \alpha \in \mathbb{Z} + \kappa \mathcal{O} \). Then set \( \text{CL}_\kappa(K) = I_\kappa / P_\kappa \). The elements of \( \text{CL}_\kappa(K) \) correspond bijectively to proper equivalence classes of positive definite quadratic forms of discriminant \( D = \kappa^2 d \). If the proper equivalence class of \( Q \) corresponds to the ideal class \( \mathfrak{c} \), then by [3], page 219, we have

\[
r(Q, n) = \sum_{\mathfrak{c} \mid \kappa} w((\kappa/\mathfrak{c})^2 d) J(\mathfrak{c}, n/r^2)
\]

where

\[
w(D) = \begin{cases} 
6 & \text{if } D = -3 \\
4 & \text{if } D = -4 \\
2 & \text{otherwise}.
\end{cases}
\]

Also \( J(\mathfrak{c}, n) \) is the number of integral ideals of norm \( n \) in the class \( \mathfrak{c} \) where \( \mathfrak{c} \) is the image of \( \mathfrak{c} \) under the natural homomorphism \( \text{CL}_\kappa(K) \to \text{CL}_\kappa(K) \). For the form \( Q(x, y) = x^2 + Ny^2 \) where \( -N \equiv 1 \mod 4 \), the conductor \( \kappa = 2 \) and so we have

\[
r_{2,N}(n) = w(-4N)J(\mathfrak{c}, n) + w(-N)J(\mathfrak{c}_2, n/4) \\
= 2J(\mathfrak{c}, n) + w(-N)J(\mathfrak{c}_2, n/4)
\]

where \( \mathfrak{c}_2 \) is the image under \( \text{CL}_2(K) \to \text{CL}_1(K) \), that is, \( \mathfrak{c}_2 \) is a class in the ideal class group of \( K = \mathbb{Q} (\sqrt{-N}) \).

We now discuss a classical result of Rankin [11] and Selberg [12] which estimates the size of Fourier coefficients of a modular form. Specifically, if \( f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z} \) is a nonzero cusp form of weight \( k \) on \( \Gamma_0(N) \), then

\[
\sum_{n \leq x} |a(n)|^2 = \alpha(f, f)x^k + O(x^{k-\delta})
\]

where \( \alpha > 0 \) is an absolute constant and \( \langle f, f \rangle \) is the Petersson scalar product. In particular, if \( f \) is a cusp form of weight 1, then \( \sum_{n \leq x} |a(n)|^2 = O(x) \). One can adapt their result to say the following. Given two cusp forms of weight \( k \) on a suitable congruence subgroup of \( \Gamma = SL_2(\mathbb{Z}) \), say \( f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z} \) and \( g(z) = \sum_{n=1}^{\infty} b(n) e^{2\pi i n z} \), then
\[
\sum_{n \leq x} a(n)\overline{b(n)}n^{-k} = Ax + O(x^{\frac{1}{2}})
\]

where \(A\) is a constant. In particular, if \(f\) and \(g\) are cusp forms of weight 1, then
\[
\sum_{n \leq x} a(n)\overline{b(n)} = O(x).
\]

We conclude this section with a relationship between genus characters of generalized ideal class groups and the poles of the Rankin-Selberg convolution of L-functions. Recall that a group homomorphism \(\chi : I_2 \to S^1\) is an ideal class character if it is trivial on \(P_2\), i.e.
\[
\chi(\langle a \rangle) = 1
\]

for \(a \equiv 1 \mod 2\). Thus an ideal class character is a character on the generalized class group \(I_2/P_2\). Recall also that a genus character (see Chapter 12, section 5 in [5]) is an ideal class character of order at most two.

Let us also recall the notion of the Rankin-Selberg convolution of two L-functions. For squarefree \(N\), consider two ideal class characters \(\chi_1, \chi_2\) for \(CL_2(K)\), the generalized ideal class group of \(K = \mathbb{Q}(\sqrt{-N})\) and their associated Hecke L-series
\[
L_2(s, \chi_1) = \sum_{(a,2)=1} \frac{\chi_1(a)}{N(a)^s}
\]

\[
L_2(s, \chi_2) = \sum_{(a,2)=1} \frac{\chi_2(a)}{N(a)^s}
\]

which converge absolutely in some right half-plane. We form the convolution L-series by multiplying the coefficients,
\[
L_2(s, \chi_1 \otimes \chi_2) = \sum_{(a,2)=1} \frac{\chi_1(a)\chi_2(a)}{N(a)^s}
\]

The following result describes a relationship between genus characters \(\chi\) and the orders of poles of \(L_2(s, \chi \otimes \chi)\). The proof is similar to that of Proposition 2.4 in [10].

**Proposition 2.1.** Let \(\chi\) be an ideal class character for \(CL_2(K)\), \(-N \equiv 1 \mod 4\), and \(L_2(s, \chi)\) the associated Hecke L-series. Then \(\chi\) is a genus character if and only if \(L_2(s, \chi \otimes \chi)\) has a double pole at \(s = 1\).

**Remark 2.2.** By Proposition 2.1, if \(\chi\) is a non-genus character, then \(L_2(s, \chi \otimes \chi)\) has at most a simple pole at \(s = 1\).

3. **Proof of Theorem 1.4**

**Proof.** As the proof is similar to that of Theorem 1.3 in [10], we sketch the relevant details. If \(-N \equiv 1 \mod 4\), then the discriminant of \(K = \mathbb{Q}(\sqrt{-N})\) is \(-N\). We also assume that \(t\) is the number of distinct prime factors of \(N\) and so the discriminant \(-N\) also has \(t\) distinct prime factors. For \(K = \mathbb{Q}(\sqrt{-N})\), consider the zeta function
\[
\zeta_K(s, 2) = \sum_{(a,2)=1} \frac{1}{N(a)^s}
\]

where the sum is over those ideals \(a\) of \(O_K\) prime to 2. We now split up \(\zeta_K(s, 2)\), according to the classes \(c_i\) of the generalized ideal class group \(CL_2(K)\), into the partial zeta functions (see page 161 of [7])
\[ \zeta_c(s) = \sum_{a \in c_i} \frac{1}{N(a)^s} \]

so that \( \zeta_K(s, 2) = \sum_{i=0}^{h_2-1} \zeta_c(s) \) where \( h_2 \) is the order of \( \text{CL}_2(K) \).

Let \( c \) be the ideal class in \( \text{CL}_2(K) \) which corresponds to the proper equivalence class of \( Q(x, y) = x^2 + Ny^2 \). Now let \( \chi \) be an ideal class character of \( \text{CL}_2(K) \) and consider the Hecke L-series for \( \chi \), namely

\[ L_2(s, \chi) = \sum_{(a,2)=1} \frac{\chi(a)}{N(a)^s}. \]

We may now rewrite the Hecke L-series as

\[ L_2(s, \chi) = \sum_{i=0}^{h_2-1} \chi(c_i) \zeta_c(s). \]

And so summing over all ideal class characters of \( \text{CL}_2(K) \), we have

\[ \sum_{\chi} \overline{\chi(c)} L_2(s, \chi) = \sum_{i=0}^{h_2-1} \zeta_c(s) \left( \sum_{\chi} \overline{\chi(c)} \chi(c_i) \right). \]

The inner sum is nonzero precisely when \( c = c_i \). Thus we have

\[ \zeta_c(s) = \frac{1}{h_2} \sum_{\chi} \overline{\chi(c)} L_2(s, \chi) \]

and so

\[ \zeta_c(s) = \frac{1}{h_2} (L_2(s, \chi_0) + \overline{\chi_1(c)} L_2(s, \chi_1) + \cdots + \overline{\chi_{h_2-1}(c)} L_2(s, \chi_{h_2-1})). \]

As \( \chi_0 \) is the trivial character, \( L_2(s, \chi_0) = \zeta_K(s, 2) \). Comparing \( n^{th} \) coefficients, we have

\[ J(c, n) = \frac{1}{h_2} (a_n + b_1(n) + \cdots + b_{h_2-1}(n)). \]

where \( a_n \) is the number of integral ideals of \( \mathcal{O}_K \) prime to 2 and of norm \( n \) and the \( b_i \)'s are coefficients of weight 1 cusp forms (see [2]). Recall we also have

\[ r_{2,N}(n) = 2J(c, n) + w(-N)J(c_2, n/4) \]

and so

\[ r_{2,N}(n) = \frac{2}{h_2} (a_n + b_1(n) + \cdots + b_{h_2-1}(n)) + w(-N)J(c_2, n/4). \]

Thus

\[ \sum_{n \leq x} r_{2,N}(n)^2 = \frac{4}{h_2^2} \left( \sum_{n \leq x} a_n^2 + \sum_{n \leq x} b_1(n)^2 + 2 \sum_{n \leq x} a_n b_1(n) + \sum_{i \neq j \atop n \leq x} b_i(n) b_j(n) \right) + \frac{4}{h_2} \sum_{n \leq x} (a_n + b_1(n) + \cdots + b_{h_2-1}(n)) w(-N)J(c_2, n/4) + \sum_{n \leq x} w(-N)^2 J(c_2, n/4)^2. \]
Assume $-N \equiv 1 \mod 8$. Applying the main theorem in \cite{6} to the Dirichlet series
\[ \sum_{n=1}^{\infty} \frac{a_n^2}{n^s} \] we obtain
\[ \sum_{n \leq x} a_n^2 \sim Ax \log x \]
where $A = \frac{1}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p|N} \frac{p}{p+1}$. As $-N$ has $t$ distinct prime factors, we have $2^t$ genus characters for $CL(K)$ where $K = \mathbb{Q}(-\sqrt{-N})$. By \cite{7} (see Theorem 1, page 127), we have $2^t$ genus characters for $\text{CL}_2(K)$. We now must estimate $\sum_{n \leq x} b_i(n)^2$. Let us now assume that the first $2^t - 1$ terms arise from L-functions associated to genus characters. By Proposition 2.1 and an application of Perron’s formula, we obtain
\[ \sum_{n \leq x} b_i(n)^2 \sim Ax \log x. \]
As this estimate holds for each $i$ such that $1 \leq i \leq 2^t - 1$, the term $Ax \log x$ appears $2^t$ times in the estimate of $\sum_{n \leq x} r_{2,N}(n)^2$. By Remark 2.2 and the Rankin-Selberg estimate, the remaining terms are all $O(x)$. Thus
\[ \sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{4}{h^2} \left( \frac{2^t}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p|N} \frac{p}{p+1} \right) x \log x. \]
By \cite{4}, we have $L(1, \chi_{-N}) = \frac{h\pi}{\sqrt{-N}}$ where $h$ is the class number of $K$ and $h_2 = h$. Thus
\[ \sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p|2N} \frac{2p}{p+1} \right) x \log x. \]
For $-N \equiv 5 \mod 8$, we have $h_2 = 3h$ and again by \cite{6},
\[ \sum_{n \leq x} a_n^2 \sim \left( \frac{9}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p|N} \frac{p}{p+1} \right) x \log x. \]
Thus
\[ \sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p|2N} \frac{2p}{p+1} \right) x \log x. \]

\[ \square \]

**Remark 3.1.** We would like to mention another approach which confirms Theorems 1.3 and 1.4. Let $Q \in \mathbb{Z}^{2 \times 2}$ be a non-singular symmetric matrix with even diagonal entries and $q(x) = \frac{1}{2}Q|x| = \frac{1}{2}x^TQx$, $x \in \mathbb{Z}^2$, the associated quadratic form in two variables. Let $r(Q, n)$ denote the number of representations of $n$ by the quadratic form $Q$. Now consider the theta function
\[ \theta_Q(z) = \sum_{x \in \mathbb{Z}^2} e^{\pi iz Q|x|}. \]
The Dirichlet series associated with the automorphic form $\theta_Q$ is
\[(4\pi)^{-1/2} \zeta_Q \left( \frac{1}{2} + s \right)\]
where
\[\zeta_Q(s) = \sum_{n=1}^{\infty} \frac{r(Q,n)}{n^s} = \sum_{x \in \mathbb{Z}^2 \setminus \{0\}} q(x)^{-s}\]
for \(\Re(s) > 1\). A careful and involved application of the Rankin-Selberg method to the above Dirichlet series (see Theorems 2.1 and 5.1 in [8] and Theorem 5.2 in [9]) combined with a Tauberian argument yields the following (see Theorem 6.1 in [8])
\[\sum_{n \leq x} r(Q,n)^2 \sim A_Q x \log x\]
where
\[A_Q = 12 \frac{A(q)}{q} \prod_{p|q} \left( 1 + \frac{1}{p} \right)^{-1}.\]
Here \(q = \det Q\) and \(A(q)\) denotes the multiplicative function defined by
\[A(p^e) = 2 + \left( 1 - \frac{1}{2} \right)(e - 1)\]
where \(p\) is an odd prime, \(e \geq 1\), and
\[A(2^e) = \begin{cases} 
1 & \text{if } e \leq 1, \\
2 & \text{if } e = 2, \\
en-1 & \text{if } e \geq 3.
\end{cases}\]

Let us now turn to our situation. Consider \(q(x) = x^2 + Ny^2 = \frac{1}{2}x^TQx\) where \(Q = \begin{pmatrix} 2 & 0 \\ 0 & 2N \end{pmatrix}\), \(N\) squarefree. Thus \(q = 4N\). Suppose \(N\) has \(t\) distinct prime factors. Then \(A(4N) = 2^{t+1}\) and so
\[A_Q = \frac{3}{N} 2^{t+1} \prod_{p|2N} \left( 1 + \frac{1}{p} \right)^{-1} = \frac{3}{N} \prod_{p|2N} \frac{2p}{p+1}.
\]

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