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A REMARK ON A CONJECTURE OF BORWEIN AND CHOI

ROBERT OSBURN

Abstract. We prove the remaining case of a conjecture of Borwein and Choi concerning an estimate on the square of the number of solutions to \( n = x^2 + Ny^2 \) for a squarefree integer \( N \).

1. Introduction

We consider the positive definite quadratic form \( Q(x, y) = x^2 + Ny^2 \) for a squarefree integer \( N \). Let \( r_{2,N}(n) \) denote the number of solutions to \( n = Q(x, y) \) (counting signs and order). In this note, we estimate

\[
\sum_{n \leq x} r_{2,N}(n)^2.
\]

A positive squarefree integer \( N \) is called solvable (or more classically “numerus idoneus”) if \( x^2 + Ny^2 \) has one form per genus. Note that this means the class number of the form class group of discriminant \(-4N\) equals the number of genera, \( 2^t \), where \( t \) is the number of distinct prime factors of \( N \). Concerning \( r_{2,N}(n) \), Borwein and Choi [1] proved the following:

Theorem 1.1. Let \( N \) be a solvable squarefree integer. Let \( x > 1 \) and \( \epsilon > 0 \). We have

\[
\sum_{n \leq x} r_{2,N}(n)^2 = \frac{3}{N} \left( \prod_{p \mid 2N} \frac{2p}{p+1} \right) (x \log x + \alpha(N)x) + O(N^{\frac{4}{3} + \epsilon}x^{\frac{11}{3} + \epsilon})
\]

where the product is over all primes dividing \( 2N \) and

\[
\alpha(N) = -1 + 2\gamma + \sum_{p \mid 2N} \frac{\log p}{p+1} + \frac{2L'(1, \chi_{-4N})}{L(1, \chi_{-4N})} - \frac{12}{\pi^2} \zeta'(-2)
\]

where \( \gamma \) is the Euler-Mascheroni constant and \( L(1, \chi_{-4N}) \) is the \( L \)-function corresponding to the quadratic character mod \(-4N\).

Based on this result, Borwein and Choi posed the following:

Conjecture 1.2. For any squarefree \( N \),

\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p \mid 2N} \frac{2p}{p+1} \right) x \log x
\]

The main result in [10] was the following.

Theorem 1.3. Let \( Q(x, y) = x^2 + Ny^2 \) for a squarefree integer \( N \) with \(-N \not\equiv 1 \mod 4\). Let \( r_{2,N}(n) \) denote the number of solutions to \( n = Q(x, y) \) (counting signs and order). Then

\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p \mid 2N} \frac{2p}{p+1} \right) x \log x.
\]

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In this note, we settle the conjecture in the remaining case, namely

**Theorem 1.4.** For $-N \equiv 1 \mod 4$, we have

$$\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p|2N} \frac{2p}{p+1} \right) x \log x.$$  

2. **Preliminaries**

Let $Q(x, y) = ax^2 + bxy + cy^2$ denote a positive definite integral quadratic form with discriminant $D = b^2 - 4ac$ and $\gcd(a, b, c) = 1$. Given $Q$, let $\kappa$ be the largest positive integer with $D/\kappa^2$ an integer congruent to 0 or 1 modulo 4. We call $\kappa$ the conductor of $Q$ and set $d = D/\kappa^2$. Let $r(Q, n)$ be the number of representations of the integer $n$ by the form $Q$. We now relate $r(Q, n)$ to counting the number of integral ideals of norm $n$ in a given class in a generalized ideal class group.

Given $D = \kappa^2d$ we consider ideals in $\mathcal{O}_K$ where $K = \mathbb{Q}(\sqrt{d})$. Let $I_\kappa$ be the group of fractional ideals of $\mathcal{O}_K$ which are quotients of ideals coprime to $\kappa$ and $P_\kappa$ be the subgroup of fractional ideals which are quotients of principal ideals $(\alpha) \in I_\kappa$ where $\alpha \in \mathbb{Z} + \kappa \mathcal{O}$. Then set $\text{CL}_\kappa(K) = I_\kappa/P_\kappa$. The elements of $\text{CL}_\kappa(K)$ correspond bijectively to proper equivalence classes of positive definite quadratic forms of discriminant $D = \kappa^2d$. If the proper equivalence class of $Q$ corresponds to the ideal class $\mathfrak{c}$, then by [3], page 219, we have

$$r(Q, n) = \sum_{r|\kappa} w((\kappa/r)^2d) J(\mathfrak{c}_r, n/r^2)$$

where

$$w(D) = \begin{cases} 
6 & \text{if } D = -3 \\
4 & \text{if } D = -4 \\
2 & \text{otherwise.}
\end{cases}$$

Also $J(\mathfrak{c}_r, n)$ is the number of integral ideals of norm $n$ in the class $\mathfrak{c}_r$ where $\mathfrak{c}_r$ is the image of $\mathfrak{c}$ under the natural homomorphism $\text{CL}_\kappa(K) \rightarrow \text{CL}_{\kappa/r}(K)$. For the form $Q(x, y) = x^2 + Ny^2$ where $-N \equiv 1 \mod 4$, the conductor $\kappa = 2$ and so we have

$$r_{2,N}(n) = w(-4N)J(\mathfrak{c}, n) + w(-N)J(\mathfrak{c}_2, n/4)$$

$$= 2J(\mathfrak{c}, n) + w(-N)J(\mathfrak{c}_2, n/4)$$

where $\mathfrak{c}_2$ is the image under $\text{CL}_2(K) \rightarrow \text{CL}_1(K)$, that is, $\mathfrak{c}_2$ is a class in the ideal class group of $K = \mathbb{Q}(\sqrt{-N})$.

We now discuss a classical result of Rankin [11] and Selberg [12] which estimates the size of Fourier coefficients of a modular form. Specifically, if $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ is a nonzero cusp form of weight $k$ on $\Gamma_0(N)$, then

$$\sum_{n \leq x} |a(n)|^2 = \alpha(f, f)x^k + O(x^{k-\delta})$$

where $\alpha > 0$ is an absolute constant and $\langle f, f \rangle$ is the Petersson scalar product. In particular, if $f$ is a cusp form of weight 1, then $\sum_{n \leq x} |a(n)|^2 = O(x)$. One can adapt their result to say the following. Given two cusp forms of weight $k$ on a suitable congruence subgroup of $\Gamma = \text{SL}_2(\mathbb{Z})$, say $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ and $g(z) = \sum_{n=1}^{\infty} b(n)e^{2\pi inz}$, then
\[
\sum_{n \leq x} a(n)b(n)n^{1-k} = Ax + O(x^{\frac{1}{2}})
\]
where \( A \) is a constant. In particular, if \( f \) and \( g \) are cusp forms of weight 1, then
\[
\sum_{n \leq x} a(n)b(n) = O(x).
\]

We conclude this section with a relationship between genus characters of generalized ideal class groups and the poles of the Rankin-Selberg convolution of \( L \)-functions. Recall that a group homomorphism \( \chi : I_2 \to S^1 \) is an ideal class character if it is trivial on \( P_2 \), i.e.
\[
\chi([a]) = 1
\]
for \( a \equiv 1 \mod (2) \). Thus an ideal class character is a character on the generalized class group \( I_2/P_2 \). Recall also that a genus character (see Chapter 12, section 5 in [5]) is an ideal class character of order at most two.

Let us also recall the notion of the Rankin-Selberg convolution of two \( L \)-functions. For squarefree \( N \), consider two ideal class characters \( \chi_1, \chi_2 \) for \( CL_2(K) \), the generalized ideal class group of \( K = \mathbb{Q}(\sqrt{-N}) \) and their associated Hecke \( L \)-series
\[
L_2(s, \chi_1) = \sum_{(a,2)=1} \frac{\chi_1(a)}{N(a)^s}
\]
\[
L_2(s, \chi_2) = \sum_{(a,2)=1} \frac{\chi_2(a)}{N(a)^s}
\]
which converge absolutely in some right half-plane. We form the convolution \( L \)-series by multiplying the coefficients,
\[
L_2(s, \chi_1 \otimes \chi_2) = \sum_{(a,2)=1} \frac{\chi_1(a)\chi_2(a)}{N(a)^s}
\]
The following result describes a relationship between genus characters \( \chi \) and the orders of poles of \( L_2(s, \chi \otimes \chi) \). The proof is similar to that of Proposition 2.4 in [10].

**Proposition 2.1.** Let \( \chi \) be an ideal class character for \( CL_2(K) \), \(-N \equiv 1 \mod 4\), and \( L_2(s, \chi) \) the associated Hecke \( L \)-series. Then \( \chi \) is a genus character if and only if \( L_2(s, \chi \otimes \chi) \) has a double pole at \( s = 1 \).

**Remark 2.2.** By Proposition 2.1, if \( \chi \) is a non-genus character, then \( L_2(s, \chi \otimes \chi) \) has at most a simple pole at \( s = 1 \).

### 3. Proof of Theorem 1.4

**Proof.** As the proof is similar to that of Theorem 1.3 in [10], we sketch the relevant details. If \(-N \equiv 1 \mod 4\), then the discriminant of \( K = \mathbb{Q}(\sqrt{-N}) \) is \(-N\). We also assume that \( t \) is the number of distinct prime factors of \( N \) and so the discriminant \(-N\) also has \( t \) distinct prime factors. For \( K = \mathbb{Q}(\sqrt{-N}) \), consider the zeta function
\[
\zeta_K(s, 2) = \sum_{(a,2)=1} \frac{1}{N(a)^s}
\]
where the sum is over those ideals \( a \) of \( O_K \) prime to 2. We now split up \( \zeta_K(s, 2) \), according to the classes \( \mathfrak{c}_i \) of the generalized ideal class group \( CL_2(K) \), into the partial zeta functions (see page 161 of [7])
\[ \zeta_{\epsilon}(s) = \sum_{a \in \epsilon_i} \frac{1}{N(a)^s} \]

so that \( \zeta_K(s, 2) = \sum_{i=0}^{h_2-1} \zeta_{\epsilon_i}(s) \) where \( h_2 \) is the order of \( CL_2(K) \).

Let \( \epsilon \) be the ideal class in \( CL_2(K) \) which corresponds to the proper equivalence class of \( Q(x, y) = x^2 + Ny^2 \). Now let \( \chi \) be an ideal class character of \( CL_2(K) \) and consider the Hecke L-series for \( \chi \), namely

\[ L_2(s, \chi) = \sum_{(a, 2) = 1}^{\chi(a)} \frac{\chi(a)}{N(a)^s}. \]

We may now rewrite the Hecke L-series as

\[ L_2(s, \chi) = \sum_{i=0}^{h_2-1} \chi(\epsilon_i) \zeta_{\epsilon_i}(s). \]

And so summing over all ideal class characters of \( CL_2(K) \), we have

\[ \sum_{\chi} \overline{\chi}(\epsilon) L_2(s, \chi) = \sum_{i=0}^{h_2-1} \zeta_{\epsilon_i}(s) \left( \sum_{\chi} \overline{\chi}(\epsilon) \chi(\epsilon_i) \right). \]

The inner sum is nonzero precisely when \( \epsilon = \epsilon_i \). Thus we have

\[ \zeta_{\epsilon}(s) = \frac{1}{h_2} \sum_{\chi} \overline{\chi}(\epsilon) L_2(s, \chi) \]

and so

\[ \zeta_{\epsilon}(s) = \frac{1}{h_2} (L_2(s, \chi_0) + \overline{\chi}(\epsilon) L_2(s, \chi_1) + \cdots + \overline{\chi}_{h_2-1}(\epsilon) L_2(s, \chi_{h_2-1})). \]

As \( \chi_0 \) is the trivial character, \( L_2(s, \chi_0) = \zeta_K(s, 2) \). Comparing \( n^{th} \) coefficients, we have

\[ J(\epsilon, n) = \frac{1}{h_2} (a_n + b_1(n) + \cdots + b_{h_2-1}(n)). \]

where \( a_n \) is the number of integral ideals of \( \mathcal{O}_K \) prime to 2 and of norm \( n \) and the \( b_i \)’s are coefficients of weight 1 cusp forms (see [2]). Recall we also have

\[ r_{2, N}(n) = 2 J(\epsilon, n) + w(-N) J(\epsilon_2, n/4) \]

and so

\[ r_{2, N}(n) = \frac{2}{h_2} \left( a_n + b_1(n) + \cdots + b_{h_2-1}(n) \right) + w(-N) J(\epsilon_2, n/4). \]

Thus

\[ \sum_{n \leq x} r_{2, N}(n)^2 = \frac{4}{h_2^2} \left( \sum_{n \leq x} a_n^2 + \sum_{i \leq x} b_i(n)^2 + \sum_{i, j \leq x} a_n b_i(n) b_j(n) + \sum_{i \neq j, n \leq x} b_i(n) b_j(n) \right) \]

\[ + \frac{4}{h_2} \sum_{n \leq x} \left( a_n + b_1(n) + \cdots + b_{h_2-1}(n) \right) w(-N) J(\epsilon_2, n/4)^2 + \sum_{n \leq x} w(-N)^2 J(\epsilon_2, n/4)^2. \]
Assume \(-N \equiv 1 \mod 8\). Applying the main theorem in [6] to the Dirichlet series
\[ \sum_{n=1}^{\infty} \frac{a_n^2}{n^s}, \]
we obtain
\[ \sum_{n \leq x} a_n^2 \sim Ax \log x \]
where
\[ A = \frac{1}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p|N} \frac{p}{p+1}. \]
As \(-N\) has \(t\) distinct prime factors, we have \(2^t\) genus characters for \(CL(K)\) where \(K = \mathbb{Q}(\sqrt{-N})\). By [7] (see Theorem 1, page 127), we have \(2^t\) genus characters for \(CL_2(K)\). We now must estimate
\[ \sum_{i \leq x} b_i(n)^2. \]
Let us now assume that the first \(2^t - 1\) terms arise from L-functions associated to genus characters. By Proposition 2.1 and an application of Perron’s formula, we obtain
\[ \sum_{n \leq x} b_i(n)^2 \sim Ax \log x. \]
As this estimate holds for each \(i\) such that \(1 \leq i \leq 2^t - 1\), the term \(Ax \log x\) appears \(2^t\) times in the estimate of \(\sum_{n \leq x} r_{2,N}(n)^2\). By Remark 2.2 and the Rankin-Selberg estimate, the remaining terms are all \(O(x)\). Thus
\[ \sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{4}{h_2^2} \left( \frac{2^t}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p|N} \frac{p}{p+1} \right) x \log x. \]
By [4], we have \(L(1, \chi_{-N}) = \frac{h}{\sqrt{N}}\) where \(h\) is the class number of \(K\) and \(h_2 = h\). Thus
\[ \sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p|2N} \frac{2p}{p+1} \right) x \log x. \]
For \(-N \equiv 5 \mod 8\), we have \(h_2 = 3h\) and again by [6],
\[ \sum_{n \leq x} a_n^2 \sim \left( \frac{9}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p|N} \frac{p}{p+1} \right) x \log x. \]
Thus
\[ \sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p|2N} \frac{2p}{p+1} \right) x \log x. \]

**Remark 3.1.** We would like to mention another approach which confirms Theorems 1.3 and 1.4. Let \(Q \in \mathbb{Z}^{2 \times 2}\) be a non-singular symmetric matrix with even diagonal entries and \(q(x) = \frac{1}{2}Q|x|^2 = \frac{1}{2}x^TQx\), \(x \in \mathbb{Z}^2\), the associated quadratic form in two variables. Let \(r(Q, n)\) denote the number of representations of \(n\) by the quadratic form \(Q\). Now consider the theta function
\[ \theta_Q(z) = \sum_{x \in \mathbb{Z}^2} e^{\pi izQ|x|}. \]
The Dirichlet series associated with the automorphic form \(\theta_Q\) is
\[(4\pi)^{-1/2} \zeta_Q \left( \frac{1}{2} + s \right) \]

where

\[ \zeta_Q(s) = \sum_{n=1}^{\infty} \frac{r(Q, n)}{n^s} = \sum_{x \in \mathbb{Z}^2 \setminus \{0\}} q(x)^{-s} \]

for \( \Re(s) > 1 \). A careful and involved application of the Rankin-Selberg method to the above Dirichlet series (see Theorems 2.1 and 5.1 in [8] and Theorem 5.2 in [9]) combined with a Tauberian argument yields the following (see Theorem 6.1 in [8])

\[ \sum_{n \leq x} r(Q, n)^2 \sim A_Q x \log x \]

where

\[ A_Q = 12 \frac{A(q)}{q} \prod_{p \mid q} \left( 1 + \frac{1}{p} \right)^{-1}. \]

Here \( q = \det Q \) and \( A(q) \) denotes the multiplicative function defined by

\[ A(p^e) = 2 + (1 - \frac{1}{2})(e - 1) \]

where \( p \) is an odd prime, \( e \geq 1 \), and

\[ A(2^e) = \begin{cases} 1 & \text{if } e \leq 1, \\ 2 & \text{if } e = 2, \\ e - 1 & \text{if } e \geq 3. \end{cases} \]

Let us now turn to our situation. Consider \( q(x) = x^2 + Ny^2 = \frac{1}{2} x^T Q x \) where \( Q = \begin{pmatrix} 2 & 0 \\ 0 & 2N \end{pmatrix} \), \( N \) squarefree. Thus \( q = 4N \). Suppose \( N \) has \( t \) distinct prime factors. Then \( A(4N) = 2^{t+1} \) and so

\[ A_Q = \frac{3}{N} 2^{t+1} \prod_{p \mid 2N} \left( 1 + \frac{1}{p} \right)^{-1} = \frac{3}{N} \prod_{p \mid 2N} \frac{2p}{p+1}. \]

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