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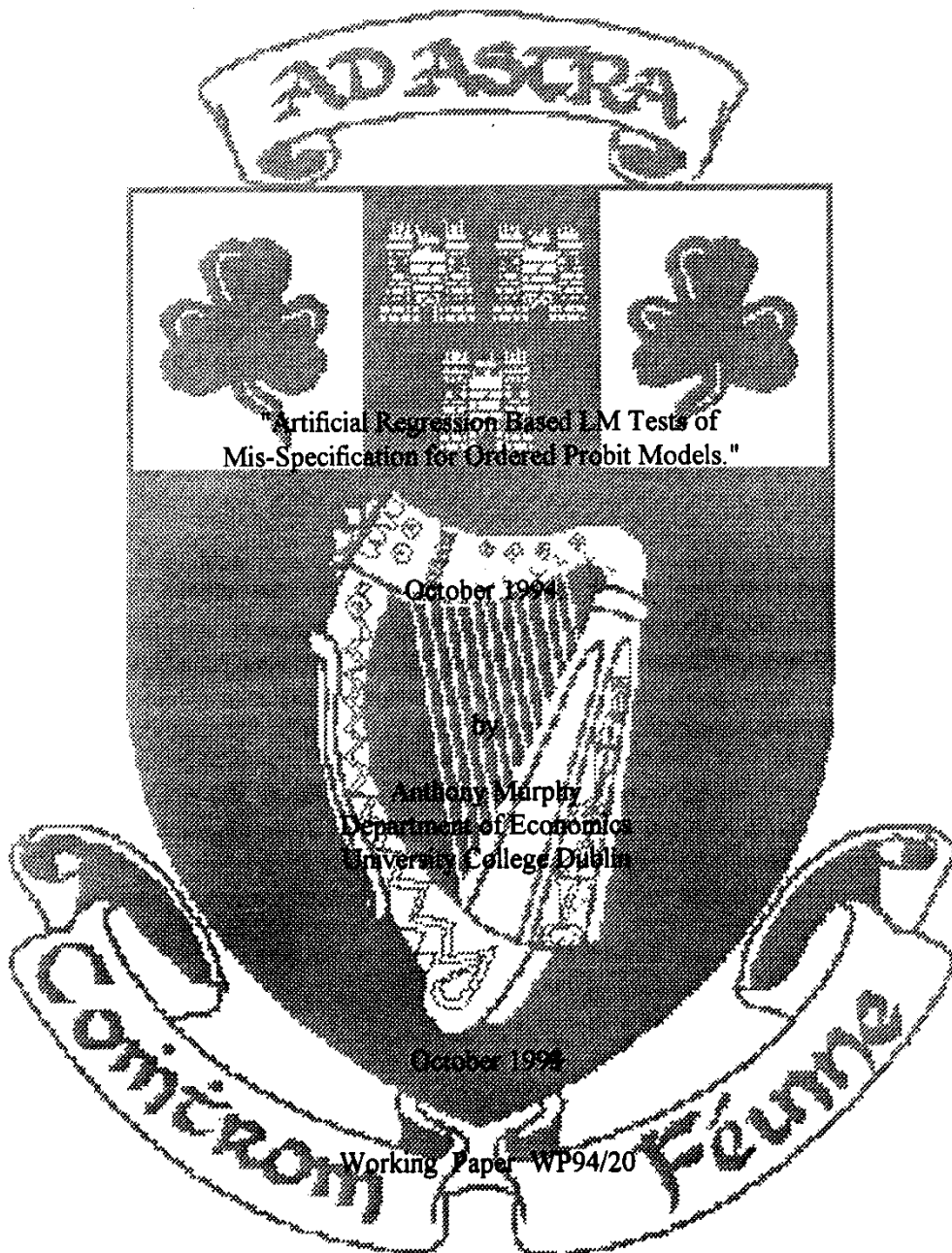
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ARTIFICIAL REGRESSION BASED LM TESTS OF MIS-SPECIFICATION FOR ORDERED PROBIT MODELS

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Abstract: Lagrange Multiplier (LM) tests for omitted variables, heteroscedasticity, incorrect functional form, and non-normality in the ordered probit model may be readily calculated using an artificial regression. The proposed artificial regression is both convenient and likely to have better small sample properties than the more common outer product gradient (OPG) form.

Keywords: Ordered Probit, Lagrange Multiplier Tests, Mis-specification, Artificial Regression, Outer Product Gradient.

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Introduction

The ordered probit model is probably the most popular ordered discrete choice model in econometrics¹. However mis-specification of the model generally results in inconsistent parameter estimates². Thus estimated ordered probit models should be subjected to a range of mis-specification tests. In this paper convenient artificial regression based Lagrange Multiplier (LM) tests for a range of mis-specifications in ordered probit models are derived. These regression based LM tests are particularly attractive since they do not use the outer product gradient (OPG) form to calculate the LM test statistic. Unlike OPG based tests, the proposed tests should perform reasonably in small samples. They are compared with some other mis-specification tests of the ordered probit model.

Background

Davidson and MacKinnon (1984b) derive Lagrange Multiplier (LM) tests for

¹ For details of the ordered probit model see Becker and Kennedy (1992), Greene (1993) or Maddala (1983).

² Kiefer and Skoog (1984) and Yatchew and Griliches (1984) show that omitted uncorrelated explanatory variables and neglected heterogeneity both generate inconsistent parameter estimates in logit and probit models, unlike classical regression models.

omitted variables and neglected heterogeneity in probit models³. Their LM tests are convenient since they are based on artificial regressions which have good small sample properties since they do not use the OPG form is not used⁴. Instead the information matrix is calculated as the expectation of the outer product of the matrix of contributions to the score and is not just approximated by this outer product.

Murphy (1994a) shows how artificial regression based LM tests for general discrete choice models may be derived. These tests are also likely to have good small sample properties since they are not based on the OPG form of the LM test. In this paper this approach is used to derive convenient artificial regression based LM tests for a range of mis-specifications such as omitted variables, neglected heterogeneity, incorrect functional form and non-normality in ordered probit models. The test for incorrect functional form is based on a suggestion in Pagan and Vella (1989) who propose a RESET like test for the omission of powers of x/β . The test for non-normality uses the Gram Charlier type A alternative⁵.

The proposed tests are compared with tests of the ordered probit model in Holden (1993) and some other tests based on Chesher and Irish (1987). Holden derives tests for neglected heteroscedasticity and non-normality. His tests for non-normality extend the results of Bera, Jarque and Lee (1982) to the ordered probit model. Bera et. al. use a Pearson distribution based alternative to the normal distribution to derive an LM test of non-normality in the probit model. Chesher and Irish derive LM tests for omitted variables, neglected heteroscedasticity and non-normality in the grouped and censored normal regression. Their tests may be readily extended to the ordered probit case.

The Ordered Probit Model

Some notation is required. Consider a general discrete choice model with N individuals, denoted by subscript i , and $J+1$ ordered alternatives. These alternatives are denoted by subscript j and numbered from 0 to J . Let y_{ij} be an indicator variable for individual i and alternative j . Thus y_{ij} equals one if individual i selects alternative j ; otherwise y_{ij} equals zero. Let p_{ij} be the probability that i selects alternative j . p_{ij} is a function of a vector of unknown parameters. The true parameter vector is assumed to be in the interior of the relevant parameter space. The sum of the y_{ij} 's and the p_{ij} 's over the $J+1$ alternatives both sum to one

³ Engle (1982,1984) and Godfrey (1989) survey the use of LM tests in econometrics.

⁴ Engle (1984) also suggests using non-OPG artificial regression based LM tests of binary choice models. Davidson and MacKinnon (1983,1984a,1993), Godfrey (1988) and MacKinnon (1992), inter alia, discuss the poor sample performance of OPG based LM tests.

⁵ Klefer and Salmon (1983) use the Gram Charlier type A expansion to derive an LM test for non-normality in the linear regression model. Murphy (1994b) uses the Gram Charlier type A expansion to derive an LM test for non-normality in the probit model. The proposed test is compared with the Bera, Jarque and Lee (1982) and Chesher and Irish (1987) tests for non-normality in the probit model.

for any individual.

The ordered choice model is built upon the latent regression:

$$y_i^* = x_i'\beta + u_i \quad (1)$$

where x_i and β are vectors of explanatory variables and coefficients and u_i is a random error term which, in the ordered probit model, has a standard normal distribution. The standard normal distribution and density are denoted by

Φ and ϕ . The latent variable y_i^* is unobserved. Instead the indicator variables y_{ij} 's are observed where:

$$\begin{aligned} y_{i0} = 1 &\leftrightarrow y_i^* \leq 0 \\ y_{i1} = 1 &\leftrightarrow 0 < y_i^* \leq \mu_1 \\ y_{i2} = 1 &\leftrightarrow \mu_1 < y_i^* \leq \mu_2 \\ &\vdots \\ y_{ij} = 1 &\leftrightarrow \mu_{j-1} < y_i^* \leq \mu_j \\ &\vdots \\ y_{iJ} = 1 &\leftrightarrow \mu_{J-1} < y_i^* \end{aligned} \quad (4)$$

and the μ 's are the J-1 unobserved thresholds. The probabilities of the J+1 ordered alternatives are:

$$\begin{aligned} p_{i0} &= \Phi(-x_i'\beta) \\ p_{i1} &= \Phi(\mu_1 - x_i'\beta) - \Phi(-x_i'\beta) \\ &\vdots \\ p_{ij} &= \Phi(\mu_j - x_i'\beta) - \Phi(\mu_{j-1} - x_i'\beta) \quad j = 2 \dots J-1 \\ &\vdots \\ p_{iJ} &= 1 - \Phi(\mu_{J-1} - x_i'\beta) \end{aligned}$$

Then the parameters to be estimated are β and μ where $\mu' = (\mu_1 \dots \mu_{J-1})$.

Omitted Variables, Neglected Heteroscedasticity and Non-Normality

Letting:

$$\Phi_{ij} = \Phi(\mu_j - x_i'\beta) = \text{prob}(y_i^* \geq \mu_j)$$

$$\phi_{ij} = \phi(\mu_j - x_i'\beta)$$

the derivatives of Φ_{ij} are:

$$\frac{\delta\Phi_{ij}}{\delta\beta} = -\phi_{ij} x_i$$

$$\frac{\delta\Phi_{ij}}{\delta\mu_j} = \phi_{ij}$$

When there are omitted variables, Φ_{ij} is replaced by:

$$F_{ij} = \Phi(\mu_j - x_i'\beta - z_i'\gamma)$$

and:

$$\frac{\delta F_{ij}}{\delta\gamma} = -\phi_{ij} z_i$$

under the null $\gamma = 0$. Pagan and Vella (1989) propose a RESET like test of incorrect functional form⁶. They suggest testing for the omission of powers of $x_i'\beta$ which is just a special case of omitted variables.

With neglected heteroscedasticity let:

$$\sigma_i^2 = \exp(w_i'\eta)^2$$

so that Φ_{ij} is replaced by:

$$F_{ij} = \Phi\left(\frac{\mu_j - x_i'\beta}{e^{w_i'\eta}}\right)$$

Then:

$$\frac{\delta F_{ij}}{\delta\eta} = -(\mu_j - x_i'\beta)\phi_{ij} w_i$$

⁶ See also MacKinnon and Magee (1990) and Davidson and MacKinnon (1993).

under the null $\eta = 0$. More generally suppose

$$\sigma_1^2 = f(w/\eta)$$

where $f(0) = 1$. Then $\delta F_{ij} / \delta \eta$ includes the term $f'(0)$ under the null. This washes out when calculating the LM test statistic since, as shown below, this may be expressed as the explained sum of squares (ESS) in an artificial regression.

Using a Gram Charlier type A alternative to capture non-normality⁷, the density ϕ_{ij} is replaced by:

$$\phi_{ij} \left\{ 1 + \frac{\kappa_3}{3!} H_2(\mu_j - x_i/\beta) + \frac{\kappa_4}{4!} H_3(\mu_j - x_i/\beta) \right\}$$

and the distribution Φ_{ij} is replaced by:

$$F_{ij} = \Phi_{ij} - \phi_{ij} \left\{ \frac{\kappa_3}{3!} H_2(\mu_j - x_i/\beta) + \frac{\kappa_4}{4!} H_3(\mu_j - x_i/\beta) \right\}$$

where κ_3 and κ_4 are the third and fourth order cumulants and H_2 to H_4 are the second, third and fourth order Hermite polynomials respectively:

$$\begin{aligned} \kappa_3 &= E x^3 \\ \kappa_4 &= E x^4 - 3E x^2 = E x^4 - 3 \\ H_2(x) &= x^2 - 1 \\ H_3(x) &= x^3 - 3x \\ H_4(x) &= x^4 - 6x^2 + 3 \end{aligned}$$

The derivatives of F_{ij} with respect to κ_3 and κ_4 are:

$$\frac{\delta F_{ij}}{\delta \kappa_3} = - \frac{1}{3!} H_2(\mu_j - x_i/\beta) \phi_{ij}$$

$$\frac{\delta F_{ij}}{\delta \kappa_4} = - \frac{1}{4!} H_3(\mu_j - x_i/\beta) \phi_{ij}$$

under the null $\kappa_3 = \kappa_4 = 0$ ie. no asymmetry or excess kurtosis which is true when the error term in the latent regression is normal.

⁷ See Kendall and Stuart (1977) or Ord (1972) for example.

Log Likelihood, Score and Information Matrix

With a random sample of N individuals the log likelihood is:

$$l = \sum_{i=1}^N \sum_{j=0}^J y_{ij} \ln p_{ij}(\theta)$$

where $\mu' = (\mu_1, \dots, \mu_{J-1})$ and $\theta' = (\beta', \mu', \gamma', \eta', \kappa_3, \kappa_4)$ is the vector of parameters. Under the null hypotheses of no omitted variables, no neglected heteroscedasticity or non-normality $\gamma=0, \eta=0, \kappa_3=0$ and $\kappa_4=0$. The score is:

$$\frac{\delta l}{\delta \theta} = \sum_{i=1}^N \sum_{j=0}^J \frac{y_{ij}}{p_{ij}} \frac{\delta p_{ij}}{\delta \theta}$$

which may be recast as:

$$\begin{aligned} \frac{\delta l}{\delta \theta} &= \sum_i \sum_j \left(\frac{y_{ij} - p_{ij}}{\sqrt{p_{ij}}} \right) \left(\frac{1}{\sqrt{p_{ij}}} \frac{\delta p_{ij}}{\delta \theta} \right) \\ &= \sum_i \sum_j u_{ij} v_{ij} \end{aligned}$$

since $\sum_j \delta p_{ij} / \delta \theta = 0$. The u_{ij} 's may be thought of as standardised residuals.

They have zero means, variances equal to $1 - p_{ij}$ and covariances equal to $-(p_{ij} p_{ik})^{1/2}$ when $j \neq k$. The information matrix is:

$$\begin{aligned} I_{\theta\theta} &= \lim_{N \rightarrow \infty} E \frac{1}{N} \frac{\delta l}{\delta \theta} \frac{\delta l}{\delta \theta'} \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \sum_j \frac{1}{p_{ij}} \frac{\delta p_{ij}}{\delta \theta} \frac{\delta p_{ij}}{\delta \theta'} \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \sum_j v_{ij} v_{ij}' \end{aligned}$$

since the sample is random, $E y_{ij}^2 = E y_{ij} = p_{ij}$ and $E y_{ij} y_{ik} = 0$ when $j \neq k$. The information matrix is assumed to be non-singular in the neighbourhood of the true parameter value. The derivatives of $p_{i0} \dots p_{iJ}$ with respect to θ , which are required to calculate the score and information matrix, are given in the Appendix.

Under the null hypotheses, the observed score is:

$$\begin{aligned}\frac{\delta l}{\delta \tilde{\theta}} &= \sum_i \sum_j \frac{y_{ij} \delta p_{ij}}{\tilde{p}_{ij} \delta \tilde{\theta}} \\ &= \sum_i \sum_j \left(\frac{y_{ij} - \tilde{p}_{ij}}{\sqrt{\tilde{p}_{ij}}} \right) \left(\frac{1}{\sqrt{\tilde{p}_{ij}}} \frac{\delta p_{ij}}{\delta \tilde{\theta}} \right) \\ &= \sum_i \sum_j \tilde{u}_{ij} \tilde{v}_{ij}\end{aligned}$$

and the observed information matrix is:

$$\begin{aligned}I_{\tilde{\theta}\tilde{\theta}}^o &= \frac{1}{N} E \frac{\delta l}{\delta \tilde{\theta}} \frac{\delta l}{\delta \tilde{\theta}'} \\ &= \frac{1}{N} \sum_i \sum_j \tilde{p}_{ij} \frac{\delta p_{ij}}{\delta \tilde{\theta}} \frac{\delta p_{ij}}{\delta \tilde{\theta}'} \\ &= \frac{1}{N} \sum_i \sum_j \tilde{v}_{ij} \tilde{v}_{ij}'\end{aligned}$$

where $\tilde{\theta}' = (\tilde{\beta}', \tilde{\mu}', 0', 0', 0, 0)$ is the restricted vector of parameter estimates and \tilde{p}_{ij} is the predicted probability evaluated at $\tilde{\theta}$ etc. Note that $\frac{\delta l}{\delta \tilde{\beta}} = \frac{\delta l}{\delta \tilde{\mu}} = 0$.

Artificial Regression Based LM Test Statistic

The LM test statistic of the null $\gamma=0, \eta=0, \kappa_3=0$ and $\kappa_4=0$ is:

$$LM = \frac{1}{N} \frac{\delta l}{\delta \tilde{\theta}'} I_{\tilde{\theta}\tilde{\theta}}^{o-1} \frac{\delta l}{\delta \tilde{\theta}}$$

which, under standard weak regularity conditions, has a chi-squared distribution with degrees of freedom equal to the number of restrictions. As shown in Murphy (1994b), the LM test statistic may be calculated as:

$$LM = \left(\sum_i \sum_j \tilde{u}_{ij} \tilde{v}_{ij} \right)' \left(\sum_i \sum_j \tilde{v}_{ij} \tilde{v}_{ij}' \right)^{-1} \left(\sum_i \tilde{u}_{ij} \tilde{z}_{ij} \right)$$

Thus the LM test statistic is simply the explained sum of squares from the

uncentered auxiliary regression of the \hat{u}_{ij} 's on the \hat{v}_{ij} 's across all J+1 alternatives and N individuals⁸. In the case of the ordered probit model, the \hat{u}_{ij} 's and the \hat{v}_{ij} 's are easily calculated given the restricted parameter estimates and the results in the Appendix. Single or joint tests for omitted variables, neglected heterogeneity, incorrect functional form and non-normality may also be performed.

Comparison With Other Tests of the Ordered Probit Model

It is useful to compare the proposed tests in this paper with some other available tests. Since the information matrix is not block diagonal one must include the scores for β and μ when calculating the LM statistics, although they are identically zero at the restricted parameter estimates. The LM statistics may be calculated using artificial regression of either the OPG form or the expectation of the outer product of the matrix of contributions to the score, which is the approach used in this paper.

Holden (1993) derives LM tests for neglected homoscedasticity and non-normality for the ordered probit model. He considers OPG and non-OPG artificial regression based tests. The regressand in his non-OPG artificial regression is $y_i / \sqrt{\hat{p}_i}$, which is a component of \hat{u}_i , and the regressor is \hat{v}_{ij} . Since:

$$\sum_i \sum_j \frac{y_{ij}}{\sqrt{\hat{p}_{ij}}} \hat{v}_{ij} = \sum_i \sum_j \hat{u}_{ij} \hat{v}_{ij}$$

Holden's LM test statistic is numerically equal to the LM test statistic proposed here. His test for homoscedasticity is the same as the test in this paper since, apart from the $f'(0)$ scaling, the scores are identical. His test for non-normality, which is based on the Pearson distribution alternative and uses results from Bera, Jarque and Lee (1982), is different from the test in this paper.

Chesher and Irish (1987) use a heuristic argument to derive, what are in effect, conditional moment based tests for omitted variables, neglected heteroscedasticity and non-normality, inter alia, in the grouped normal regression model. These tests are easily extended to the ordered probit model by including likelihood equations for the unobserved thresholds:

⁸ The LM test statistic is also asymptotically equal to NJ times the R² from this auxiliary regression. See Murphy (1994b) for details.

$$\frac{\delta l}{\delta \mu_j} = 0$$

and dropping the likelihood equation for the constant variance, which is not identified in the ordered probit case.

Chesher and Irish calculate various "moment residuals" but these are not required in the approach used in this paper. Their conditional moment based approach naturally leads to the use of the OPG form of LM test although the tests may also be calculated using the artificial regression in this paper. The scores for omitted variables and heteroscedasticity are the same as in this paper. The scores for non-normality are linear combinations of those in Holden so the two LM test statistics are numerically equal if both are calculated in the same way.

The Holden test for non-normality may be compared with the new test proposed in this paper. In the case of the probit model, Murphy(1994b) shows that the two tests examine different scores, to see if they are significantly different from zero. Since the test statistics may be expressed as the ESS 's from artificial regressions, one may ignore signs and scaling and only consider linear combinations of the scores. Apart from scaling the score for κ_3 is the same as one of the scores in Holden. However the score for κ_4 is not. Thus the small sample performance of the two tests may be worth investigating using Monte Carlo methods. However the results reported in Murphy suggest that the two tests are likely to perform equally well.

Conclusion

LM test statistics for omitted variables, heteroscedasticity, incorrect functional form and non-normality in the ordered probit model may be readily calculated using artificial regressions. However the form of the artificial regressions are different to those used in the (binary) probit model. This artificial regression approach is both convenient and likely to have better small sample properties than the more common outer product gradient (OPG) form of LM test statistic.

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Appendix
The Derivatives Of $p_{i0} \dots p_{iJ}$ With Respect to θ Under The Null

The derivatives of $p_{i0} \dots p_{iJ}$ with respect to the elements of θ under the null are given below. These are required to calculate the observed u_{ij} 's and v_{ij} 's used in the regression which produces the LM test statistic.

The derivatives with respect to β are:

$$\begin{aligned} \frac{\delta p_{i0}}{\delta \beta} &= -\phi_{i0} x_i \\ \frac{\delta p_{ij}}{\delta \beta} &= -(\phi_{ij} - \phi_{ij-1}) x_i \quad j=1 \dots J-1 \\ \frac{\delta p_{iJ}}{\delta \beta} &= \phi_{iJ-1} x_i \end{aligned} \tag{A1}$$

where $\phi_{i0} = \phi(-x_i' \beta)$. The derivatives with respect to γ are the same as in (A1) except that z_i replaces x_i .

The non-zero derivatives with respect to μ are:

$$\begin{aligned} \frac{\delta p_{i,1}}{\delta \mu_1} &= \phi_{i1} \\ \frac{\delta p_{ij}}{\delta \mu_{j-1}} &= -\phi_{ij-1} \quad j=2 \dots J-1 \\ \frac{\delta p_{ij}}{\delta \mu_j} &= \phi_{ij} \quad j=2 \dots J-1 \\ \frac{\delta p_{iJ}}{\delta \mu_{J-1}} &= -\phi_{iJ-1} \end{aligned} \tag{A2}$$

All the other derivatives with respect to μ are zero.

The derivatives with respect to η are:

$$\begin{aligned}\frac{\delta p_{10}}{\delta \eta} &= x_1' \beta \phi_{10} w_1 \\ \frac{\delta p_{1j}}{\delta \eta} &= -\{(\mu_j - x_1' \beta) \phi_{1j} - (\mu_{j-1} - x_1' \beta) \phi_{1j-1}\} w_1 \quad j=1 \dots J-1 \\ \frac{\delta p_{1J}}{\delta \eta} &= (\mu_{J-1} - x_1' \beta) \phi_{1J-1} w_1\end{aligned}\tag{A3}$$

where $\mu_0 = 0$.

The derivatives with respect to κ_3 are:

$$\begin{aligned}\frac{\delta p_{10}}{\delta \kappa_3} &= -\frac{1}{3!} H(-x_1' \beta) \phi_{10} \\ \frac{\delta p_{1j}}{\delta \kappa_3} &= -\frac{1}{3!} \{H(\mu_j - x_1' \beta) \phi_{1j} - H(\mu_{j-1} - x_1' \beta) \phi_{1j-1}\} \quad j=1 \dots J-1 \\ \frac{\delta p_{1J}}{\delta \kappa_3} &= \frac{1}{3!} H(\mu_{J-1} - x_1' \beta) \phi_{1J-1}\end{aligned}\tag{A4}$$

The derivatives with respect to κ_4 are similar.