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# Adaptive Universal Portfolios

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## Abstract

The purpose of this paper is to develop a stock selection algorithm with similar properties as Cover's Universal Portfolio[2], but providing superior early growth. Cover's Universal Portfolio generates a growth rate asymptotically equal to the best achievable growth rate over the set of constant rebalanced portfolios. However, Cover's Universal Portfolio is empirically seen to generate poor early growth.

While much research has been conducted in relation to Cover's Universal Portfolio, much of this has focused on efficient implementation of the algorithm and considerations of market frictions. As such, there remains a significant research gap in addressing the issue of poor early growth generated by Cover's strategy. With this in mind we develop the Adaptive Universal Portfolio, a sequential portfolio selection algorithm with similar asymptotic properties as Cover's Universal Portfolio but providing greater early growth.

In this paper we provide an analysis of the growth generated by the two algorithms. Furthermore we present empirical evidence of the superior early growth generated by the Adaptive Universal Portfolio. Finally we discuss possible criticisms of the Adaptive Universal Portfolio, including evidence of momentum following and vulnerability to individual stock risks, and provide an insight into possible future work in this area.

Keywords: Portfolio Choice, Universal Portfolios, Bayesian Analysis, Capital Growth Models, Kelly Criterion.

JEL Classification: C11; G11

## 1. Introduction

The goal of the rational investor is to maximize the return on his capital investment. To this end when given a set of possible investments, he seeks to maximize his return with respect to some underlying assumptions on the assets, or some other strategy constraints. As mathematicians have taken a firmer interest in thoughts of a financial nature, many portfolio strategies have been developed based on assumed market conditions and structures, with varying theoretical and empirical results. Many of these strategies have become widely accepted and acclaimed, furthermore these strategies have led to much debate and research within both the academic and financial world. A notable such strategy is that of the Modern Portfolio Theory proposed by Harry Markowitz in 1952 [10], although Markowitz refers to it as simply "Portfolio Theory" since "there's nothing modern about it"!

Modern Portfolio Theory [10,11,12,13,14] assumes investors are risk averse and invest their wealth so as to maximize expected return for a given level of risk, or minimize risk for a given level of expected

return, where the risk of a portfolio is given by its standard deviation. One advantage of Modern Portfolio Theory is that of the diversification effect in which an investor can reduce exposure to individual asset risk, by holding a well-diversified portfolio of assets. However a significant issue with Modern Portfolio Theory is that of parameter estimation. Modern Portfolio Theory requires knowledge of the future expected returns and the future covariance matrix of the assets in the portfolio. These values can not be exactly known for market traded assets which form an investors portfolio. Therefore, an investor must make estimations of these model parameters, either through use of historical price data or using thier own judgement, and errors in these estimations can have significant repercussions on the performance of the model.

Another notable investment strategy which has been subject to much interest, both in the academic world and in industry, is the optimization of expected log value of an investors wealth, which was introduced by Kelly [8], and has duly become known as the Kelly Criterion [1,8,15,16], otherwise known as Kelly Betting to many. An investor applying the Kelly Criterion seeks to maximize his log wealth. The Kelly Criterion has two significant benefits. Firstly the optimization of expected log wealth is shown to minimize the expected time for an investors wealth to grow to a suitably large level. Secondly, the growth generated using the Kelly Criterion is shown to exceed the growth generated using anyother investment strategy, that is the Kelly Criterion maximizes wealth, and an investor using the Kelly Criterion will become the richest.

However, significant issues arise when implementing the Kelly Criterion. Since the probability structure of market traded assets can not be known exactly, and must therefore be estimated. This can lead to significant issues of investing more than the true Kelly Bet, the proportion of wealth invested which maximizes expected log wealth, which leads to a reduction in the probability of beneficial results, and an increases in the probability of detrimental results.

For example, let us consider a game where we may bet on the outcome a biased coin toss where we have a 60% probability of winning and a 40% probability of loosing. At even odds, i.e. an investment of 1 unit returns 2 units if we win, or 0 units if we loose. We observe that an investment of a proportion of our wealth,  $c$ , in the game results in the expected log wealth of

$$\frac{3}{5}\log(1+c) + \frac{2}{5}\log(1-c) \tag{1}$$

Differentiating (1) with respect to  $c$  we observe that the optimal investment in the game is 20% of our wealth. However, as we see in Figure 1 below, investment of more than the 20% optimal investment can have detrimental results on our expected log wealth. Moreover since our optimal investment value, or Kelly Criterion for the game, will be based on observations of imperfect information in real world cases, misspecification of the probability structure of underlying processes can have detrimental effects on the growth of our wealth.

These two examples demonstrate how model assumptions and parameter estimation can lead an investor to his detriment if reality does not align with his perception of the world. Fortunately, there exist portfolio strategies which display favorable properties without making assumptions as to the mechanisms which drive the prices, and thus returns and growths, of the assets in a portfolio. We call such strategies assumption free, or alternatively parameterless models. A notable such assumption free trading strategy for investing in a portfolio of equities is the Universal Portfolio as proposed by Cover [2]. Cover's portfolio selection algorithm is shown to be universal over the set of constant rebalanced portfolios and is shown to be competitive with the best constant rebalanced portfolio, in a sense which will become clearer presently.

The goal of Cover's Universal Portfolio is to form a sequential portfolio selection algorithm with the aim of performing as well as an investor who has hindsight knowledge, i.e. an investor who has complete knowledge of the future prices of the stocks under consideration. Cover wishes to form a sequential portfolio selection algorithm without making any assumptions about the market or the return dynamics of the stocks

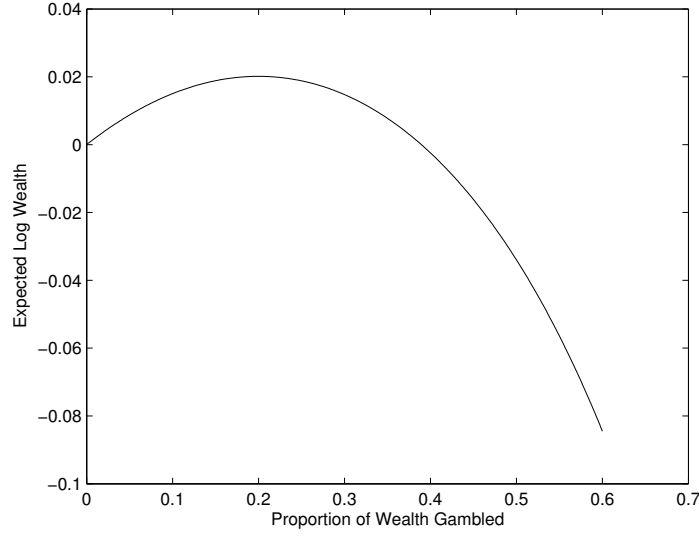


Figure 1: Expected Log Wealth Resulting from the above stochastic game.

under consideration. The goal is to form a portfolio which performs as well as the best constant rebalanced portfolio.

Consider an investor considering investing in  $m$  equities. Let  $\mathbf{x} = (x_1, \dots, x_m)' \geq 0$  denote a stock market vector for one investment period, where we set  $x_i$  to be the price relative of the  $i^{\text{th}}$  stock over the investment period. A portfolio is an  $m$ -dimensional vector,  $b = (b_1, \dots, b_m)'$  such that  $b_i \geq 0, i = 1, \dots, m$  and  $\sum_{i=1}^m b_i = 1$ , where  $b_i$  is the proportion of the investors wealth invested in equity  $i$ . The increase in wealth obtained over the investment period by holding the portfolio  $b$  is given by

$$S(b) = b_1x_1 + \dots + b_mx_m = \sum_{i=1}^m b_ix_i = b'\mathbf{x} \quad (2)$$

Moving to a multi-period investment horizon we consider an (arbitrary) sequence of stock market vectors  $\mathbf{x}_1, \dots, \mathbf{x}_t$ , denoting the stock vector for the  $i^{\text{th}}$  investment period by  $\mathbf{x}_i$ . A constant rebalanced portfolio strategy is a strategy whereby trading occurs at the end of each investment period, so as to reset the proportion of the investors wealth held in each stock to the proportion held at the beginning of the investment period. Therefore, the constant rebalanced portfolio,  $b$ , held over  $t$  investment periods increases by a factor of

$$S_t(b) = \prod_{i=1}^t b'\mathbf{x}_i \quad (3)$$

where we normalize  $S_0(b) = 1 \forall b$ . The best constant rebalanced portfolio over  $t$ -periods is defined by

$$b_t^* = \underset{b}{\operatorname{argmax}} S_t(b) \quad (4)$$

This leads to a natural definition of the greatest achievable wealth growth over the  $t$  investment periods by a

constant rebalanced portfolio strategy as

$$S_t^* = \max_b S_t(b) = S_t(b_t^*) \quad (5)$$

With  $S_t^*$  as the target wealth, Cover defines his sequential portfolio selection by

$$\begin{aligned} \hat{b}_1 &= \left(\frac{1}{m}, \dots, \frac{1}{m}\right) \\ \hat{b}_{k+1} &= \frac{\int_{\mathfrak{B}} b S_k(b) db}{\int_{\mathfrak{B}} S_k(b) db} \end{aligned} \quad (6)$$

where  $\mathfrak{B} = \{b \in \mathbb{R}^m, b_i \geq 0 \text{ for } i = 1, \dots, m, \sum_{i=1}^m b_i = 1\}$ .

Therefore Cover's Universal Portfolio invests uniformly over each of the possible portfolios initially, and then invests more heavily in those portfolios which have performed relatively better as time goes by. The wealth growth of the Universal Portfolio over  $t$  investment periods is given by

$$\hat{S}_t = \prod_{i=1}^t \hat{b}_i' \mathbf{x}_i \quad (7)$$

Cover [2] has shown that for a bounded sequence of stock vectors  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_t$

$$\frac{1}{t} \ln \hat{S}_t - \frac{1}{t} \ln S_t^* \rightarrow 0 \quad (8)$$

Therefore  $\hat{S}_t$  and  $S_t^*$ , have the same exponent to first order. However, most industry practitioners find Cover's Universal Portfolio to have one significant flaw. The algorithm is empirically seen to provide poor early growth, as it requires a long time horizon for the initial conditions of the algorithm to wash out and the fitter portfolios to be recognized, as Cover [4] has stated "the performance of the algorithm, although good relative to the best portfolio in hindsight, is still slow in responding in an absolute sense. It sometimes requires hundreds of days before the initial conditions wash out, leaving the 'fittest' rebalanced portfolio dominating the performance. It's guiding thinking, but no one's making money off it yet." The focus of this paper is to develop a causal investment algorithm with similar properties to Cover's Universal Portfolio but which provides significantly better returns for an investor with a short to medium investment horizon. The sequential portfolio selection algorithm developed in this paper is similar in structure to that of Cover's Universal Portfolio, but with greater emphasis given to the portfolios which historically performed better. We define the Adaptive Universal Portfolio by

$$\begin{aligned} \hat{b}_1 &= \left(\frac{1}{m}, \dots, \frac{1}{m}\right) \\ \hat{b}_{k+1} &= \frac{\int_{\mathfrak{B}} \alpha_k(b) b S_k(b) db}{\int_{\mathfrak{B}} \alpha_k(b) S_k(b) db} \end{aligned} \quad (9)$$

where  $\mathfrak{B} = \{b \in \mathbb{R}^m, b_i \geq 0 \text{ for } i = 1, \dots, m, \sum_{i=1}^m b_i = 1\}$  and  $\int_{\mathfrak{B}} \alpha_k(b) db = 1 \forall k$ . Thus, the algorithm can also be seen as being a weighted average, where the function  $\alpha_k$  gives greater weighting to the portfolios which have performed better up to time  $k$ , and penalizes the portfolios which have performed badly.

The remainder of this paper is laid out as follows. In section 2 we give a brief introduction to Cover's Universal Portfolio and state many of its properties. In section 3 we introduce portfolio discriminate functions

and the Adaptive Universal Portfolio. In section 4 we discuss two methods of algorithmic implementation for the Adaptive Universal Portfolio and Cover's Universal Portfolio, namely quantization methods and Monte Carlo methods for definite integral calculation. In section 5 we apply the quantization method to a variety of pairs-of-stocks over a variety of short-to-medium term time horizons to investigate the performance of the Adaptive Universal Portfolio and Cover's Universal Portfolio. Finally, in section 6 we give our concluding remarks, and a brief insight into possible future work in the area.

## 2. Cover's Universal Portfolio Algorithm

In the previous section we introduced the nature and goals of the work undertaken in this paper. In this section we introduce Cover's Universal Portfolio [2] and state its main properties and goals. We begin by considering an investor who is considering investing his wealth in a portfolio of  $m$  different assets. We begin by introducing some basic definitions related to the performance of a particular portfolio. We denote the growth rate of the portfolio  $b$  with respect to the distribution of the stock market vector  $F(\mathbf{x})$  by

$$W(b, F) = \int \ln(b' \mathbf{x}) dF(\mathbf{x}) \quad (10)$$

We denote the maximum achievable growth rate by

$$W^*(F) = \max_b W(b, F) \quad (11)$$

Denoting the empirical distribution of the realized (arbitrary) stock market sequence  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_t$  by  $F_t$ , i.e.  $F_t$  places a mass of  $\frac{1}{t}$  at each of the points  $\mathbf{x}_i$  for  $i = 1, 2, \dots, t$ . We now note the following properties of the target growth rate  $S_t^*$  as defined by (5) above.

**Proposition 1 (Cover):** The target wealth exceeds the best stock.

$$S_t^* \geq \max_{j=1,2,\dots,m} S_t(e_j) \quad (12)$$

where  $(e_j)_{j=1}^m$  is the canonical basis for  $\mathbb{R}^m$ .

**Proof:**  $S_t^*$  is a maximization of  $S_t(b)$  over the simplex, whereas  $\max_{j=1,2,\dots,m} S_t(e_j)$  is the maximization of  $S_t(b)$  over the vertices of the simplex.

**Proposition 2 (Cover):** The target wealth exceeds the value line.

$$S_t^* \geq \left( \prod_{j=1}^m S_t(e_j) \right)^{1/m} \quad (13)$$

where  $(e_j)_{j=1}^m$  is the canonical basis for  $\mathbb{R}^m$

**Proof:**  $S_n^* \geq S_n(e_j)$  for each  $j = 1, \dots, m$ , implies  $S_n^{*m} \geq \prod_{j=1}^m S_n(e_j)$ , which gives  $S_t^* \geq (\prod_{j=1}^m S_t(e_j))^{1/m}$ .

**Proposition 3 (Cover):** The target wealth exceeds arithmetic mean

$$S_t^* \geq \sum_{j=1}^m \alpha_j S_t(e_j) \quad (14)$$

where  $(e_j)_{j=1}^m$  is the canonical basis for  $\mathbb{R}^m$ ,  $\alpha_j \geq 0$  for  $j = 1, \dots, m$  and  $\sum_{j=1}^m \alpha_j = 1$ .

**Proof:**  $S_t^* \geq S_t(e_j)$ ,  $j = 1, \dots, m$ . Thus  $\sum_{j=1}^m \alpha_j S_t^* \geq \sum_{j=1}^m \alpha_j S_t(e_j)$ . Therefore we have  $S_t^* \geq \sum_{j=1}^m \alpha_j S_t(e_j)$ .

We further note that  $S_t^*(\mathbf{x}_1, \dots, \mathbf{x}_t)$  is invariant under permutation of  $\mathbf{x}_1, \dots, \mathbf{x}_t$ . We will now discuss Cover's Universal Portfolio algorithm and its properties. Cover defines the Universal Portfolio as follows

$$\begin{aligned} \hat{b}_1 &= \left(\frac{1}{m}, \dots, \frac{1}{m}\right) \\ \hat{b}_{k+1} &= \frac{\int_{\mathfrak{B}} b S_k(b) db}{\int_{\mathfrak{B}} S_k(b) db} \end{aligned} \quad (15)$$

A sequential portfolio selection algorithm,  $b_\bullet$ , is said to be a causal portfolio strategy if the choice of  $b_i$  depends only on  $\mathfrak{F} = \sigma(\mathbf{x}_1, \dots, \mathbf{x}_{i-1})$ , i.e. the choice of  $b_\bullet$  is independent of future stock prices, therefore Cover's Universal Portfolio is a causal portfolio strategy.

The accumulated value of Cover's Universal Portfolio after  $t$  investment periods is therefore given by

$$\hat{S}_t = \prod_{i=1}^t \hat{b}'_i \mathbf{x}_i \quad (16)$$

We note that  $\hat{b}_{k+1} = \frac{\int_{\mathfrak{B}} b S_k(b) db}{\int_{\mathfrak{B}} S_k(b) db}$  implies the following proposition.

**Proposition 4 (Cover):**

$$\hat{S}_n = \prod_{k=1}^n \hat{b}'_k \mathbf{x}_k = \frac{\int_{\mathfrak{B}} S_n(b) db}{\int_{\mathfrak{B}} db} \quad (17)$$

where

$$S_n(b) = \prod_{k=1}^n b' \mathbf{x}_k \quad (18)$$

Thus the wealth resulting from the Universal Portfolio,  $\hat{S}_n$ , is the average of  $S_n(b)$  over the portfolio simplex.

**Proof:**

$$\hat{b}'_k \mathbf{x}_k = \frac{\int_{\mathfrak{B}} b' \mathbf{x}_k \prod_{i=1}^{k-1} b' \mathbf{x}_i db}{\int_{\mathfrak{B}} \prod_{i=1}^{k-1} b' \mathbf{x}_i db} = \frac{\int_{\mathfrak{B}} \prod_{i=1}^k b' \mathbf{x}_i db}{\int_{\mathfrak{B}} \prod_{i=1}^{k-1} b' \mathbf{x}_i db} \quad (19)$$

Therefore

$$\hat{S}_n = \prod_{k=1}^n \hat{b}'_k \mathbf{x}_k = \frac{\int_{\mathfrak{B}} \prod_{i=1}^n b' x_i db}{\int_{\mathfrak{B}} db} = \frac{\int_{\mathfrak{B}} S_n(b) db}{\int_{\mathfrak{B}} db} \quad (20)$$

Furthermore the following proposition holds

**Proposition 5 (Cover):** The Universal Portfolio exceeds the value line index

$$\hat{S}_n \geq \left( \prod_{k=1}^m S_n(e_j) \right)^{1/m} \quad (21)$$

**Proof:** With  $F_t$  the empirical distribution function of  $\mathbf{x}_1, \dots, \mathbf{x}_t$ . We have

$$\hat{S}_t = \frac{\int_{\mathfrak{B}} S_t(b) db}{\int_{\mathfrak{B}} db} = \mathbb{E}_b S_t(b) = \mathbb{E}_b e^{tW(b, F_t)} \geq e^{t\mathbb{E}_b W(b, F_t)} = e^{t\mathbb{E}_b \int \ln b' \mathbf{x} dF_t(\mathbf{x})} \quad (22)$$

Thus

$$\hat{S}_t = e^{t\mathbb{E}_b \int \ln(\sum_{j=1}^m b_j e'_j \mathbf{x}) dF_t(\mathbf{x})} \quad (23)$$

Therefore

$$\hat{S}_t \geq e^{t\mathbb{E}_b \sum_{j=1}^m b_j \int \ln(e'_j \mathbf{x}) dF_t(\mathbf{x})} = e^{n((1/m)\sum \int \ln(e'_j \mathbf{x}) dF_t(\mathbf{x}))} = \left( \prod_{j=1}^m S_n(e_j) \right)^{1/m} \quad (24)$$

where the two inequalities above result from Jensen's Inequality.

We further note that the wealth achieved under Cover's Universal Portfolio,  $\hat{S}_t(\mathbf{x}_1, \dots, \mathbf{x}_t)$  may be given by

$$\hat{S}_t = \frac{\int_{\mathfrak{B}} S_t(b) db}{\int_{\mathfrak{B}} db} = \frac{\int_{\mathfrak{B}} \prod_{k=1}^t b' \mathbf{x}_k db}{\int_{\mathfrak{B}} db} \quad (25)$$

and we observe that the above integrals are invariant under permutations of  $(\mathbf{x}_1, \dots, \mathbf{x}_t)$ .

We now define the sensitivity matrix function  $J(b)$  with respect to the distribution of the stock market vector  $dF(\mathbf{x})$ .  $J(b)$  gives a measure of the curvature of  $S_t$  about its maximum. This curvature accounts for the second order behavior of  $\hat{S}_t$  with respect to  $S_t^*$ .

The sensitivity matrix function  $J(b)$  of a market with respect to the distribution  $dF(\mathbf{x})$ , where  $\mathbf{x} \in \mathbb{R}_+^m$ , is the  $(m-1) \times (m-1)$  matrix given by

$$J_{ij}(b) = \int \frac{(x_i - x_m)(x_j - x_m)}{(b' \mathbf{x})^2} dF(\mathbf{x}), \quad 1 \leq i, j \leq m-1 \quad (26)$$

Moreover the matrix  $J^*$  is defined by  $J(b^*)$ , where  $b_i^*$  is given by (3), and is given by

$$J_{ij}^* = - \frac{\partial^2 W((b_1^*, b_2^*, \dots, b_{m-1}^*, 1 - \sum_{k=1}^{m-1} b_k^*), F)}{\partial b_i \partial b_j} \quad (27)$$

We now give the main theorem of Cover's Universal Portfolios [2].

**Theorem 1 (Cover):** Suppose  $\mathbf{x}_1, \mathbf{x}_2, \dots \in [a, c]^m$ ,  $0 < a \leq c < \infty$ , and at a subsequence of times  $n_1, n_2, \dots$ ,  $W_n(b) \nearrow W(b)$  for  $b \in \mathfrak{B}$ ,  $J_t^* \rightarrow J^*$ ,  $b_t^* \rightarrow b^*$ , where  $W(b)$  is strictly concave, the third partial derivatives of  $W$  are bounded on  $\mathfrak{B}$ , and  $W(b)$  achieves its maximum at  $b^*$  in the interior of  $\mathfrak{B}$ . Then

$$\frac{\hat{S}_t}{S_t^*} \sim \left( \sqrt{\frac{2\pi}{t}} \right)^{m-1} \frac{(m-1)!}{|J^*|^{1/2}} \quad (28)$$

in the sense that the ratio of the right and left hand sides converge to 1 along the subsequence

**Proof:** See Cover [2].

A portfolio strategy,  $b_\bullet$ , is said to be Universal over the set of possible portfolio strategies,  $\mathfrak{B}$ , if the difference between the mean growth rate,  $G_t(\bullet)$ , of  $b_\bullet$  and  $b^*$  vanishes asymptotically, that is if we have

$$\limsup_{t \rightarrow \infty} (G_t(b^*) - G_t(b_\bullet)) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \frac{S_t(b^*)}{S_t(b_\bullet)} = 0 \quad (29)$$

where we have  $G_t(b_\bullet) = \frac{1}{t} \sum_{k=1}^t \log(b'_k x_k)$ . Therefore Cover's Universal Portfolio is universal over the space of constant rebalanced portfolio.

An alternative understanding of Cover's Universal Portfolio can be obtained as follows. Consider the problem of calculating  $b^*$ , the best constant rebalanced portfolio, based on the observation of  $S_t(b)$ , the performance of each constant rebalanced portfolio at time  $t$ . We can make use of Bayesian methods to calculate an expected value for  $b^*$ . Making no assumption on the portfolio performances, one would use the uniform distribution as the prior distribution. Therefore

$$\mathbb{P}(b^* = \check{b}) = \frac{1}{\int_{\mathfrak{B}} db} \quad (30)$$

where  $\mathfrak{B} = \{b \in \mathbb{R}^m | b'1 = 1, b \geq 0\}$ .

Using the likelihood probability given by

$$\mathbb{P}(S_t | b^* = \check{b}) = \frac{S_t(\check{b})}{\int_{\mathfrak{B}} S_t(b) db} \quad (31)$$

We obtain the following

$$\begin{aligned}
\mathbb{P}(b^* = \check{b} | S_t) &= \frac{\mathbb{P}(b^* = \check{b}) \mathbb{P}(S_t | b^* = \check{b})}{\int_{\mathfrak{B}} (b^* = b) \mathbb{P}(S_t | b^* = b) db} \\
&= \frac{1 \quad S_t(\check{b})}{\int_{\mathfrak{B}} db \int_{\mathfrak{B}} S_t(b) db} \\
&= \frac{1 \quad S_t(\check{b})}{\int_{\mathfrak{B}} \frac{1 \quad S_t(b)}{\int_{\mathfrak{B}} db \int_{\mathfrak{B}} S_t(b) db} db} \\
&= \frac{1 \quad S_t(\check{b})}{\int_{\mathfrak{B}} db \int_{\mathfrak{B}} \frac{S_t(b)}{\int_{\mathfrak{B}} S_t(b) db} db}
\end{aligned} \tag{32}$$

Rearranging the integrals we observe

$$\begin{aligned}
\mathbb{P}(b^* = \check{b} | S_t) &= \frac{1 \quad S_t(\check{b})}{\int_{\mathfrak{B}} db \int_{\mathfrak{B}} S_t(b) db} \\
&= \frac{1}{\int_{\mathfrak{B}} db \int_{\mathfrak{B}} S_t(b) db} \int_{\mathfrak{B}} S_t(b) db \\
&= \frac{S_t(\check{b})}{\int_{\mathfrak{B}} S_t(b) db}
\end{aligned} \tag{33}$$

And so

$$\begin{aligned}
\mathbb{E}(b^* | S_t) &= \int_{\mathfrak{B}} b \mathbb{P}(b^* = b | S_t(b)) db \\
&= \int_{\mathfrak{B}} b \frac{S_t(b)}{\int_{\mathfrak{B}} S_t(b) db} db \\
&= \frac{\int_{\mathfrak{B}} b S_t(b) db}{\int_{\mathfrak{B}} S_t(b) db}
\end{aligned} \tag{34}$$

Noting that the above is the exact form of Cover's Universal Portfolio, we see that Cover's Universal Portfolio is the expected value of the best constant rebalanced portfolio based on Bayesian probabilities assuming a uniform prior distribution and likelihood function given by portfolio relative performance.

In this section we have discussed many of the properties of Cover's Universal Portfolio, the most important of all being the universal nature of Cover's algorithm over the space of constant rebalanced portfolio strategies. In the next section we introduce the notion of a portfolio discriminant function, which can be used to tractably rank portfolios based on historical performance. With the use of a portfolio discriminant function, we will propose a competing causal sequential portfolio selection algorithm, which we show will perform at least as well as a finite fraction of the wealth obtained through Cover's Universal Portfolio algorithm. Therefore, through application of Theorem 1, the proposed algorithm performs at least as well as a finite fraction of  $S_t^*$ , so that we may conclude that the proposed sequential portfolio selection algorithm is universal.

### 3. Adaptive Universal Portfolio

Previously we have discussed the need for parameterless models, in light of the possible issues of parameter misestimation in parameter dependent models such as those discussed in the introduction of this paper. In the previous section we discussed one such parameter free model, Cover's Universal Portfolio.

In this section we propose a second causal sequential portfolio selection algorithm. Cover's Universal Portfolio strategy has the following rationale. Given an array of portfolio managers, each investing according to a unique constant rebalanced portfolio strategy. If we give each of the portfolio managers a portion of our wealth,  $db / \int_{\mathfrak{B}} db$ , with each portfolio manager accumulating  $S_n(b) = e^{nW(b, dF_t)} / \int db$ , at the exponential rate  $W(b, dF_t)$ , where  $b$  is the unique constant rebalanced portfolio strategy implemented by the portfolio manager. Aggregation of the different portfolio manager's accumulated wealths is carried out theoretically, resulting in the portfolio selection strategy we refer to as Cover's Universal Portfolio.

Under suitable smoothness conditions, the average of exponentials has the same asymptotic growth rate as the maximum, therefore Cover's Universal Portfolio achieves almost as much growth as the growth achieved by the best constant rebalanced portfolio. Our strategy seeks to place greater weighting in better performing portfolio managers, and penalize the worse performing portfolio managers, which appears intuitively reasonable.

Furthermore, observing that Cover's Universal Portfolio is guaranteed to obtain at least a finite fraction as the growth as the best constant rebalanced portfolio, by holding a positive finite amount of wealth in every constant rebalanced portfolio, so too does the Adaptive Universal Portfolio, which ensures it is also universal. A further comparison of the wealths achieved by the two portfolios is given through Theorem 2.

We propose the following sequential portfolio selection algorithm, which we refer to as the *Adaptive Universal Portfolio*, denoted by  $\check{b}_k$  at time  $k$

$$\check{b}_1 = \left( \frac{1}{m}, \dots, \frac{1}{m} \right)$$

$$\check{b}_{k+1} = \frac{\int_{\mathfrak{B}} b \alpha_k(b) S_k(b) db}{\int_{\mathfrak{B}} \alpha_k(b) S_k(b) db} \quad (35)$$

where

$$\alpha_k(b) > 0 \quad \forall k, \forall b \in \mathfrak{B} \quad (36)$$

$$\int_{\mathfrak{B}} \alpha_k(b) db = 1 \quad \forall k \quad (37)$$

$$\forall b_i, b_j \text{ s.t. } S_k(b_i) > S_k(b_j) \Rightarrow \alpha_k(b_i) > \alpha_k(b_j) \quad (38)$$

We refer to the function  $\alpha_k$  as being a *portfolio discriminant function*, which ranks the constant rebalanced portfolios according to their historical performance as described by (3). The accumulated growth of the Adaptive Universal Portfolio is given by

$$\check{S}_t = \prod_{k=1}^t \check{b}'_k \mathbf{x}_k \quad (39)$$

The following theorem characterizes the behavior of  $\check{S}_t$  with respect to  $\hat{S}_t$ .

**Theorem 2:** Suppose  $\mathbf{x}_1, \mathbf{x}_2, \dots \in [a, c]^m, 0 < a \leq c < \infty$ , and at a subsequence of times  $n_1, n_2, \dots, W_n(b) \nearrow W(b)$  for  $b \in \mathfrak{B}, J_t^* \rightarrow J^*, b_t^* \rightarrow b^*$ , where  $W(b)$  is strictly concave, the third partial derivatives of  $W$  are bounded on  $\mathfrak{B}$ , and  $W(b)$  achieves its

maximum at  $b^*$  in the interior of  $\mathfrak{B}$ , and with  $\alpha_k$  being a suitably smooth portfolio discriminant function of  $k$ , moreover it is assumed  $\alpha_t \rightarrow \alpha$ . Then

$$\check{S}_t \geq (1 - \varepsilon)^{t-1} \hat{S}_t \quad (40)$$

for some small positive value  $\varepsilon$ .

**Proof:**

$$\check{b}_k = \frac{\int_{\mathfrak{B}} \alpha_{k-1}(b) b S_{k-1}(b) db}{\int_{\mathfrak{B}} \alpha_{k-1}(b) S_{k-1}(b) db} \quad (41)$$

Therefore

$$\check{b}'_k \mathbf{x}_k = \frac{\int_{\mathfrak{B}} \alpha_{k-1}(b) b' \mathbf{x}_k S_{k-1}(b) db}{\int_{\mathfrak{B}} \alpha_{k-1}(b) S_{k-1}(b) db} \quad (42)$$

Therefore

$$\check{b}'_k \mathbf{x}_k = \frac{\int_{\mathfrak{B}} \alpha_{k-1}(b) b' \mathbf{x}_k \prod_{i=1}^{k-1} b' \mathbf{x}_i db}{\int_{\mathfrak{B}} \alpha_{k-1}(b) \prod_{i=1}^{k-1} b' \mathbf{x}_i db} = \frac{\int_{\mathfrak{B}} \alpha_{k-1}(b) \prod_{i=1}^k b' \mathbf{x}_i db}{\int_{\mathfrak{B}} \alpha_{k-1}(b) \prod_{i=1}^{k-1} b' \mathbf{x}_i db} \quad (43)$$

Thus

$$\check{S}_t = \prod_{k=1}^t \check{b}'_k \mathbf{x}_k = \prod_{k=1}^t \frac{\int_{\mathfrak{B}} \alpha_{k-1}(b) \prod_{i=1}^k b' \mathbf{x}_i db}{\int_{\mathfrak{B}} \alpha_{k-1}(b) \prod_{i=1}^{k-1} b' \mathbf{x}_i db} = \prod_{k=1}^t \frac{\int_{\mathfrak{B}} \alpha_{k-1}(b) S_k(b) db}{\int_{\mathfrak{B}} \alpha_{k-1}(b) S_{k-1}(b) db} \quad (44)$$

Resulting in the following

$$\check{S}_t = \frac{\int_{\mathfrak{B}} \alpha_t(b) S_t(b) db}{\int_{\mathfrak{B}} db} \prod_{i=1}^{t-1} \frac{\int_{\mathfrak{B}} \alpha_i(b) S_i(b) db}{\int_{\mathfrak{B}} \alpha_{i+1}(b) S_i(b) db} \quad (45)$$

Under suitable smoothness conditions of  $\alpha_k$  we have

$$\frac{\int_{\mathfrak{B}} \alpha_i(b) S_i(b) db}{\int_{\mathfrak{B}} \alpha_{i+1}(b) S_i(b) db} \leq (1 - \delta) \quad (46)$$

where  $\delta > 0$  is small. With (45) and (46) we can describe  $\check{S}_t$  as

$$\check{S}_t = \frac{\int_{\mathfrak{B}} \alpha_t S_t(b) db}{\int_{\mathfrak{B}} db} (1 - \varepsilon)^{t-1} \quad (47)$$

where  $\varepsilon > 0$  is small. Moreover, since  $\alpha_t \rightarrow \alpha$  we have that for all but finitely many values of  $i$

$$\frac{\int_{\mathfrak{B}} \alpha_i(b) S_i(b) db}{\int_{\mathfrak{B}} \alpha_{i+1}(b) S_i(b) db} = 1 \quad (48)$$

Based on the nature of the function  $\alpha$ , in particular from (38) we have

$$\int_{\mathfrak{B}} \alpha_t S_t(b) db \geq \int_{\mathfrak{B}} S_t(b) db \quad (49)$$

Therefore

$$\check{S}_t \geq \frac{\int_{\mathfrak{B}} S_t(b) db}{\int_{\mathfrak{B}} db} (1 - \varepsilon)^{t-1} \quad (50)$$

From (17) and (50) we have

$$\check{S}_t \geq \hat{S}_t (1 - \varepsilon)^{t-1} \quad (51)$$

which proves the theorem.

In essence Theorem 2 states that the wealth accumulated by the Adaptive Universal Portfolio is at least as great as a finite fraction of the wealth accumulated by Cover’s Universal Portfolio. Therefore, as Cover’s Universal Portfolio accumulates almost as much wealth as the best constant rebalanced portfolio, we conclude that the Adaptive Universal Portfolio accumulates a value at least as great as a finite fraction of the (assumed infinite) wealth accumulated by the best constant rebalanced portfolio, and therefore the Adaptive Universal Portfolio is a universal portfolio strategy as given by (29). The Adaptive Universal Portfolio has two other significant properties which we wish to discuss.

The form of the Adaptive Universal Portfolio algorithm mimics the behavior of carrying out Cover’s Universal Portfolio calculation over a subset of the portfolio simplex consisting of historically better performing portfolios. More precisely, the portfolio discriminant function gives a greater measure to portfolios which have historically performed better, and lessens the measure given to historically poor performing portfolios. Therefore, in the integral the better performing portfolios are given a greater measure, and poor performing portfolios are given a lesser measure. This mimics the act of reducing the integration space from the entire portfolio simplex, to the subset of ‘fittest’ portfolios. However, while the Adaptive Universal Portfolio maintains some investment in every portfolio, albeit a significantly greater investment in the ‘fittest’ portfolios, which ensures the strategy is universal, carrying out Cover’s Universal Portfolio over a reduced region of the simplex does not appear to maintain the universal property.

Secondly, given the conditions of Theorem 1, the portfolio discriminant function’s act of giving a larger measure to historically better performing portfolios, and reducing the measure given to historically poor performing portfolios implies that for all but a finite number of trading periods, the Adaptive Universal Portfolio will outperform Cover’s Universal Portfolio. Therefore, we conclude the Adaptive Universal Portfolio will asymptotically outperform Cover’s Universal Portfolio.

In this section we have proposed a new causal sequential portfolio selection strategy, and shown that it accumulates at least as much wealth as a finite fraction of Cover’s Universal Portfolio. In the next section we will describe 2 possible schemes for implementation of both Cover’s Universal Portfolio and the Adaptive Universal Portfolio. In subsequent sections we will carry out empirical tests on a variety of data samples obtained from CRSP.

#### 4. Algorithm Implementation

In previous sections we have proposed two causal sequential portfolio selection algorithms, and we observe that both of these algorithms require the calculation of definite integrals over the space  $\mathfrak{B} = \{b \in \mathbb{R}^m \text{ such that } b_i \geq 0 \text{ for } i = 1, \dots, m \text{ and } \sum_{i=1}^m b_i = 1\}$ . In this section we propose two possible schemes for numerical approximation of the definite integrals, namely, quantization of the space  $\mathfrak{B}$  and Monte Carlo Estimation of the definite integrals.

To help motivate quantization of the space  $\mathfrak{B}$ , we define on the space  $\mathfrak{B} \subseteq \mathbb{R}^m$  the following metric,  $d$ . For portfolios  $x, y$  and  $z$  from  $\mathfrak{B}$  with

$$x = (\epsilon_1, \dots, \epsilon_m), y = (\eta_1, \dots, \eta_m), z = (\zeta_1, \dots, \zeta_m) \quad (52)$$

we define

$$d(x, y) = \sum_{i=1}^m |\epsilon_i - \eta_i| \quad (53)$$

**Proposition 5:**  $(\mathfrak{B}, d)$  is a metric space.

**Proof:** From Kriezlig [8] we can see that the first three requirements of a metric space are obviously fulfilled by  $(\mathfrak{B}, d)$ . For the final requirement, proof of the triangle inequality, we note

$$d(x, y) = \sum_i^m |\varepsilon_i - \eta_i| = \sum_i^m |\varepsilon_i - \zeta_i + \zeta_i - \eta_i| \leq \sum_i^m |\varepsilon_i - \zeta_i| + \sum_i^m |\zeta_i - \eta_i| = d(x, z) + d(z, y) \quad (54)$$

We now introduce a related concept. An investor is said to be  $\delta$ -indifferent if whenever offered two portfolios  $x$  and  $y$  from  $\mathfrak{B}$ , such that  $d(x, y) < \delta$ , he is indifferent between the two portfolios. We now consider the difference in wealth accrued by two portfolios,  $x$  and  $y$ , such that  $d(x, y) < \delta$ , where the stock vector for the investment period is  $\mathbf{x} = (x_1, \dots, x_m) \in [0, c]^m$ . We have

$$\begin{aligned} |S(x) - S(y)| &= \left| \sum_{i=1}^m \varepsilon_i x_i - \eta_i x_i \right| \\ &= \left| \sum_{i=1}^m (\varepsilon_i - \eta_i) x_i \right| \\ &\leq \sum_{i=1}^m |\varepsilon_i - \eta_i| |x_i| \\ &< \sum_{i=1}^m \delta c \\ &= m\delta c \end{aligned} \quad (55)$$

Therefore, setting  $\delta$  suitably small, the difference between the wealth accumulated by the two portfolios can be made negligible. Therefore, a  $\delta$ -indifferent investor considers not the entire space,  $\mathfrak{B}$ , but rather the subset of  $\mathfrak{B}$ , denoted by  $\tilde{\mathfrak{B}}$ , defined as the set of all pairwise  $\delta$ -different portfolios.

**Proposition 6:** The space  $\tilde{\mathfrak{B}}$  is countable.

**Proof:**  $\tilde{\mathfrak{B}}$  is the subset of points of  $\mathfrak{B}$  such that the points are pairwise distanced by at least a distance  $\delta$ . Therefore, for any  $\delta > 0$  the set  $\tilde{\mathfrak{B}}$  can be described by a set of rational numbers. The countability of the rationals ensures that any of its subsets is also countable, and therefore the set  $\tilde{\mathfrak{B}}$  is a countable.

We now consider a special case of  $\delta$ -indifference, where an investor is indifferent between two portfolios which have a difference in individual component beyond a certain level of decimal expansion. To be more precise the investor considers the two holdings,  $b$  and  $\hat{b}$ , in an individual asset as being indifferent at the  $p^{\text{th}}$  level of decimal expansion if

$$b = b_0.b_1 \dots b_p b_{p+1} \dots \text{ and } \hat{b} = \hat{b}_0.\hat{b}_1 \dots \hat{b}_p \hat{b}_{p+1} \dots \quad (56)$$

where

$$b_i, \hat{b}_i \in 0, 1, \dots, 9, \text{ such that } b_i = \hat{b}_i \text{ for } i = 0, 1, \dots, p \quad (57)$$

Clearly an investor who is indifferent between two portfolios at the  $p^{th}$  level of decimal expansion, has a finite space of possible portfolio choices. As an example consider an investor who is indifferent between portfolios at the  $2^{nd}$  level of decimal expansion and who is considering investing in two different equities. The space of portfolios under consideration for such an investor can be given as

$$\tilde{\mathfrak{B}} = \{(1,0), (0.99,0.01), \dots, (0.01,0.99), (0,1)\} \quad (58)$$

With the above as motivation we propose a first scheme for the implementation of our two proposed sequential portfolio algorithms. For a fixed  $p$  take the space of all unique portfolios at the  $p^{th}$  level of decimal expansion, which we denote by  $\tilde{\mathfrak{B}}$ . We give Cover's Universal Portfolio Algorithm, for investment in a market consisting of  $m$  stocks as

$$\begin{aligned} \hat{b}_1 &= \left(\frac{1}{m}, \dots, \frac{1}{m}\right) \\ \hat{b}_{k+1} &= \frac{\sum_{i=1}^{|\tilde{\mathfrak{B}}|} b_i S_k(b_i)}{\sum_{i=1}^{|\tilde{\mathfrak{B}}|} S_k(b_i)} \end{aligned} \quad (59)$$

where  $|\tilde{\mathfrak{B}}|$  denotes the cardinality of  $\tilde{\mathfrak{B}}$ .

We can give the Adaptive Universal Portfolio Algorithm, for investment in a market consisting of  $m$  stocks as

$$\begin{aligned} \check{b}_1 &= \left(\frac{1}{m}, \dots, \frac{1}{m}\right) \\ \check{b}_{k+1} &= \frac{\sum_{i=1}^{|\tilde{\mathfrak{B}}|} \alpha_k(b_i) b_i S_k(b_i)}{\sum_{i=1}^{|\tilde{\mathfrak{B}}|} \alpha_k(b_i) S_k(b_i)} \end{aligned} \quad (60)$$

where

$$\sum_{i=1}^{|\tilde{\mathfrak{B}}|} \alpha_k(b_i) = 1, \alpha_k(b_i) > 0 \forall k, \forall b_i \in \tilde{\mathfrak{B}} \quad (61)$$

such that

$$\forall b_i, b_j \in \tilde{\mathfrak{B}} \text{ s.t. } S_k(b_i) > S_k(b_j) \Rightarrow \alpha_k(b_i) > \alpha_k(b_j) \quad (62)$$

We now give details of a second possible scheme for numerical solution to the two proposed algorithms. Monte Carlo methods is a mathematical tool for approximation of definite integrals, which can be applied to all definite integrals, regardless as to the existence of an analytical solution to the integral. For a more detailed account of Monte Carlo methods than that which will be given here please refer to Glasserman [5].

As per Cover and Ordentlich [3], we apply to the space  $\mathfrak{B}$  the distribution law for  $b \in \mathfrak{B}$ , which we will denote by  $F(b)$ . Therefore the two proposed algorithms may be given as

$$\hat{b}_{k+1} = \frac{\int_{\mathfrak{B}} b S_k(b) dF(b)}{\int_{\mathfrak{B}} S_k(b) dF(b)} \quad (63)$$

and

$$\check{b}_{k+1} = \frac{\int_{\mathfrak{B}} \alpha_k(b) b S_k(b) dF(b)}{\int_{\mathfrak{B}} \alpha_k(b) S_k(b) dF(b)} \quad (64)$$

where the two representations refer to Cover's Universal Portfolio and the Adaptive Universal Portfolio respectively. We assume a Dirichlet law for the distribution of  $b$  on  $\mathfrak{B}$ . Moreover, we observe that (58) and (59) imply the following identities

$$\hat{b}_{k+1} = \frac{\mathbb{E}_F[bS_k(b)]}{\mathbb{E}_F[S_k(b)]} \quad (65)$$

and

$$\check{b}_{k+1} = \frac{\mathbb{E}_F[\alpha_k(b)bS_k(b)]}{\mathbb{E}_F[\alpha_k(b)S_k(b)]} \quad (66)$$

Therefore we can numerically calculate the values of  $\hat{b}_{k+1}$  and  $\check{b}_{k+1}$  by drawing portfolio samples,  $\{b_i\}_{i=1}^n$ , from the distribution  $F$  and calculating the arithmetic mean of the above identities. More formally we set

$$(\hat{b};n)_{k+1} = \frac{\frac{1}{n}\sum_{i=1}^n b(i)S_k(b)}{\frac{1}{n}\sum_{i=1}^n S_k(b)} = \frac{\sum_{i=1}^n b(i)S_k(b)}{\sum_{i=1}^n S_k(b)} \quad (67)$$

and

$$(\check{b};n)_{k+1} = \frac{\frac{1}{n}\sum_{i=1}^n \alpha_k(b_i)b(i)S_k(b)}{\frac{1}{n}\sum_{i=1}^n \alpha_k(b_i)S_k(b)} = \frac{\sum_{i=1}^n b(i)S_k(b)}{\sum_{i=1}^n S_k(b)} \quad (68)$$

and observe that the properties of the Monte Carlo Integral imply

$$\lim_{n \rightarrow \infty} (\hat{b};n)_{k+1} = \hat{b}_{k+1}, \quad \lim_{n \rightarrow \infty} (\check{b};n)_{k+1} = \check{b}_{k+1} \quad (69)$$

Therefore, we can carry out the numerical estimation of  $\hat{b}$  and  $\check{b}$  following the steps given above so long as we can draw portfolio samples from the appropriate Dirichlet distribution, for details on how to draw samples from the Dirichlet distribution see Ishjima [7].

## 5. Empirical Results

In the previous sections we presented two schemes for implementation of the given causal sequential portfolio stock selection algorithms. In this section we will demonstrate the competitiveness of the proposed Adaptive Universal Portfolio with respect to Cover's Universal Portfolio in a two asset environment. We shall use the quantization method as described in the previous section. Quantization is carried out at the percentage level, therefore  $\mathfrak{B}$  is given by (58).

We have investigated the use of a number of candidate performance discriminant functions including the following

- $\alpha_k(b) = \frac{S_k(b)}{\int_{\mathfrak{B}} S_k(b)db}$ , the identity performance discriminant function.
- $\alpha_k(b) = \frac{e^{S_k(b)}}{\int_{\mathfrak{B}} e^{S_k(b)}db}$ , the exponential portfolio discriminant function.
- $\alpha_k(b) = \left(\frac{1}{2}\right)^{\text{rank}(b)}$ , the discrete portfolio discriminant function.

Empirical studies have shown that for a short term investor, i.e. an investor investing for 3 to 7 years, the discrete performance discriminant function based Adaptive Universal Portfolio shows significant increases in performance over Cover’s Universal Portfolio, and in what remains of this paper we have carried out the Adaptive Universal Portfolio using the discrete portfolio discriminant function.

Figure 2 shows the performance of the Adaptive Universal Portfolio and Cover’s Universal Portfolio based on the pair of stocks International Business Machines and Ford Motor Company for the period 1/1/1970 to 1/1/1973 using data obtained from CRSP.

Figure 3 shows the performance of the Adaptive Universal Portfolio and Cover’s Universal Portfolio based on the pair of stocks Boeing and General Motors for the period 1/1/1970 to 1/1/1973 using data obtained from CRSP.

Figure 4 shows the performance of the Adaptive Universal Portfolio and Cover’s Universal Portfolio based on the pair of stocks Coca Cola and Johnson and Johnson for the period 1/1/1970 to 1/1/1973 using data obtained from CRSP.

Moving to a longer investment horizons Figure 5 shows the performance of the Adaptive Universal Portfolio and Cover’s Universal Portfolio based on the pair of stocks International Business Machines and Boeing for the period 1/1/1980 to 1/1/1987 using data obtained from CRSP.

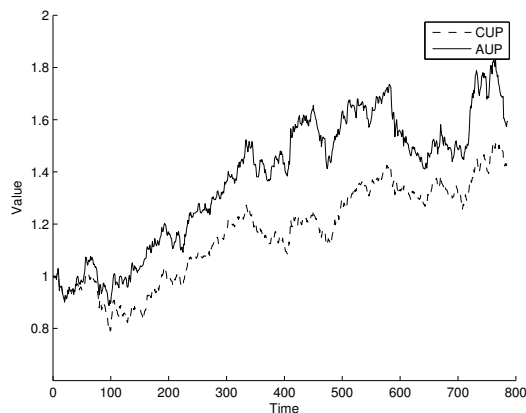


Figure 2: Wealth Growth from 1 unit initial investment.

In Figure 6 we show the performance of the Adaptive Universal Portfolio and Covers Universal Portfolio based on the pair of stocks Du Pont and International Business Machines for the period 1/1/1990 to 1/1/1996 using data obtained from CRSP.

As these examples show the Adaptive Universal Portfolio significantly improves early growth for the investor, and as shown in Theorem 2 previously, we have that the wealth achieved by the Adaptive Universal Portfolio is lower bounded by a finite fraction of Cover’s Universal Portfolio and as such the asymptotic universal property of Cover’s Universal Portfolio is maintained.

However, our empirical research has also revealed an issue with the Adaptive Universal Portfolio. Extending our analysis of the stocks in Figure 5, International Business Machines and Boeing, to include the market crash of 1987, using data for the period 1/1/1980 to 1/1/1989, we discover that the market crash causes a larger drawdown for the Adaptive Universal Portfolio than for Cover’s Universal Portfolio. In fact, we see that the terminal accumulated value of Cover’s Universal Portfolio is slightly better than that of the

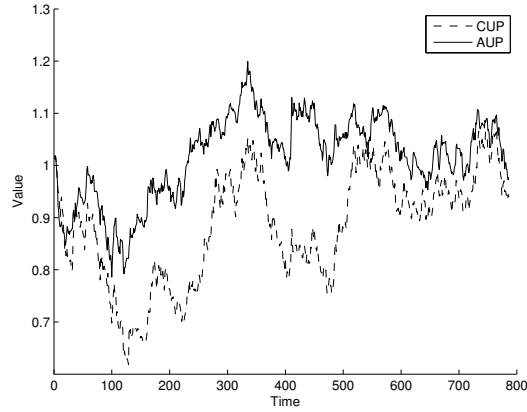


Figure 3: Wealth Growth from 1 unit initial investment.

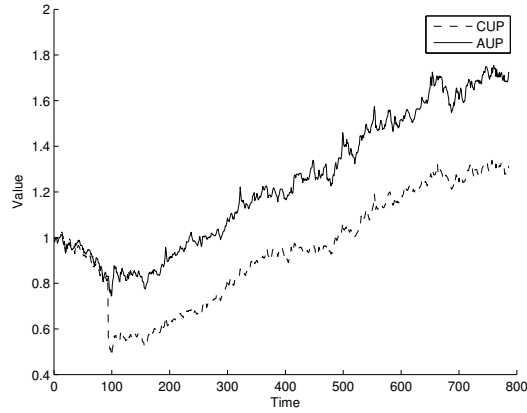


Figure 4: Wealth Growth from 1 unit initial investment.

#### Adaptive Universal Portfolio.

We may explain the above finding by noting that the portfolio discriminant function carries with it a path dependency, by which the value of  $\alpha_k(b)$  is not invariant under permutation of the market vectors as we move through sample, however, we do note that the  $\alpha_k(b)$  is invariant under permutation of  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{k-1}$ . This through-the-sample invariance carries through to the wealth accumulated by the Adaptive Universal Portfolio as we observe by (45). This contrasts with Cover's Universal Portfolio, which by (25), accumulates a wealth invariant under permutation of the market vectors. While this invariance property is beneficial, as a means of acting as a buffer against market crashes, since Cover's Universal Portfolio accumulates the same wealth as if the 'down days' of a market crash were to have had occurred uniformly throughout the investment window, we feel that it is too restricting, and may be a significant factor in causing Cover's Universal Portfolio to demonstrate such low early returns. Moreover, it appears quite a restricting property when one considers

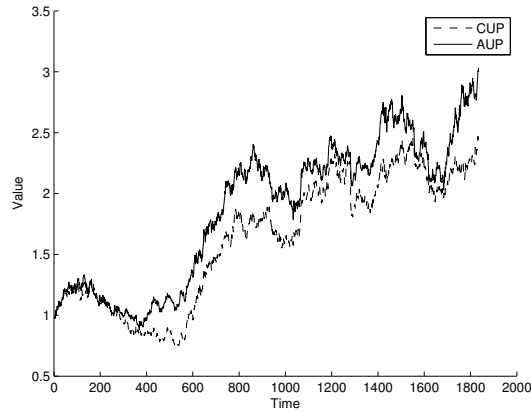


Figure 5: Wealth Growth from 1 unit initial investment.

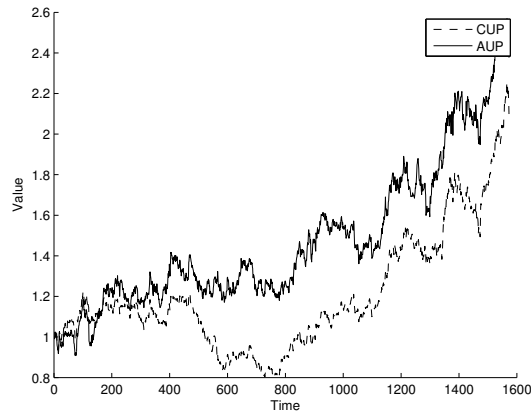


Figure 6: Wealth Growth from 1 unit initial investment.

that markets have been shown to exhibit serial autocorrelation, and as such we can choose assets based on properties of the time series of their returns to which application of both Cover’s Universal Portfolio and the Adaptive Universal Portfolio are more appropriate.

Another factor which may explain why the Adaptive Universal Portfolio exhibits larger downturns is the increased memory and momentum following demonstrated by the Adaptive Universal Portfolio when compared with Cover’s Universal Portfolio. The increased weighting in historically well performing portfolios puts the investor at risk of over exposure to ‘bubble’ stocks. Moreover the investor is at risk of investing long-term in past winners, which tend to be future losers as found in Fama [5].

In this section we have shown the competitiveness of the Adapted Universal Portfolio over Cover’s Universal Portfolio for investors with short term goals in mind. We have also shown that there appears to be a trade-off with respect to short term gains and long term risk by use of the Adaptive Universal Portfolio.

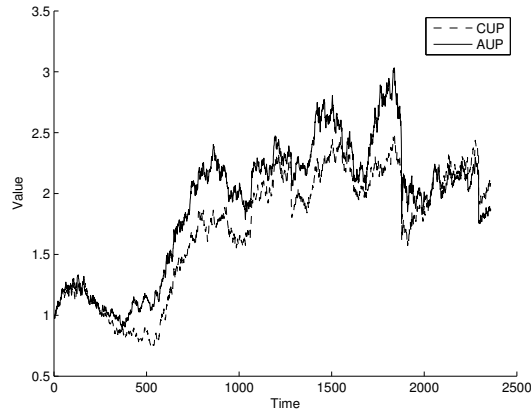


Figure 7: Wealth Growth from 1 unit initial investment.

## 6. Conclusion

In this paper we have introduced the portfolio discriminant function as a means of tractably ranking the performance of portfolios. We introduced the Adaptive Universal Portfolio, which can be seen as an extension of Cover's Universal Portfolio [2] which gives greater weighting to better performing portfolios, while penalizing poor performing portfolios.

Our research has shown the Adaptive Universal Portfolio to provide significantly increased returns over the short-to-medium term, when compared with Cover's Universal Portfolio, while maintaining the universal property of Cover's Algorithm. We feel this is a significant step in addressing industries main criticism of Cover's Universal Portfolio [4].

Our empirical findings have also indicated that this increased early performance comes at a cost to the investor. While the Universal Portfolio tracks better performing portfolios and can be susceptible to large drawdowns, particularly in times of market changes, such as moving from bull to bear market conditions, the Adaptive Universal Portfolio, such as the algorithm based on the discrete performance discriminant function, aggressively tracks better performing portfolios to an extent that leaves the investor at risk of increased drawdowns during times of market change and stock price declines.

It may be that this increased risk, exhibited as exaggerated drawdowns for the Adaptive Universal Portfolio, may be the price the investor must pay for increased early performance, just as in Modern Portfolio Theory [9,10,11,12,13] an investor must accept a higher risk for a higher expected return. However, the author feels incorporating a time dependency structure to the Adaptive Universal Portfolio may be able to reduce the effect of the long memory exhibited by the Adaptive Universal Portfolio, which is felt to be the cause of the large drawdown. This time dependency structure, along with investigating the fittest form of the portfolio discriminant function will form the focus of future work in this area.

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