



Title	Experimental modal analysis of violin and similar thin plates by added point masses
Authors(s)	O'Connor, William, Hanson, Thomas J.
Publication date	2012
Publication information	O'Connor, William, and Thomas J. Hanson. "Experimental Modal Analysis of Violin and Similar Thin Plates by Added Point Masses" [Forthcoming] (2012).
Publisher	Elsevier
Item record/more information	http://hdl.handle.net/10197/4007
Publisher's statement	This is the author's version of a work that was accepted for publication in Journal of Sound and Vibration. Changes resulting from the publishing process, such as peer review, editing, corrections structural formatting, and other quality control mechanisms may not be reflected in this document. Changes may have been made to this work since it was submitted for publication. Elsevier Ltd.

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Experimental modal analysis of violin and similar thin plates by added point masses

William J O'Connor
School of Mechanical and Materials Engineering
University College Dublin, UCD Belfield, Dublin 4, Ireland
e-mail: william.oconnor@ucd.ie

Thomas J Hanson
Former Honorary President of Dublin Amateur Luthiers Society
18 Oakely Grove, Blackrock, County Dublin, Ireland.
e-mail: hansomt@eircom.net

Abstract

Novel methods are proposed to measure the modal properties of thin plates, such as free violin plates (prior to assembly), simply and inexpensively, by measuring certain changes when a small mass is added to the resonating plate. Iso-amplitude contours and mode shapes can easily be plotted. Modal mass, stiffness and damping can also be inferred. Underlying theory is developed, and experimental and numerical modelling methods of validation are briefly outlined.

Key words: Experimental modal analysis; vibration of thin plates; violin plate tuning; modal mass; modal stiffness; modal damping

1. Introduction, context and motivation

In the art and craft of violin making, a crucial stage is the so-called tuning of the top and back plates of the violin soundbox, prior to assembly [1-13]. The tuning consists of a careful, final thinning of selected areas of a plate to achieve a desired vibration response. Because every piece of wood is different, even when taken from the same tree, simply reproducing the dimensions of a superb violin does not guarantee a desired response. Luthiers (string instrument makers and repairers) have developed many techniques and craft-based rules to guide them in this critical process. In addition to manually testing the local stiffness using experienced, sensitive fingers, they also use techniques which amount to reduced forms of modal analysis.

One technique involves holding the plate loosely along a nodal line of a desired mode and tapping it at an antinode, in effect getting the impulse response for that vibration mode. In these tap tones the trained ear can detect the modal frequency, the amplitude (or liveliness) of the initial response, and the rate of decay (or damping) [1,10-12]. A more developed method involves Chladni patterns, where a horizontal plate is sprinkled with particles such as light sand, tea-leaves or tinsel, and then made to vibrate, sometimes by a violin bow, rubbing a glass rod, or more often by a loudspeaker driven by a variable frequency supply. The plate is typically mounted on three or four small, soft supports, perhaps of sponge, located somewhere along the nodal lines, so that they cause minimal interference with the plate vibration and simply support the plate's weight. When excited at a resonant frequency, the resulting vibrations cause the particles to gather along the nodal lines of that vibration mode. The location, shape and sharpness of these lines, and the frequency at which they occur, then guide the luthier in his or her plate thinning.

There is wide consensus that, to achieve a good plate, the focus should be on three modes, known as modes 1, 2 and 5, sometimes designated +, x and 0 modes, corresponding to their shapes. They are considered to be decisively important. The frequencies and shapes of the modes are important, as well as the modal damping, not forgetting the total plate mass. Luthiers often try to achieve three

outcomes: that the three dominant modal frequencies have a target inter-relationship (usually octave) [10-12]; that each mode shape be as close as possible to the widely agreed, ideal shape; and that the frequencies for mode 2 for top and back plates be equal [1]. They work towards these targets by thinning the plate, that is, by removing small amounts of wood in carefully chosen locations. Thinning at a specific area generally has multiple effects on the modes. The removal of wood reduces the local plate inertia in proportion to the thickness reduction, but affects the bending stiffness in proportion to the inverse of the cube of the thickness. So where bending is high relative to the amplitude of the mode shape, the dominant effect of thinning is to reduce the modal stiffness more than the modal mass, thereby reducing the resonant frequency. And vice versa: where the modal amplitude is large and the bending small, thinning there will increase the frequency of that mode.

A complication therefore is that thinning in a specific area typically will have different effects on the frequency and shapes of different modes: a nodal area for one mode can be an antinodal zone for another. Luthiers have developed rules of thumb to interpret the Chladni patterns and so to guide them in their challenging task of optimally tuning a given plate. Typically, strategic thinning can improve a given plate's response up to a point when further thinning causes sudden deterioration. So part of the art is knowing when to stop.

The modal analysis tools usually available to luthiers are limited, and few luthiers go beyond determining the resonant frequencies and the shapes of the nodal lines as described. There is certainly much to be deduced from these, but this information is limited when compared with complete modal analysis. It is also possible to rely on it too much. For example, the nodal line distribution does not give the shape (bending) throughout the plate, away from nodal lines. Furthermore there is an impression among some violin makers that there is no bending at nodal lines, which is true only for bending in moving along the nodal line: the bending (rate of change of plate angular displacement) perpendicular to nodal lines may be arbitrarily large. It is also seldom appreciated that the location adopted by the nodal lines is such as to create a balance of momentum between the areas of the plate vibrating in anti-phase: an insight which is helpful for plate graduation.

This paper proposes simple ways, believed to be novel, which allow the luthier to achieve a much more complete picture of a plate's vibration response, prior to each successive tuning stage. The methods are based on simply adding a small mass to the vibrating plate at points of interest. The instruments involved are inexpensive and correspond to what modern luthier workshops would typically already have, avoiding the need for expensive instruments such as laser-doppler scanners and associated hardware and software. In addition, the method gives further important modal analysis information, namely the modal mass, stiffness and damping, values of which are not directly deducible from laser scanning techniques.

Regarding measuring vibration amplitudes, attaching accelerometers to the plate is generally avoided because of mounting complications and dynamic loading (even though very light accelerometers are now available). Instead a small microphone can be used, held close to the plate. This is inexpensive, rapid, non-disturbing and very easy to move around, although the distance from the plate should be kept constant. Nevertheless the authors have found that it can give misleading amplitude results when approaching the edge of a plate, for two reasons. Firstly, there is a boundary effect in the acoustic pressure signal near the plate edge involving a more complex pressure radiation pattern. Secondly, there is the problem of leakage, where the microphone picks up not only the signal from the plate motion (as desired) but also sound directly from the driving loudspeaker (not desired) which can be in, or out of, phase with the local plate vibration. To minimise this effect the driving loudspeaker will usually be on the side of the plate opposite to that of the microphone, but near the plate edges the leakage increases.

The paper is the result of years of research on violin plate tuning. Many aspects of the work are believed to be novel. While the primary context is violin plate tuning, it is believed that the ideas will be of more general interest in areas such as vibration testing and experimental modal analysis, especially of light structures. This paper concentrates on the more theoretical aspects of the techniques.

From the perspective of mechanics, the problem is that of vibration of thin, non-uniform, non-isotropic plates, of complex shape, with free boundaries [14-17]. The nature of wood causes the non-uniformities. Young's modulus along the grain is usually 10 to 15 times higher than across the grain, so the plate is often considered to be orthotropic [14-19]. The plates are arched rather than flat. The cutting of the grain, for example to achieve plate arching, causes local variations in Young's modulus, in addition to the grain orientation effects. The mass density is also non-uniform, although less non-uniform than the flexural stiffness. In addition to the arching, the plate boundaries (edges) have the constantly changing curvature characteristic of the shape of violin top and back plates, making analytical solutions difficult or impossible.

Section 2 outlines the assumptions of this work. It then presents a theorem and proof that the addition of a small point mass to a plate, vibrating at or near a clear resonance, has negligible effect on the bending distribution (or mode shape). Section 3, the heart of the paper, outlines the three methods of determining the local plate vibration amplitudes based on the change in the resonance frequency, the change in the peak amplitude and the change in phase. The theory of each method is also outlined. Section 4 gives a phasor interpretation of the theory and alternative proofs of some key results. Section 5 briefly describes numerical and experimental validation methods. Section 6 interprets some of the work in the light of Maxwell's reciprocity theorem. Section 7 has some conclusions.

2. Theoretical framework

2.1 Assumptions and theorem

In the theory and experimental work, the following conditions apply.

- a) For all measurements the plate is excited at, or close to, the resonant frequency of the mode of interest.
- b) The added mass is small in comparison with the plate mass causing, at most, a small change in the resonant frequency.
- c) The plate has no other vibration modes whose resonant frequencies are close to those of the mode shape being measured (usually modes 1, 2 or 5).

Under these assumptions it can be shown that the added mass leaves mode shapes virtually unchanged. More precisely, the rate of change of the mode shape (as distinct from the mode amplitude) with respect to added mass is zero (in the limit of zero added mass). This point may be counterintuitive so we provide a proof and some comments. For brevity, the proof will be given for uniform, isotropic plates. It is a simple matter to extend the proof to orthotropic or generally non-uniform plates, provided superposition remains applicable.

2.2 Proof

The dynamics of small, free vibrations of a thin, uniform isotropic plate can be described by

$$\frac{Eh^3}{12(1-\nu^2)} \left(\frac{\partial^4 z}{\partial x^4} + 2 \frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} \right) + \rho h \ddot{z} = 0 \quad (1)$$

where $z = z(x, y)$ is the small departure of the plate from the neutral (equilibrium) position in a direction perpendicular to the plate, x and y are coordinates defining points on the plate, E is Young's modulus, ν Poisson's ratio, h is the thickness of the plate and ρ the mass density [14, 15, 20].

Assuming synchronous motion, and after separation of space and time variables, the equation becomes

$$\frac{Eh^3}{12(1-\nu^2)} \left(\frac{\partial^4 z}{\partial x^4} + 2 \frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} \right) = \rho h \omega^2 z \quad (2)$$

This equation and accompanying boundary conditions have an infinite series of solutions for $z(x, y)$ and associated ω . These are the eigenfunctions or mode shapes, $z_i(x, y)$, and corresponding eigenvalues, ω_i^2 . The natural frequency for the i th mode is ω_i . Finding these solutions is not easy (certainly without numerical modelling or without simplifying the plate geometry), but for present purposes it is only necessary to know that they exist. The mode shapes have an arbitrary amplitude, so if $z_i(x, y)$ is a solution, the function $a z_i(x, y)$ will also be a solution, where a is an arbitrary scaling factor. When the plate is driven at a frequency ω_i (without an added mass) to steady state, the envelope of the shape it adopts will be described by $\pm z_i(x, y)$.

Now the small mass is added at a point $P(x_p, y_p)$ and the plate is driven at the same frequency, ω_i . Assume that there is a change, $d(x, y)$, in the shape then adopted by the plate, so that the new shape is given by $z_i(x, y) + d(x, y)$. This new shape should also satisfy the same Eq.(2) at every point (except perhaps the point P , the only point where there is a mass change) and the same boundary conditions. Therefore

$$\frac{Eh^3}{12(1-\nu^2)} \left(\frac{\partial^4 (z+d)}{\partial x^4} + 2 \frac{\partial^4 (z+d)}{\partial x^2 \partial y^2} + \frac{\partial^4 (z+d)}{\partial y^4} \right) = \rho h \omega^2 (z + d) \quad (3)$$

at every point on the plate, except perhaps at P .

Assuming linearity and superposition for small displacements, one can subtract Eq.(2) from Eq.(3), giving

$$\frac{Eh^3}{12(1-\nu^2)} \left(\frac{\partial^4 d}{\partial x^4} + 2 \frac{\partial^4 d}{\partial x^2 \partial y^2} + \frac{\partial^4 d}{\partial y^4} \right) = \rho h \omega^2 d \quad (4)$$

for all points except perhaps P

Therefore the function, $d(x, y)$, describing the change in shape, is itself also a mode shape corresponding to the natural frequency of the unmodified plate, at all points throughout the plate. The point P must now be included for reasons of continuity with the surrounding plate, because an isolated discontinuity in the shape functions, or their lower spatial derivatives, is not physically reasonable. Therefore all three functions $z_i(x, y)$, $d(x, y)$ and $z_i(x, y) + d(x, y)$ have the same shape (bending distribution) and differ only by a scaling factor (positive or negative). They can each be expressed as a suitable constant times a common shape function, such as $\phi_i(x, y)$. When normalised in the same way the differences between them become negligible.

2.3 Comments

The result suggests that the effect of adding a small point mass to a plate, driven at its resonance frequency, is not local but extends to the entire plate. One might relate this result to how a cantilever beam responds under a point load, applied, say, at the tip. Mechanical work is done by the load force at the point of application, but the resulting stored elastic potential energy is distributed throughout the beam. Furthermore, the shape of this distribution does not change as the load changes: only the magnitude changes. A given change in load causes a proportional change at every point along the beam. Something similar happens with the addition of a point mass (or inertial

load) to a plate driven at or near resonance: a local cause has a global effect. The amplitude changes proportionally everywhere, leaving the shape unchanged.

Another view is as follows. When the small point mass is added somewhere on a plate which is being excited at or close to resonance, a new, extra (oscillating) force is required if it is to have the same acceleration as the point on the plate where it is attached. Conceivably, this new force could arise from increased local bending at the point, while otherwise keeping the vibration amplitude and frequency unchanged. Instead the plate reacts by reducing the acceleration amplitude everywhere, including at the point, by reducing the amplitude (and so the bending) everywhere, proportionally, by just the right amount, but leaving the *bending distribution* almost unchanged. Again it is seen that the extra, point load is shared across the entire plate while keeping the bending distribution as before.

Note that the above assumes no change in the forcing frequency. If instead the plate is vibrating freely, the addition of the point mass will in general cause a change in the resonance frequency, so that the frequencies in Equations (2) and (3) above, for example, will no longer be identical. If the added mass is small, however, the frequency difference will be small. Furthermore, because all parts of the plate will be vibrating at the new common frequency, the *distribution* of the inertial load throughout the plate will not otherwise change. Thus the resulting changes in the mode shape due to the resonance frequency adjustment will be an order of magnitude lower than the effects on the amplitude, phase, or on the resonance frequency itself.

Regarding energy, for a given excitation, while the addition of the point mass reduces the total elastic potential energy, leaving the *distribution* unchanged, by contrast, the distribution of kinetic energy does undergo a strong local change where the point mass is applied. If the plate is otherwise uniform, the added point mass produces a sharp peak in the kinetic energy distribution at the point.

The result also is linked to the observation that when driving a plate near one of its resonance frequencies, it can be driven (physically forced) at a particular point on the plate (e.g. by a violin bow, a small loudspeaker, or a small shaker), rather than being driven in a suitably distributed way over the entire plate. Yet the entire plate then adopts the modal shape, as if the inertia were being driven everywhere. Such a point (or localised) driving force is assisting the plate's bending force at the application point, to give the required acceleration to the local inertia. This is dynamically equivalent to reducing the plate's effective inertia (or mass) at the point. So again, a local change in the (effective) mass is observed to leave the mode shape unchanged.

3. The proposed methods

3.1 Common aspects and practical considerations

All the methods involve vibrating the plate at or close to the resonance frequency of the mode of interest, adding a point mass successively to different points on the plate, and then inferring, from simple measurements, the amplitude of vibration at these points. Other characteristics of the mode can also be inferred, namely modal mass, stiffness and damping. The method of supporting and driving the plate can be standard, for example as used in classical Chladni pattern analysis. The plate can be held approximately horizontally over a loudspeaker driven by a variable frequency source. The supports should be soft and should be located along nodal lines. (As the exact position of the nodal lines will not be known initially, the position of the support points can be iteratively adjusted to keep them on the nodal lines.)

The point mass can be added in various ways. One is by simply placing a small mass, such as a metal nut, on the plate. A practical consideration is that it is better if the metal is non-magnetic, to avoid

interaction with the driving loudspeaker's magnetic field. If, however, the amplitude or frequency of the vibrations is large, so that the acceleration approaches that of gravity, such an added mass may rattle, thereby reducing the added inertia. Furthermore it may move about during the measurement, a tendency which becomes even greater where the plate arching causes a significant slope in the plate. So a second method is to add a lump of soft, malleable plastic, which sticks to the wood. It solves the jumping problem, but is more difficult to move around and locate exactly.

A third technique, specially developed for this work, is to use an L-shaped, non-magnetic, metal rod, one end-point of which, on the shorter leg, touches the plate perpendicularly, while the other end-point has a pivot point around a small handle, so that the vibration of the plate causes a rotational vibration around this pivot. The rotational inertia of this system is seen by the plate as an added, local inertia at the contact point. This method allows very accurate and convenient positioning, and subsequent moving, of the added, point inertia. The size of the effective mass can be adjusted by locating a small lumped mass on the L-rod arm and then changing its position along the arm as required. Again plate accelerations close to that of gravity should be avoided to ensure accurate and consistent results.

In mode shapes, only the relative (rather than absolute) vibration amplitudes of points throughout the plate are of interest. It is therefore appropriate to have a reference vibration amplitude to normalise measurements. In the present work, this was obtained by placing a microphone at or near an antinodal area, and monitoring the measured signal level in the microphone. The microphone sound level can be measured after amplification, using any kind of voltage meter or oscilloscope. This microphone signal is also used as a way to monitor changes in mode vibration amplitudes and to find a resonance frequency (by changing the loudspeaker driving frequency until the microphone signal is at a local maximum).

Note that, for the proposed techniques, the microphone reading does not have to be calibrated absolutely. Nor does it need to be accurate, so it does not matter if it also picks up leakage noise directly from the loudspeaker, provided the leakage signal does not change. The reading simply has to be repeatable, to ensure that all other readings are taken at a common excitation level. A further feature of the proposed methods is that the sound level in the driving loudspeaker can be set very low, thereby almost eliminating the sound leakage problem, whether or not it matters here. The largest signal and excitation level will obviously be obtained at anti-nodal areas for any particular mode.

If the signal generator driving the loudspeaker does not have an accurate frequency control and display, a suitable frequency meter can be used.

To measure the amplitude at any point on the plate when excited at a resonant frequency, the point mass is added there. The added mass causes various, measurable changes: the resonance frequency shifts downwards; the amplitude of vibration of the entire plate falls; and the phase lag between the output (response) and the input (the driver) increases. To a first order approximation, the finite changes (taken as estimates of the rate of change) are proportional to the product of the square of the amplitude at the point by the added mass. By knowing the size of the added mass, the amplitudes can be inferred. In this way, mode shape can be determined. In addition, if required, the modal mass, stiffness and damping can be inferred. If it is desired to determine simply iso-amplitude contours (without quantifying them), then one can simply map out points where adding a mass has the same effect on any of the changing quantities. Finally, as a special case, finding the nodal lines corresponds to plotting points where an added point mass causes no change in the relevant quantities.

Because the change in the measured quantity depends on the product of the square of the amplitude and the added mass, where the amplitude is low the sensitivity of the method can be increased by increasing the added mass, and vice-versa.

The rate of change of the measurand with added mass should ideally be measured in the limit of zero added mass. If greater accuracy is required, the rate of change at zero mass can be inferred to very good accuracy by adding say 3 known increments of mass, and taking 3 corresponding measurements, and extrapolating backwards, perhaps with a simple polynomial fit whose slope at zero added mass can be calculated. Such accuracy is not generally required, and even if it is considered desirable it need be done only once for a particular contour.

3.2 Method based on changing resonance frequency with added point mass

The first method is as follows. The plate is driven at the resonant frequency of the mode of interest. This is most conveniently achieved by varying the driving frequency until the microphone signal is at a local maximum. This maximum signal value is noted, as a reference. Subsequently the microphone position is not changed throughout the test. Then the small mass is added to the plate at a point of interest and the change in resonant frequency is noted. As will be shown below, to a first order approximation, the change in the square of the resonant frequency is proportional to the square of the vibration amplitude at the point where the mass is added.

A variation of this procedure is easier and faster in practice. The mass is added and the new resonant frequency found. The driver is then left at this new, lower frequency. Then the point mass is moved around the plate, finding points at which the measured response is a maximum, that is, points giving the same (lower) resonant frequency. In this way, iso-amplitude contours can quickly be found. Different contours can be found by changing the value of the point mass, and/or the change in frequency from the original resonant frequency.

3.2.1 Theory

Rayleigh's quotient expresses the square of the resonance frequency as the ratio of the maximum potential energy per cycle to a term related to the maximum kinetic energy [20]. It has the form of

$$R = \omega_i^2 = \frac{\iint [\text{function of } (E, \nu, h, (\nabla^2 z_i)^2)] dx dy}{\iint \rho h z_i^2 dx dy} \quad (5)$$

Taking the normalised mode shape to be $\phi(x, y)$, with $z_i(x, y) = a \cdot \phi(x, y)$, where a is a constant scaling factor, then

$$\omega_i^2 = \frac{(\text{constant}) a^2 \iint (\nabla^2 \phi_i)^2 dx dy}{a^2 \iint \rho h \phi_i^2 dx dy} = \frac{k_i}{m_i} \quad (6)$$

where k_i denotes the modal stiffness, m_i the modal mass of the i th mode and the "(constant)" term will involve E , ν and h . How the mode shape is normalised to get $\phi(x, y)$ does not matter, provided one method is used consistently: one might simply scale to set the maximum value to unity, for example. The square of the amplitude scaling factor, a^2 , appearing in both numerator and denominator, obviously cancels, leaving the modal stiffness divided by the modal mass.

Now a concentrated mass, value Δm , is attached to the plate at P . If the exciting force is unchanged, this will cause the amplitude, a , to fall but have a negligible effect on $\phi(x, y)$ (see theorem above), and therefore on $\nabla^2 \phi_i$. The only significant change in the Rayleigh quotient therefore will be due to

the effect of the added mass on the modal mass (the denominator). The denominator integral is the limiting sum of the plate mass elements by the square of the vibration amplitudes of those elements. The addition of Δm will therefore increase this integral by an amount

$$\delta m_i = \phi_i^2(x_p, y_p) \cdot \Delta m \quad (7)$$

The change in the square of the resonance frequency of the i th mode will therefore be

$$\Delta(\omega_i^2) = \frac{k_i}{m_i + \delta m_i} - \frac{k_i}{m_i} = -\frac{k_i \delta m_i}{(m_i)(m_i + \delta m_i)} \quad (8)$$

For convenience, noting δm_i is small in comparison with m_i , this can be expressed as

$$\Delta(\omega_i^2) \approx -\frac{\omega_i^2}{m_i} \delta m_i = -\frac{\omega_i^2}{m_i} \phi_i^2(x_p, y_p) \cdot \Delta m \quad (9)$$

The result can be left in this form, giving a proportionality constant of $\frac{\omega_i^2}{m_i} \Delta m$ between the change in the square of the resonant frequency and the square of the amplitude at the point P . Alternatively, by dividing by Δm and taking the limit, an exact result is

$$\frac{\partial \omega_i^2}{\partial m} = -\frac{\omega_i^2}{m_i} \phi_i^2(x_p, y_p) = -(\text{constant}) \cdot \phi_i^2(x_p, y_p) \quad (10)$$

(This result can also be obtained by direct differentiation of Eq.(6) with respect to the added mass, m , using Eq.(7) to get the rate of change of the denominator.)

To get the mode shape there is no need to evaluate the proportionality constant, $\frac{\omega_i^2}{m_i}$, as the shape is independent of any multiplicative constant. However, if the proportionality constant is noted in the experimental work, it gives a way of estimating the modal mass, m_i , and hence the modal stiffness, k_i , via ω_i .

Note that the same normalising or scaling will apply to both ϕ_i and m_i , making their ratio independent of the scaling. The results in Eqs.(9) & (10) are therefore independent of the driving force or of the amplitude of vibration of the plate, so it can be as small (or as large) as desired to suit the experimental set-up and measurement methods.

3.3 Method based on changes in resonance amplitude

As has been seen, when the point mass is added to any point, not on a nodal line, of a plate being excited at a resonance frequency, it causes the vibration amplitude to fall. The drop is a non-linear function of (among other things) the added mass and of the square of the amplitude of the point where the mass is added. Iso-amplitude contours can therefore be found by plotting the points on the plate where the adding of a fixed mass produces a specific new amplitude.

Going further, because the nature of the non-linear function is well known [20], it can be used to quantify these vibration amplitude contours if required. When driven at a frequency ω close to a resonance frequency ω_i , the vibration amplitude throughout the plate, $z_i(x, y)$ can be well approximated by

$$z_i(x, y) = \frac{F^*}{\sqrt{[1-(\omega/\omega_i)^2]^2 + [2\zeta_i\omega/\omega_i]^2}} \phi_i(x, y) = \mathcal{M} \phi_i(x, y) \quad (11)$$

where F^* is a scalar which is proportional to the amplitude of the driving force (at frequency ω), ζ_i is a damping ratio for that mode, and \mathcal{M} has the role of a dynamic magnifier (with a scaling factor built in). If the mode shape has dimensions of displacement (meters), then, F^* is dimensionless, as is \mathcal{M} . Its actual value will depend on a series of variables in the experimental set-up, including the amplifier output power, the loudspeaker size and its proximity to the plate, and the effective modal stiffness. At resonance, the term multiplying the mode shape, \mathcal{M} , becomes simply $F^*/2\zeta_i$, so if the modal damping ratio is known, F^* can be measured. (The modal damping ratio can be obtained, for example, using the 3dB bandwidth method or the logarithmic decrement method.)

The addition of the small mass has the primary effect of shifting ω , by a small amount, causing a small change in $(\omega/\omega_i)^2$ as described above, which in turn causes the magnifier, \mathcal{M} , of the mode shape to fall. The shape of this function can easily be plotted once the damping ratio is known. Using this curve the appropriate conversion can be made between the drop in value, the change in resonance frequency, and ultimately the value of $\phi_i^2(x_p, y_p) \cdot \Delta m$, to within an arbitrary scaling factor.

Alternatively, one can get the rate of change of \mathcal{M} wrt $(\omega/\omega_i)^2$ at resonance, which is

$$\frac{\partial \mathcal{M}}{\partial (\omega/\omega_i)^2} = -\frac{F^*}{2\sqrt{2}\zeta_i} \quad (12)$$

But

$$\frac{\partial (\omega/\omega_i)^2}{\partial (\delta m_i)} = \frac{\omega^2}{k_i} \approx \frac{1}{m_i} \quad (13)$$

So the rate of change of the magnitude wrt to δm_i is

$$\frac{\partial \mathcal{M}}{\partial (\delta m_i)} = -\frac{F^*}{2m_i\sqrt{2}\zeta_i} \quad (14)$$

which again is a constant. This implies

$$\Delta \mathcal{M} = -\frac{F^*}{2m_i\sqrt{2}\zeta_i} \Delta m \phi_i^2(x_p, y_p) \quad (15)$$

Again the proportionality constant between $\Delta \mathcal{M}$ and $\phi_i^2(x_p, y_p)$ need not be evaluated to get the mode shape. It is sufficient that it remain constant. But again, once experimentally determined, together with the modal damping ratio ζ_i , it gives another way to get the modal mass, m_i , and so the modal stiffness, k_i .

Note that the sensitivity of this method is greatest when the modal damping is lightest. In other words the higher the resonance quality factor, or the sharper the peak, the greater the relative drop in amplitude. And conversely. Because the resonance peaks in a good violin plate are generally very sharp, this method has high sensitivity.

3.4 Method based on phase changes with added mass

At resonance, the phase difference, θ , between the driving force and the plate motion is $\pi/2$. Adding the point mass will cause this phase to change. So again, iso-amplitude contours can be found by using a fixed added mass and finding points of constant phase shift. The phase can be measured by

the change in the lag between signals from the driving loudspeaker current and the microphone response.

Again, if it is desired to quantify the contours, the starting point can be the relationship

$$\tan \theta = \frac{-2\zeta_i \omega / \omega_i}{1 - (\omega / \omega_i)^2} \quad (16)$$

The rate of change of θ with respect to (ω / ω_i) , evaluated at $\omega = \omega_i$, is then

$$\frac{\partial \theta}{\partial (\frac{\omega}{\omega_i})} = -\frac{\omega}{\zeta_i \omega_i} \cong -\frac{1}{\zeta_i} \quad (17)$$

The rate of change of (ω / ω_i) with respect to the change in the modal mass, δm_i , is

$$\frac{\partial (\omega / \omega_i)}{\partial (\delta m_i)} = \frac{\partial}{\partial (\delta m_i)} \left(\omega \sqrt{\frac{m_i + \delta m_i}{k_i}} \right) = \omega \left(-\frac{1}{2} \right) \sqrt{\frac{1}{k_i}} \sqrt{\frac{1}{m_i + \delta m_i}} \approx -\frac{1}{2m_i} \quad (18)$$

So

$$\frac{\partial \theta}{\partial (\delta m_i)} = \frac{1}{2\zeta_i m_i} \quad (19)$$

and the change in phase, $\Delta\theta$, due to the added mass is proportional to the square of the vibration amplitude at the point, and the proportionality constant is $\frac{\Delta m}{2\zeta_i m_i}$. If evaluated, it yields another method to get the modal mass, m_i , and the modal stiffness, k_i , using ω_i .

Note again, the result is independent of the driving amplitude. Also it is most sensitive when the modal damping is light and so the resonance is sharp.

4. A phasor interpretation

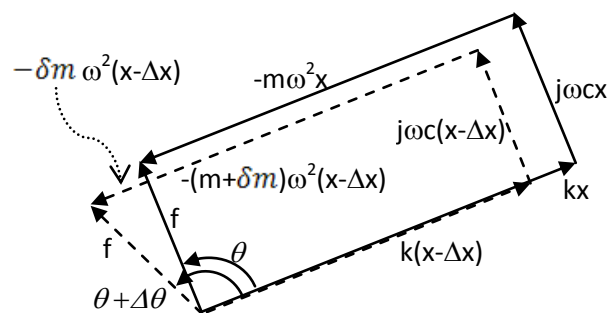


Fig.1 Phasor diagram close to resonance

The phasor diagram in Fig.1 applies to a 1-DOF forced vibrating system, such as a lumped mass-spring-damper, m , k , c , with displacement x . But it can also be applied at, or close to, resonance for a particular mode, where “ x ” can be taken as a representative amplitude within the mode, and m , k and c are related to the modal mass, stiffness and damping respectively. The solid phasors describe the situation at resonance, when angles are all $\pi/2$. At resonance, the phasor associated with the elastic displacement force, kx , is equal and opposite to that associated with mass-acceleration, $-m\omega^2 x$. The applied force f is simply offsetting the damping losses $j\omega c x$.

Now the mass, Δm , is added at some point on the plate, keeping the driving force f and the frequency unchanged. This causes the effective, modal mass to become $m + \delta m$. Due to the drop in motion amplitude, x , the elastic force drops to $k(x - \Delta x)$, shown as a dotted phasor in the diagram. The damping force drops by the same proportion, that is to $j\omega c(x - \Delta x)$. The mass-acceleration force changes in two ways due to two opposing effects. The drop in x causes a proportional drop in the acceleration, which therefore falls to a magnitude equal to $k(x - \Delta x)$. At the same time, the increase in mass causes a rise in the mass-acceleration term, so that the total force is

$$-(m + \delta m) \omega^2 (x - \Delta x) = k(x - \Delta x) + \delta m \omega^2 (x - \Delta x). \quad (20)$$

The dotted phasor diagram shows the new situation with the added mass. From the phasor diagram, the phase, θ , must increase by $\Delta\theta$ above $\pi/2$, so that one component of f , $f \sin(\Delta\theta)$, is now driving the extra mass-acceleration, $\delta m \omega^2 (x - \Delta x)$, and the other component, $f \cos(\Delta\theta)$, is overcoming the (reduced) damping losses, $\omega c(x - \Delta x)$. This gives

$$\tan(\Delta\theta) = \frac{\delta m \omega (x - \Delta x)}{c \omega (x - \Delta x)} = \delta m \frac{\omega}{c} = \delta m \frac{\omega}{2m\omega\zeta} = \frac{1}{2\zeta} \frac{\delta m}{m} \quad (21)$$

In returning from the lumped system to the plate dynamics, the last terms should be taken in the modal sense, with m and ζ becoming the modal mass, m_i and modal stiffness ζ_i respectively, and δm_i becoming the change in the modal mass due to the addition of the point mass Δm at P, as in Eq.(7). For small $\Delta\theta$, the result in Eq.(21), with variables taken in the modal sense, becomes identical to Eq.(19).

Alternatively, after the point mass has been added, one could then reduce the driving frequency to find the new resonant frequency, when all the angles in the phasor diagram would again be right angles. If then the driving level is adjusted to get the amplitudes back to their original values, the final phasor diagram will coincide with the original. The kx phasor would be as before, therefore, but the mass-acceleration phasor ($= -kx$) would have undergone two effects which must therefore cancel. Thus, the mass will have increased and the (square of the) frequency will have decreased, in such a way that

$$-(m + \delta m) \cdot (\omega^2 - \Delta(\omega^2)) \cdot x = -m\omega^2 x \quad (22)$$

This gives

$$\delta m = \frac{m \cdot \Delta(\omega^2)}{\omega^2 - \Delta(\omega^2)} \approx \frac{m \cdot \Delta(\omega^2)}{\omega^2} \quad (23)$$

Again, re-interpreting these variables in their modal sense, with m becoming the modal mass, m_i and δm the change in the modal mass, or $\Delta m \phi_i^2$, the result gives

$$\phi_i^2(x_p, y_p) \cdot \Delta m = \frac{m_i}{\omega_i^2} \cdot \Delta(\omega_i^2) \quad (24)$$

which is the same as Eq.(9).

5. Experimental and numerical verification

The focus of this paper is theoretical rather than practical. Nevertheless some aspects of the experimental work have been mentioned above and aspects of verification will now be briefly considered. Naturally, the accuracy of experimental techniques is limited by the accuracy of the

measurements. For example, if resonance frequencies are determined by finding the frequency giving the maximum response, the final accuracy will depend on the resolution of the signal generator frequency, the frequency meter, and the ability to find the exact maximum response.

A real violin plate was used to test the ideas. Results from added masses were then compared with those from a laser Doppler optical measurement system. The consistency between the added mass method and the laser measurement depended a little on the amplitude, but was of the order of +/- 3% to 6%, which is quite sufficient for violin plate graduation.

The ideas were also tested on a model of an Euler-Bernoulli beam, rather than a plate, simply for ease and speed of modelling. (For present purposes, the beam can be considered as a 1-D plate, or a long and narrow plate). The model of the beam used a finite difference, shooting method to find resonant frequencies and mode shapes. Extra masses were then added at points along the beam and the change in the square of the resonance frequency was noted. To get the rate of change of resonance frequency at zero added mass, two methods were used. The first was a three mass method and extrapolation, as described above. The second was to remove a little mass at the point and then add the same amount, so that the result was centred on the zero added mass condition. With a single added mass of about 1% of the total beam mass, the change in resonant frequency gave the correct amplitude to within 2 to 3%. The ultimate accuracy is limited only by the numerical discretisation errors and algorithm convergence criteria.

The second numerical test used a finite element model of a real violin plate (using a commercial package, SAMSEF). The numerical model was in three dimensions, capturing the arching of the plate and complex curved boundaries. The software's modal analysis was then used to get the mode frequencies and shapes. Stiffness and density values throughout the plate were then adjusted to achieve closer agreement between the modal responses of the model and plate. The addition of the point mass was simulated by increasing the local mass at particular points. The changes in the (square of the) resonance frequencies were confirmed to be proportional to the amplitude squared at the selected points, again to an accuracy of a few percent, consistent with the numerical modelling accuracy, mesh resolution, etc.

6. Maxwell reciprocity theorem

Maxwell's reciprocity theorem provides a separate confirmation of the iso-amplitude contour idea. It says that if the application of a force at point A produces a displacement at point B, then the application of the same force at B will produce the same displacement at A. One can consider the addition of the point mass as providing an extra inertia load (mass-acceleration) of magnitude

$$f_p = \Delta m \omega^2 z_p \quad (25)$$

where z_p is the vibration amplitude at that point. This causes a change in the output at a second point. Then, by the reciprocity principle, adding the same force at the second point will cause the same change in the output at the first. So points of equal change must be points where f_p is the same, which means z_p is the same, that is, they lie on iso-amplitude contours.

This argument does not yield more information about the amplitude of vibration: it simply says that the points of equal change in output will be points of equal vibration amplitude. But it does confirm that iso-amplitude contours need be quantified only once.

7. Conclusions

The paper has presented three simple methods, based on added point masses, to determine iso-amplitude contours for vibration modes of thin plates. The methods are believed to be novel, certainly within luthier communities. They all involve finding points where adding a given point mass produces the same effect on one of three measured variables, namely shift in resonance frequency, drop in amplitude and shift in phase. If desired, the iso-amplitude contours can be quantified and a full modal shape obtained. All this would also be possible with laser-doppler techniques. The technique however offers further information not readily available from laser-doppler approaches. By evaluating the rates of change of the variables with added mass, the modal mass, stiffness and damping can also be measured.

The context for this work is the tuning of violin plates. The paper presented the theory. The application to the luthier's craft, and in particular how the improved modal analysis can be used to guide plate tuning, will be considered in another paper. Meanwhile it is believed that the theoretical results should be of general interest in experimental modal analysis of thin plates and it might find application in areas beyond musical instrument construction.

Acknowledgements

The authors would like to thank Steve Jacobs of SAMSEF for guidance with the finite element modelling, and William Doogue and Patrick King for their work on the experimental and numerical validation of the theoretical results.

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