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Path dependent option pricing under Lévy processes

Applied to Bermudan options

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A model is developed that can price path dependent options when the underlying process is an exponential Lévy process with closed form conditional characteristic function. The model is an extension of a recent quadrature option pricing model so that it can be applied with the use of Fourier and Fast Fourier transforms. Thus the model possesses nice features of both Fourier and quadrature option pricing techniques since it can be applied for a very general set of underlying Lévy processes and can handle exotic path dependent features. The model is applied to European and Bermudan options for geometric Brownian motion, a jump-diffusion process, a variance gamma process and a normal inverse Gaussian process. However it must be noted that the model can also price other path dependent exotic options such as lookback and Asian options.

Key words: Fast Fourier transform, recursive, path dependent option pricing.

¹All comments welcome and encouraged.

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1. Introduction

Existing Fourier and Fast Fourier transform option pricing models are essentially closed form option pricing formula that can be applied to a wide range of underlying stochastic processes. Fourier and Fast Fourier transform models (hereafter referred to collectively as transform models) can handle general stochastic processes and can also be applied to exotic versions of the Black Scholes (BS) option pricing formula² such as barrier options, digital options etc. They cannot be applied recursively so this limits their applicability when an option includes certain exotic features, for example, if an option has a changing barrier, if an option is exercisable on a number of dates or if there is a discrete dividend during the life of the option. Pioneering articles on the application of transform techniques to option pricing include Stein and Stein (1991), Heston (1993), Bates (1996), Duffie, Pan and Singleton (2001), Carr and Madan (1999), Backshi and Madan (2000), Lewis (2000) and Lee (2004).

Existing quadrature option pricing models have been developed for a restrictive class of underlying processes, namely geometric Brownian motion, however they are recursive models and so they can accurately price options with exotic features. Option pricing using numerical integration methods was first introduced by Parkinson (1997) but because of the discontinuities involved in option pricing this has been one of the least used option pricing methods. Sullivan (2000) did successfully apply quadrature routines to price American options under geometric Brownian motion. One of the most recent of these numerical integration schemes specifically deals with the non-linearity error introduced by the discontinuities in option prices, see Andricopoulos, Widdicks, Duck and Newton (2003). In their article it was recognised that options could be priced accurately using a discounted integration of the payoff provided the payoff is segmented so the integral is only carried out over continuous segments of the payoff. They run a quadrature routine (QUAD) to estimate each section of the segmented integral. In this way the quadrature routine is only applied to functions that are continuous and have continuous higher order derivatives and thus the routine retains its accuracy. If there is more than one time step needed to price the option, an asset price lattice is set up and the price at each point in the lattice at time t is calculated recursively from the prices at time $t + 1$ using the quadrature routine. Thus, for example, in

²Transform models can also price exotic options that cannot be priced in the BS framework, for example, Asian options where the average is taken over the entire life of the option, see Backshi and Madan (2000).

a plain vanilla call option only one time step is needed and the payoff is split into two continuous regions, the segment that is greater than zero and the segment that is zero. As another example, a single barrier at time t can be included by working backwards from maturity to find the prices on the lattice at time t and then applying the barrier condition. The option price prior to time t is calculated recursively from the time t prices which are split into two continuous regions, one on either side of the barrier.

In this article the quadrature option pricing model developed by Andricopoulos et al. (2003) is extended so that it can be applied with the use of Fourier and Fast Fourier transform (FFT) techniques originated by Heston (1993) and Carr and Madan (1999) respectively. This model allows transform option pricing models to be applied recursively resulting in a model that can incorporate exotic features and can be applied for a wide range of stochastic processes, specifically those stochastic processes that result in a process with independent increments³ and whose conditional characteristic function is known in closed form. Many Lévy processes fall into this category. Examples of finite activity Lévy processes include the jump-diffusion process of Merton (1976), and the double exponential jump-diffusion model of Kou (2002). Examples of infinite activity Lévy processes (Lévy processes whose jump arrival rate is infinite) include the variance gamma process of Madan and Seneta (1990), Madan and Milne (1991) and Madan, Carr and Chang (1998), and the normal inverse Gaussian model of Barndorff-Nielsen (1998). See Geman (2002) and Carr and Wu (2004) and references therein for more on these models and other Lévy processes. The model developed in this paper can be considered as a simple alternative model to current numerical pricing methods for path dependent options under Lévy processes, such as the finite difference method of Hirt and Madan (2004), the lattice method of Kellizi and Webber (2003) and the multinomial tree method of Maller, Solomon and Szimayer (2004).

To demonstrate the accuracy of the model European options are priced and compared to existing prices under a number of different processes. These processes include geometric Brownian motion (GBM) for benchmarking purposes, Merton's jump-diffusion (JD) process as an example of a finite activity Lévy process, a variance gamma (VG) and a normal inverse Gaussian (NIG) process as examples of infinite activity Lévy processes. In the GBM case American options are

³The model developed in this paper can be used with processes that do not have independent increments, for example options can be priced under Heston's (1993) stochastic volatility process. However the model becomes slower than competing methods as the integrations must run over two dimensions: the asset price and the variance.

approximately priced using extrapolation techniques and compared with prices from existing American option pricing models to demonstrate the ability of the model to price path dependent options. For the remaining processes Bermudan options are priced for benchmarking purposes, however extrapolation techniques can be applied to approximate American prices. The remaining article is organised as follows. Section 2 covers the methodology by first examining the existing QUAD model and then extending it so that transform techniques can be applied in conjunction with the quadrature routine. Section 3 contains the implementation details for both the Fourier and the FFT approach. Section 4 contains results for European and American options priced under GBM, and European and Bermudan options priced under JD, VG and NIG processes. Section 5 concludes.

2. Methodology

2.1. Quadrature model

Let us start by considering the quadrature model of Andricopoulos et al. (2003). Assume the underlying process is geometric Brownian motion. The notation is as follows: S_t is the stock price, E is the exercise price, r is the constant risk free rate, q is the dividend yield and σ is the volatility. Let $T = t + \Delta t$ be the time at which the payoff (or option price) is known, noting that the model is recursive. Define the transformed prices as $x = \log(S_t/E)$ and $y = \log(S_{t+\Delta t}/E)$. The option price at time t can be written as

$$V(x, t) = e^{-r\Delta t} E_t^Q [V(y, t + \Delta t)] \quad (2.1)$$

where the expectation is a risk neutral (RN) expectation and where $V(y, t + \Delta t)$ is the time $t + \Delta t$ known payoff (or option price) at a transformed price y . This RN expectation can be written as

$$V(x, t) = e^{-r\Delta t} \int_{-\infty}^{\infty} V(y, t + \Delta t) f(y|x) dy \quad (2.2)$$

where $f(y|x)$ is the conditional RN probability density of attaining y given x . If the stock price process is geometric Brownian motion then y is normally distributed and

$$f(y|x) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} \exp\left(-\frac{\left(y - x - \left(r - q - \frac{1}{2}\sigma^2\right)\Delta t\right)^2}{2\sigma^2\Delta t}\right) \quad (2.3)$$

and

$$\begin{aligned}
V(x, t) &= \frac{e^{-r\Delta t}}{\sqrt{2\pi\sigma^2\Delta t}} \int_{-\infty}^{\infty} V(y, t + \Delta t) \exp\left(-\frac{(y - x - (r - q - \frac{1}{2}\sigma^2)\Delta t)^2}{2\sigma^2\Delta t}\right) dy \\
&= \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} A(x) \int_{-\infty}^{\infty} V(y, t + \Delta t) B(x, y) dy \\
&= \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} A(x) \int_{-\infty}^{\infty} F(x, y) dy
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
A(x) &= \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} \exp\left(-\frac{1}{2}kx - \frac{1}{8}k^2\sigma^2\Delta t - r\Delta t\right) \\
F(x, y) &= V(y, t + \Delta t) B(x, y) \\
B(x, y) &= \exp\left(-\frac{(y - x)^2}{2\sigma^2\Delta t} + \frac{1}{2}ky\right) \\
k &= 2\left(\frac{r - q}{\sigma^2}\right) - 1
\end{aligned} \tag{2.5}$$

Simpson's rule, or any other quadrature routine, is simply applied to each continuous segment of $F(x, y)$ to estimate the integral $\int_{-\infty}^{\infty} F(x, y) dy$. For example, Simpson's rule is carried out as follows:

$$\begin{aligned}
\int_a^b g(y) dy &\approx \frac{\delta y}{6} \left\{ g(a) + 4g\left(a + \frac{1}{2}\delta y\right) + 2g(a + \delta y) + 4g\left(a + \frac{3}{2}\delta y\right) \right. \\
&\quad \left. + 2g(a + 2\delta y) + \dots + 2g(b - \delta y) + 4g\left(b - \frac{1}{2}\delta y\right) + g(b) \right\}
\end{aligned} \tag{2.6}$$

for some continuous function $g(y)$. Thus for a plain vanilla call option Simpson's rule is applied for $y > 0$ (corresponding to $S_{t+\Delta t} > E$) to estimate the integral $\int_0^{\infty} F(x, y) dy$. For $y < 0$ (corresponding to $S_{t+\Delta t} < E$) the integral is zero everywhere i.e. $\int_{-\infty}^0 F(x, y) dy = 0$. If the option is a discrete knock-out barrier option and the time $t + \Delta t$ option prices have been calculated recursively, Simpson's rule is applied for $y > b$, where $b = \ln(B/E)$ is the transformed barrier at time $t + \Delta t$. For $y < b$ the integral is zero. See Andricopoulos et al. (2003) for more details on this method.

2.2. Quadrature transform model

Writing the option price as above results in a major part of the work being carried out analytically, however there is no need to write the option price in this manner. It can simply be written as

$$V(x, t) = e^{-r\Delta t} \int_{-\infty}^{\infty} V(y, t + \Delta t) f(y|x) dy \quad (2.7)$$

Provided the conditional RN density function $f(y|x)$ and the time $t + \Delta t$ option prices are known, Simpson's rule can be used to estimate the integral in Eq. 2.7 assuming the integral is split up into its relevant continuous segments. However $f(y|x)$ is known for only for a small number of plausible stock price processes, including geometric Brownian motion, arithmetic Brownian motion and Ornstein-Uhlenbeck processes.

Define the conditional characteristic function (CCF) of a process y given x as

$$ccf(\phi, \Delta t; \Theta, x) = E_t^Q [\exp(i\phi y) | x] = \int_{-\infty}^{\infty} \exp(i\phi y) f(y|x) dy \quad (2.8)$$

where i is the imaginary number $\sqrt{-1}$, ϕ is a transform parameter (ϕ is usually a real number but can be an imaginary number, see Lewis (2000)) and Θ is the parameter vector of the underlying process. For example, a process following geometric Brownian motion has a CCF given by

$$ccf(\phi, \Delta t; \Theta, x) = \exp \left\{ i\phi x + \left(r - q - \frac{1}{2}\sigma^2 \right) \Delta t i\phi - \frac{1}{2}\sigma^2 \Delta t \phi^2 \right\} \quad (2.9)$$

where the parameter vector is now a scalar with $\Theta = \sigma$. The conditional characteristic function is known in closed form for many processes including Heston's (1993) stochastic volatility process, jump-diffusion stochastic volatility processes, see Bates (1996), and Lévy processes, see Lewis (2001), Geman (2002) and Carr and Wu (2004). In many of these cases the CCF is known in closed form but the probability density (PDF) function is not known in closed form and it is more convenient to work in characteristic space. The CCF is a Fourier transform of the PDF and hence there exists a one-to-one relationship between the PDF and its CCF. If the CCF is known the cumulative density function (CDF) is given by the inverse Fourier transform

$$\begin{aligned} \Pr\{u < y | x\} &= \int_{-\infty}^y f(u|x) du \\ &= \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left[\frac{\exp(-i\phi y) ccf(\phi, \Delta t; \Theta, x)}{i\phi} \right] d\phi \end{aligned} \quad (2.10)$$

therefore to calculate the PDF from the conditional characteristic function take the derivative w.r.t. y of the CDF in characteristic space to yield

$$f(y|x) = \frac{1}{\pi} \int_0^\infty \operatorname{Re} [\exp(-i\phi y) \operatorname{ccf}(\phi, \Delta t; \Theta, x)] d\phi \quad (2.11)$$

To apply the quadrature option pricing model using transform techniques Eq. 2.11 is calculated at discrete points along the density function using a transform technique and these are substituted into the option pricing formula in Eq. 2.7 and the option price itself is calculated using a quadrature routine. Thus there are two sources of numerical error in this model: the error from calculating the density function using a transform technique and the error from running the quadrature routine to calculate the option price. However, if the CCF is known in closed form, the error from approximating the density function can be controlled, see Pan (2002) and Lee (2004). Thus one can insure that the density function calculations are at a particular level of accuracy before using them as inputs into the quadrature routine.

The density function can be calculated (to within a specified error tolerance) at discrete points using either of two transform techniques, a Fourier transform or a FFT. The resulting option pricing model is recursive, unlike other existing transform option pricing methods that use closed form CCFs. Hence, if required, the time dimension can be divided into a discrete number of steps and one can work backwards from maturity applying the appropriate boundary conditions at each point, rendering this method with the ability to price exotic path dependent options.

2.3. Greeks

The greeks can also be calculated in this model. A plain vanilla call options sensitivity to the underlying price, delta, is given as follows

$$\Delta = \frac{\partial V}{\partial S} = \frac{\partial V}{\partial x} \frac{\partial x}{\partial S} = \frac{\partial V}{\partial x} \frac{1}{S} \quad (2.12)$$

where

$$\frac{\partial V}{\partial x} = e^{-r\Delta t} \int_{-\infty}^{\infty} V(y, t + \Delta t) \frac{\partial f(y|x)}{\partial x} dy \quad (2.13)$$

and

$$\frac{\partial f(y|x)}{\partial x} = \frac{1}{\pi} \int_0^\infty \operatorname{Re} [i\phi \exp(-i\phi y) \operatorname{ccf}(\phi, \Delta t; \Theta, x)] d\phi \quad (2.14)$$

Thus the delta is calculated in same manner as the option price itself.

Suppose $\Theta_i \in \Theta$ is a parameter of the model. The options sensitivity to Θ_i is given by

$$\frac{\partial V}{\partial \Theta_i} = e^{-r\Delta t} \int_{-\infty}^{\infty} V(y, t + \Delta t) \frac{\partial f(y|x)}{\partial \Theta_i} dy \quad (2.15)$$

where

$$\frac{\partial f(y|x)}{\partial \Theta_i} = \frac{1}{\pi} \int_0^{\infty} \text{Re} \left[\exp(-i\phi y) \frac{\partial ccf}{\partial \Theta_i} \right] d\phi \quad (2.16)$$

and the options sensitivity to Θ_i is calculated in same manner as the option price itself.

3. Implementation

In this section the implementation details are outlined for setting up the lattice, and for applying the two different methods of calculating the density function using transform techniques.

3.1. Lattice model

The model is implemented in a similar fashion to Andricopoulos et al. (2003). First the asset price lattice is constructed. Suppose x_0 is the initial transformed price at time t . The transformed prices at time $t + \Delta t$ are given by the vector

$$\begin{aligned} \mathbf{y} &= \{y_0, y_1, \dots, y_{N-1}\}' \\ &= \{x_0 - q^*, x_0 - q^* + \Delta y, \dots, x_0 + q^* - \Delta y, x_0 + q^*\}' \end{aligned}$$

where $\Delta y = 2q^*/N$, $q^* = L \times vol(\Delta t)$ and $vol(\Delta t)$ is the volatility of the process⁴ over the time interval Δt , so in the geometric Brownian motion case $vol(\Delta t) = \sigma\sqrt{\Delta t}$. This means that the PDF is calculated L standard deviations either side of x_0 and should result in sufficient accuracy if $L = 10$ (this of course can be adjusted downwards). For $i = 0, \dots, N - 1$ each discrete point of the density function $f(y_i|x_0)$ can be calculated using a Fourier transform or the entire vector of discrete points $f(\mathbf{y}|x_0)$ can be calculated using one FFT.

⁴This volatility can be computed for stochastic processes whose CCF is known with

$$vol(\Delta t) = \sqrt{var(\Delta t)}, \quad var(\Delta t) = \frac{1}{i^2\phi^2} \frac{\partial^2 ccf}{\partial \phi^2} \Big|_{\phi=0}$$

Suppose there are two time steps needed to price the option. Let x_0 be the initial transformed price at time t , \mathbf{y}_1 be the transformed price vector at time $t + \Delta t$ and \mathbf{y}_2 be the transformed price vector at time $t + 2\Delta t$. Define \mathbf{y}_1 and \mathbf{y}_2 as follows

$$\begin{aligned}\mathbf{y}_1 &= \{x_0 - q^*, x_0 - q^* + \Delta y, \dots, x_0 + q^* - \Delta y, x_0 + q^*\}' \\ \mathbf{y}_2 &= \{x_0 - 2q^*, x_0 - 2q^* + \Delta y, \dots, x_0 + 2q^* - \Delta y, x_0 + 2q^*\}'\end{aligned}$$

Calculate $f(\mathbf{y}_1 | x_0)$ using either transform technique. If the returns process is a proportional returns process, rather than calculating $f(y_{2j} | y_{1i})$ for each $y_{2j} \in \mathbf{y}_2$ given each $y_{1i} \in \mathbf{y}_1$, one can simply use the PDF estimated for \mathbf{y}_1 given x_0 and change the location of this PDF so that it is centered at each $y_{1i} \in \mathbf{y}_1$ with a range of L standard deviations either side of y_{1i} . This will result in a reduction in computing time as the PDF only needs to be calculated once. This can only be done if the time step between successive lattice periods is equal. The method can also be easily applied to “piecewise Lévy processes” where, for example, the density function has increments that are stationary on each time interval $[t, t_1]$, $[t_1, t_2]$ and $[t_2, T]$, but the density function can change from one interval to the next, thereby increasing calibration performance with respect to market option prices over both strike price and maturity. See Eberlein and Kluge (2004) for an example of a piecewise Lévy process applied to a term structure model. Such a process would provide a better fit to implied volatility surfaces than a single Lévy process and overcome the difficulties that are encountered when using Lévy processes to price options across different maturities.

Suppose there is a discontinuity in the option price, such as a barrier, along the transformed price vector \mathbf{y} and denote this discontinuity as b . If $b = y_i$ for some $i = 0, \dots, N - 1$ the integration is split in two by running it over the two ranges $[x_0 - q^*, b]$ and $[b, x_0 + q^*]$. If $b \neq y_i$ but $y_{i-1} < b < y_i$ for some $i = 1, \dots, N - 1$, then the integration must be handled with care because the integration over the range $[x_0 - q^*, b]$ now includes a step (that from y_{i-1} to b) that is smaller in length than Δy , the same holds for the first step in the integration over the range $[b, x_0 + q^*]$. This can be done without much difficulty.

3.2. Fourier transform method

This section covers the details required to calculate the density function at discrete points of y using a Fourier transform. To do this y is discretised into N points with $\mathbf{y} = \{y_0, y_1, \dots, y_{N-1}\}'$. Then we approximate $f(y_i | x)$ using a Fourier transform

for $i = 0, \dots, N - 1$. The PDF in Eq. 2.11 can be discretised as follows

$$\begin{aligned} f(y_i | x) &= \frac{1}{\pi} \int_0^\infty \operatorname{Re} [\exp(-i\phi y_i) \operatorname{ccf}(\phi, \Delta t; \Theta, x)] d\phi \\ &\simeq \frac{1}{\pi} \sum_{j=0}^{T-1} \operatorname{Re} [\exp(-i\phi_j y_i) \operatorname{ccf}(\phi_j, \Delta t; \Theta, x)] \Delta\phi \end{aligned} \quad (3.1)$$

where the transform parameter ϕ is discretised into a vector with $\phi = \{\phi_0, \dots, \phi_{T-1}\}'$, T is the truncation point and $\Delta\phi$ is the discretisation length. The integral can be approximated using Simpson's rule or any quadrature routine. There will be a truncation and discretisation error in the approximation of the integral but these can be controlled given the closed form nature of the CCF, see Pan (2002). The option pricing model is denoted Q-FT when the density function is calculated in this manner.

3.3. Fast Fourier transform method

This section covers the details required to calculate the vector whose entries are discrete points on the density function, $f(\mathbf{y} | x)$, with *one* application of a FFT. The FFT improves computational speed and error control relative to FT methods, see Carr and Madan (1999) and Lee (2004), thus it is logical to want to use its considerable power in this setting. A FFT can be applied to compute the density function in Eq. 2.11 and furthermore the density function is an integrable function so there is no need to damp the function in this method as required in previous FFT methods. To run the method proceed as follows:

$$\begin{aligned} f(y | x) &= \frac{1}{\pi} \int_0^\infty \operatorname{Re} [\exp(-i\phi y) \operatorname{ccf}(\phi, \Delta t; \Theta, x)] d\phi \\ &= \frac{1}{\pi} \operatorname{Re} \left[\int_0^\infty \exp(-i\phi y) \operatorname{ccf}(\phi, \Delta t; \Theta, x) d\phi \right] \end{aligned} \quad (3.2)$$

A FFT of a vector v of dimension $N \times 1$ computes the following sum

$$FFT(v(p)) = \sum_{n=0}^{N-1} \exp\left(-i\frac{2\pi}{N}np\right) v(n) \text{ for } p = 0, \dots, N - 1 \quad (3.3)$$

To apply this algorithm in the current setting divide ϕ and y into N points with

$$\phi_n = (n - N/2) \Delta\phi, \quad y_p = (p - N/2) \Delta y \quad (3.4)$$

for $n, p = 0, 1, \dots, N - 1$. Write the density function as

$$\begin{aligned} f(\mathbf{y} | x) &\cong \frac{1}{\pi} \operatorname{Re} \left[\sum_{n=0}^{N-1} \exp(-i\phi_n y_p) \operatorname{ccf}(\phi_n, \Delta t; \Theta, x) \Delta\phi \right] \\ &= \frac{1}{\pi} (-1)^p \operatorname{Re} \left[\sum_{n=0}^{N-1} \exp(-i\Delta\phi \Delta y n p) (-1)^n \operatorname{ccf}(\phi_n, \Delta t; \Theta, x) \Delta\phi \right] \end{aligned} \quad (3.5)$$

Provided $\Delta\phi$ and Δy are restricted so that $\Delta\phi \times \Delta y = 2\pi/N$ the FFT can be applied with

$$f(\mathbf{y} | x) = \frac{1}{\pi} (-1)^p \operatorname{Re} [FFT(v(p))] \quad (3.6)$$

where

$$v(n) = (-1)^n \operatorname{ccf}(\phi_n, \Delta t; \Theta, x) \Delta\phi \quad (3.7)$$

and it can be applied with considerable computational speed if N is a power of 2. A trade-off exists here that is also present in other FFT option pricing models. As Δy becomes smaller $\Delta\phi$ must necessarily become larger and vice versa. So if the transformed stock price grid becomes finer the grid for computing the density function becomes coarser. After experimenting with different values of N and L it was found that $N = 512$ or 1024 and $L = 10$ provided a good compromise between speed and accuracy. The option pricing model is denoted Q-FFT when the density function is calculated in this manner.

4. Numerical results

Q-FFT is superior in accuracy and speed relative to Q-FT so in this section we only consider Q-FFT as it is by far the dominant model. First Q-FFT is used to price European and American options under GBM and prices are compared to those from existing models. Then Q-FFT is used to price European and Bermudan options under a JD process, a VG process and an NIG process and in each case the prices are benchmarked against existing models

4.1. Geometric Brownian motion

Consider a call option when the underlying follows geometric Brownian motion. Let BS represent Black-Scholes prices and Q-FFT represent Q-FFT prices where the conditional density function is calculated using a FFT. The true American

price is calculated with a binomial tree with 10,000 steps. Q-FFT R4, R10 and R20 are American options priced with a four point Richardson extrapolation scheme, and priced as $2P_{10} - P_5$ and $2P_{20} - P_{10}$ respectively, where P_n is an n -times exercisable option. Numerical results are reported in Table 1 and Table 2.

The root mean square error (RMSE) is calculated w.r.t. the Black-Scholes (BS) price for European options⁵. The RMSE is calculated w.r.t. a binomial tree with 10,000 steps for American options. Q-FFT has an RMSE of 0.0004 when used to price European options and is extremely accurate. The RMSE is 0.0122 when American options are priced with the four point scheme R4 and is 0.0062 and 0.0046 when American options are priced with R10 and R20 respectively. The results for American options were compared with the results of Table 1 in Ju (1998). The RMSE's from the American option pricing models in Ju and this paper are recorded in Table 2. In this table BT800 is a binomial tree with 800 steps, GJ4 is the Geske-Johnson method (1984), MGJ2 is the Bunch and Johnson method (1992), HYS4 and HYS6 are four and six-point methods of Huang, Subrahmanyam, and Yu (1996), LUBA is the lower and upper bound method of Broadie and Detemple (1996), RAN4 and RAN6 are the four and six-point randomization methods of Carr (1998), and EXP3 is the three-point method of Ju (1998). The model performs well in comparison to other American option pricing models. Prices from Q-FFT R20 are not as accurate as BT800, LUBA, RAN6 or EXP3 however the model performs better than the other models. This performance does not take into account the computational time required to price the options using Q-FFT. The time required increases with the higher order approximations and although it only takes a few seconds to price each option, this is vastly slower than some of the competing methods such as RAN6 or EXP3. However the real gain from using Q-FFT comes from its ability to price exotic options under general exponential Lévy processes in a relatively simple manner. This is why we examine the models ability to handle more complex stochastic processes in the next sections.

⁵For European options the RMSE is calculated as:

$$RMSE = \sqrt{\frac{\sum_{j=1}^n (V_{BS_j} - V_{Q-FFT_j})^2}{n}}$$

Table 1: Prices of European and American call options $(E = \$100, t = 0, \Delta t = 0.5 \text{ years}, L = 10 \text{ and } N = 512)$

(S, σ, r, q)	European prices		American prices			
	BS	Q-FFT	Bin Tree	Q-FFT		
			True	R4	R10	R20
(80, 0.2, 0.03, 0.07)	0.2148	0.2149	0.2194	0.2194	0.2188	0.2188
(90, 0.2, 0.03, 0.07)	1.3451	1.3444	1.3864	1.3849	1.3846	1.3854
(100, 0.2, 0.03, 0.07)	4.5778	4.5778	4.7825	4.7829	4.7794	4.7801
(110, 0.2, 0.03, 0.07)	10.4208	10.4205	11.0978	11.0832	11.0932	11.0911
(120, 0.2, 0.03, 0.07)	18.3024	18.3017	20.0004	20.0088	20.0088	20.0049
(80, 0.4, 0.03, 0.07)	2.6506	2.6511	2.6889	2.6896	2.6865	2.6875
(90, 0.4, 0.03, 0.07)	5.6221	5.6228	5.7223	5.7181	5.7184	5.7195
(100, 0.4, 0.03, 0.07)	10.0211	10.0211	10.2385	10.2431	10.2340	10.2342
(110, 0.4, 0.03, 0.07)	15.7676	15.7678	16.1812	16.1801	16.1746	16.1740
(120, 0.4, 0.03, 0.07)	22.6502	22.6509	23.3598	23.3247	23.3482	23.3491
(80, 0.3, 0.00, 0.07)	1.0064	1.0058	1.0373	1.0337	1.0353	1.0364
(90, 0.3, 0.00, 0.07)	3.0041	3.0040	3.1233	3.1278	3.1195	3.1212
(100, 0.3, 0.00, 0.07)	6.6943	6.6943	7.0354	7.0375	7.0303	7.0315
(110, 0.3, 0.00, 0.07)	12.1661	12.1666	12.9552	12.9287	12.9457	12.9481
(120, 0.3, 0.00, 0.07)	19.1555	19.1555	20.7173	20.7424	20.7346	20.7082
(80, 0.3, 0.07, 0.03)	1.6644	1.6635	1.6644	1.6616	1.6643	1.6644
(90, 0.3, 0.07, 0.03)	4.4947	4.4945	4.4947	4.4960	4.4946	4.4947
(100, 0.3, 0.07, 0.03)	9.2506	9.2506	9.2504	9.2473	9.2506	9.2506
(110, 0.3, 0.07, 0.03)	15.7975	15.7980	15.7977	15.7954	15.7975	15.7975
(120, 0.3, 0.07, 0.03)	23.7062	23.7062	23.7061	23.7032	23.7062	23.7061
RMSE		0.0004		0.0122	0.0062	0.0046

Table 1: Columns 2 - 6 represent Black-Scholes European prices, Q-FFT European prices, binomial tree American prices with 10,000 time steps, Q-FFT American prices using 4 point Richardson extrapolation and extrapolation schemes given by $2P_{10}-P_5$ and $2P_{20}-P_{10}$.

Table 2: Comparison of model accuracy

Model	RMSE	Model	RMSE
BT800	0.0012	RAN4	0.0104
GJ4	0.0090	RAN6	0.0035
MGJ2	0.0805	EXP3	0.0013
HSY4	0.0231	Q-FFT R4	0.0122
HSY6	0.0089	Q-FFT R10	0.0062
LUBA	0.0012	Q-FFT R20	0.0046

Table 2: RMSE from different American option pricing models is calculated w.r.t. a binomial tree with 10,000 steps.

4.2. Finite activity Lévy processes

Finite activity Lévy processes are commonly referred to as jump-diffusion processes and have been very popular in option pricing literature because markets frequently undergo discontinuous jumps and because jumps can explain the implied volatility smile across strike prices especially well at short time horizons. For more on jump-diffusion processes see, for example, Bates (1996) and Duffie, Pan and Singleton (2000).

Suppose the risk neutral dynamics of the asset price are given by the following process:

$$dS_t/S_t = (r - q - \lambda\mu)dt + \sigma dW_t + (e^J - 1)dN_t(\lambda) \quad (4.1)$$

where dW_t is a Brownian motion under the risk neutral volatility measure, N_t is a pure jump Poisson process with instantaneous intensity λ and μ is the risk neutral mean relative jump size with $\mu = E_t^Q[e^J - 1]$. J is the random percentage jump size with distribution ν conditional on a jump occurring. For tractability purposes the risk neutral distribution of the jump size J is taken to be normal with $\nu(J) \sim N(\mu_J, \sigma_J)$, hence $\mu = E_t^Q[e^J - 1] = e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1$. The term $\lambda\mu$ appears in the risk neutral returns process to compensate for the discontinuous jump term and ensures that the discounted stock price process is a martingale. This model was originally proposed by Merton (1976), however in this setting we can use any jump amplitude that has a tractable Fourier transform such as lognormal and exponential jumps. Letting $x_t = \log(S_t/E)$, this model has a closed form CCF given by

$$\begin{aligned} & ccf(\phi, \Delta t; \Theta, x_t) \\ &= \exp \left[i\phi x_t + i\phi \left(r - q - \frac{1}{2}\sigma^2 + w \right) \Delta t - \Psi(\phi, \Delta t; \Theta) \Delta t \right] \end{aligned} \quad (4.2)$$

where

$$\Psi(\phi, \Delta t; \Theta) = \frac{1}{2}\sigma^2\phi^2 - \lambda \left(e^{i\phi\mu_J - \frac{1}{2}\phi^2\sigma_J^2} - 1 \right) \quad (4.3)$$

and

$$w = -\lambda\mu = -\lambda \left(e^{\mu_J - \frac{1}{2}\phi^2\sigma_J^2} - 1 \right) \quad (4.4)$$

Ψ is the characteristic exponent and w is risk neutral compensator⁶ that ensures that the discounted stock price follows a martingale. Note that the parameter vector is given by $\Theta = \{\sigma, \lambda, \mu_J, \sigma_J\}'$ and that the CCF is conditioned only on the observable transformed stock price x_t .

Let us consider a put option in the jump-diffusion case. Let FT represent prices using the standard Fourier transform techniques and Q-FFT represent Q-FFT prices. Let FT2 represent twice-exercisable option prices using a FT approach extended to the case of two exercise dates⁷ and let Q-FFT2 represent twice-exercisable option prices using Q-FFT. In each twice-exercisable option the first exercise date is $t + \frac{1}{2}\Delta t$ and the second exercise date is $t + \Delta t$. The benchmark Bermudan put prices are calculated for ten evenly spaced exercise dates using an implicit lattice method based on Das (1999) with the number of time steps $N_t = 1000$ and the number of asset price steps $N_s = 300$. Q-FFT10 are 10 times exercisable Bermudan prices using Q-FFT. Numerical results are reported in Table 3.

The RMSE of Q-FFT is calculated w.r.t. the standard FT method and is 0.0001 which is extremely accurate. The RMSE of Q-FFT2 is calculated w.r.t. the twice-exercisable FT option prices and is 0.0019 which is also very accurate. The RMSE of Q-FFT is calculated w.r.t. finite difference prices for the Bermudan options and one can see that the difference between the finite difference prices and the Q-FFT prices is still very small with a RMSE of 0.0069 and certainly lies within a reasonable accuracy range.

⁶To solve for w set

$$ccf(\phi = -i) = \exp(x_t + (r - q)\Delta t) \text{ i.e. } E_t^Q[S_{t+\Delta t}] = S_t \exp((r - q)\Delta t)$$

⁷To price a twice-exercisable option, a modified version of the two-dimensional Fourier transform methods of Backshi and Madan (2000) and Dempster and Hong (2000) was used. Contact the author for further details.

Table 3: European and American put options under JD $(E = \$100, t = 0, \Delta t = 0.5 \text{ years}, L = 10, N = 512$ $r = 0.08, q = 0, \sigma = 0.10, \text{ and } \lambda = 5)$

(S, σ_J, μ_J)	European		Twice-Exercisable		Bermudan	
	HES	Q-FFT	FT2	Q-FFT2	LAT	Q-FFT10
(80, 0.02, 0.0)	16.1006	16.1006	18.0204	18.0204	19.6008	19.6008
(90, 0.02, 0.0)	6.8791	6.8792	8.2314	8.2293	9.6046	9.6012
(100, 0.02, 0.0)	1.4603	1.4603	1.6702	1.6682	1.8199	1.8342
(110, 0.02, 0.0)	0.1314	0.1315	0.1357	0.1358	0.1462	0.1466
(120, 0.02, 0.0)	0.0055	0.0055	0.0055	0.0055	0.0059	0.0058
(80, 0.02, -0.02)	16.1053	16.1054	18.0204	18.0203	19.6008	19.6008
(90, 0.02, -0.02)	6.9964	6.9960	8.2823	8.2795	9.6059	9.5989
(100, 0.02, -0.02)	1.6937	1.6937	1.9158	1.9144	2.0905	2.0963
(110, 0.02, -0.02)	0.2256	0.2256	0.2360	0.2361	0.2561	0.2550
(120, 0.02, -0.02)	0.0194	0.0194	0.0196	0.0197	0.0214	0.0210
(80, 0.02, 0.02)	16.1336	16.1336	18.0240	18.0238	19.6008	19.6008
(90, 0.02, 0.02)	7.0891	7.0888	8.3491	8.3448	9.6176	9.6132
(100, 0.02, 0.02)	1.6477	1.6477	1.8478	1.8462	1.9757	1.9952
(110, 0.02, 0.02)	0.1638	0.1639	0.1673	0.1675	0.1774	0.1784
(120, 0.02, 0.02)	0.0068	0.0068	0.0068	.0068	0.0073	0.0071
(80, 0.04, 0.0)	16.1787	16.1787	18.0321	18.0318	19.6009	19.6009
(90, 0.04, 0.0)	7.3525	7.3528	8.4990	8.4934	9.6424	9.6317
(100, 0.04, 0.0)	2.0443	2.0443	2.2526	2.2505	2.4063	2.4173
(110, 0.04, 0.0)	0.3475	0.3477	0.3613	0.3615	0.3865	0.3842
(120, 0.04, 0.0)	0.0427	0.0426	0.0433	0.0433	0.0469	0.0457
RMSE		0.0001		0.0019		0.0069

Table 3: Columns 2 - 6 represent FT and Q-FFT European prices, FT and Q-FFT twice-exercisable prices, and lattice and Q-FFT Bermudan prices.

4.3. Infinite activity Lévy processes

Infinite activity Lévy processes are those Lévy processes with an infinite jump arrival rate, that is an infinite number of jumps can occur in a finite time horizon. They have become increasingly popular in finance because of their ability to fit asset return processes and option prices across strike prices in a parsimonious manner. The fact that on very small time scales market dynamics undergo a large number of very small discrete jumps is another theoretically appealing property of infinite activity Lévy processes.

4.3.1. Variance gamma process

The variance gamma process for asset returns was first proposed by Madan and Seneta (1990), and extended by Madan and Milne (1991), Madan, Carr and Chang (1998) and Carr, Geman, Madan and Yor (2003). The process is an arithmetic Brownian motion with drift evaluated at a random time change that follows a gamma process thus the VG process is a pure jump process. European options under VG can be priced in terms of special functions. Methods for pricing path dependent options under VG processes include the finite difference method of Hirta and Madan (2004), the lattice methods of Kellizi and Webber (2003) and the multinomial tree method Maller, Solomon and Szimayer (2004).

Denote an arithmetic Brownian motion with drift θ and volatility σ as follows:

$$b(t; \theta, \sigma) = \theta t + \sigma W(t) \quad (4.5)$$

Then a VG process has a log stock price process given by

$$X(t; \sigma, \nu, \theta) = b(T_t^v; \theta, \sigma) \quad (4.6)$$

where $T_t^v = \gamma(t; 1, \nu)$ is a gamma process with unit mean and variance ν . Letting $x_t = \ln(S_t/E)$ this model has a simple closed form CCF given by

$$ccf(\phi, \Delta t; \Theta, x_t) = \exp[i\phi x_t + i\phi(r - q + w)\Delta t - \Psi(\phi, \Delta t; \Theta)\Delta t] \quad (4.7)$$

where

$$\Psi(\phi, \Delta t; \Theta) = \frac{1}{\nu} \ln \left[1 - i\theta\nu\phi + \frac{1}{2}\sigma^2\nu\phi^2 \right] \quad (4.8)$$

and

$$w = \frac{1}{\nu} \ln \left[1 - \theta\nu - \frac{1}{2}\sigma^2\nu \right] \quad (4.9)$$

As before Ψ is the characteristic exponent and w is the risk neutral compensator. The parameter vector is given by $\Theta = \{\sigma, \theta, \nu\}'$ and the CCF is conditioned only on the observable transformed stock price x_t .

Option prices from Q-FFT are compared to those from Kellezi and Webber (2003), denoted as KW, to benchmark Q-FFT with results available in the literature. Table 4 contains results for European call options and Bermudan put options with 10 evenly spaced exercise dates. The parameter vector is given by $\Theta = \{0.12, -0.14, 0.2\}'$, the current stock price is $S_t = 100$ and the exercise price takes on values $E = \{90, 95, \dots, 110\}'$.

Table 4: European and Bermudan options under VG

($t = 0$, $\Delta t = 1$ year, $r = 0.10$, $q = 0$, $L = 10$ and $N = 1024$)

E	European call prices			European and Bermudan put prices		
	Analytical	KW	Q-FFT	Q-FFT (Euro)	KW (Berm)	Q-FFT10 (Berm)
90	19.09935	19.09936	19.09937	0.53474	0.76115	0.75787
95	15.07047	15.07048	15.07049	1.03005	1.52574	1.53866
100	11.37002	11.37002	11.37003	1.85377	2.88152	2.87782
105	8.11978	8.11978	8.11986	3.12779	5.17036	5.19537
110	5.42960	5.42960	5.42963	4.96175	9.04064	9.06040
115	3.36543	3.36544	3.36523	7.42153	13.87623	13.89456
120	1.92110	1.92110	1.92078	10.50127	18.80965	18.82561
RMSE		6.5×10^{-6}	1.5×10^{-4}			0.01603

Table 4: Columns 2 - 6 represent analytical, KW and Q-FFT European call prices, Q-FFT European put prices, KW and Q-FFT Bermudan put prices

The RMSE of KW and Q-FFT are calculated w.r.t. the analytical prices for the European call prices. Both models are very accurate with KW more accurate than Q-FFT. The RMSE of Q-FFT for the Bermudan put prices is calculated w.r.t. the KW prices and is reasonably accurate. The Bermudan prices from both models are relatively close as can be seen from the table. The surface plot in figure 1 is a plot of Bermudan put prices in the VG model versus number of exercise dates, M , and exercise price, E . The plot uses the put option parameters given in table 4 with M now varying from 1 to 10. As can be seen from this plot the early exercise premium for in-the-money puts grows at a much faster rate than the premium for out-of-the-money puts.

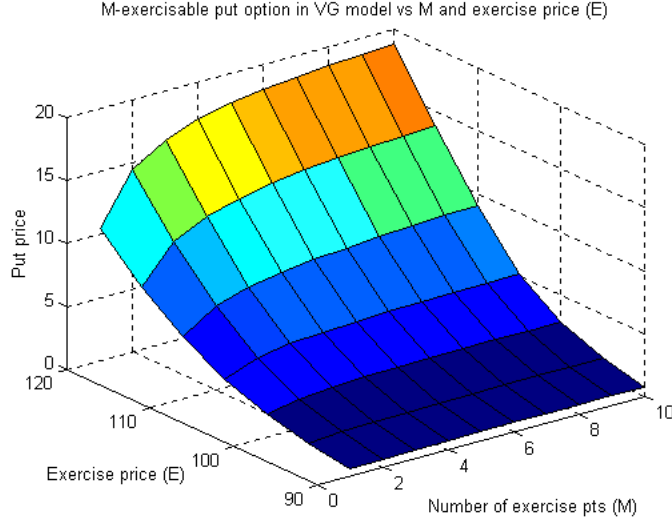


Figure 2. Put option price versus number of exercise dates and exercise price

4.3.2. Normal inverse Gaussian process

Generalized hyperbolic distributions contain as a subset the normal inverse Gaussian (NIG) distribution which has been used as a model of asset returns by Eberlein and Keller (1995) and Barndorff-Nielsen (1998) among others. European options can be priced using numerical integration or FFTs. Methods for pricing path dependent options under NIG processes include the lattice methods of Kellizi and Webber (2003) and the multinomial tree method Maller, Solomon and Szimayer (2004).

Let T_t^v be the first time that a Brownian motion with drift v reaches the positive level t . The density of T_t^v is inverse Gaussian. An NIG process is an arithmetic Brownian motion with drift θ and volatility σ evaluated at this Gaussian time and thus, like the VG process, the NIG process is a pure jump process. Denoting an arithmetic Brownian motion with drift as in Eq. 4.5 an NIG process has a log stock price process given by

$$X(t; \sigma, v, \theta) = b(T_t^v; \theta, \sigma) \quad (4.10)$$

Letting $x_t = \ln(S_t/E)$ this model has a closed form CCF given by

$$ccf(\phi, \Delta t; \Theta, x_t) = \exp[i\phi x_t + i\phi(r - q + w)\Delta t - \Psi(\phi, \Delta t; \Theta)\Delta t] \quad (4.11)$$

where

$$\Psi(\phi, \Delta t; \Theta) = \delta \left(\sqrt{\alpha^2 - (\beta + i\phi)^2} - \sqrt{\alpha^2 - \beta^2} \right) \quad (4.12)$$

and

$$w = \delta \left(\sqrt{\alpha^2 - (\beta + 1)^2} - \sqrt{\alpha^2 - \beta^2} \right) \quad (4.13)$$

As before Ψ is the characteristic exponent and w is the risk neutral compensator. The parameter vector is written as $\Theta = \{\alpha, \beta, \delta\}'$ to conform with conventional notation, where the relationship with the parameters $\{\sigma, v, \theta\}'$ is given as follows

$$\alpha^2 = \frac{v^2}{\sigma^2} + \frac{\theta^2}{\sigma^4}, \quad \beta = \frac{\theta}{\sigma^2}, \quad \delta = \sigma \quad (4.14)$$

The CCF is conditioned only on the observable transformed stock price x_t .

To benchmark results option prices from Q-FFT are compared to those from Kellezi and Webber (2003). Table 5 contains results for European call options and Bermudan put options with 10 evenly spaced exercise dates. The parameter vector is given by $\Theta = \{28.42141, -15.08623, 0.31694\}'$ ⁸, the current stock price is $S_t = 100$ and the exercise price takes on values $E = \{90, 95, \dots, 110\}'$.

As with the VG case the European options from Q-FFT are extremely accurate and the Bermudan prices are very close to the prices from Kellezi and Webber.

Table 5: European and Bermudan options under NIG

($t = 0$, $\Delta t = 1$ year, $r = 0.10$, $q = 0$, $L = 10$ and $N = 1024$)

E	European call prices			European and Bermudan put prices		
	Reference	KW	Q-FFT	Q-FFT (Euro)	KW (Berm)	Q-FFT10 (Berm)
90	19.09330	19.09330	19.09321	0.52858	0.74482	0.73000
95	15.06077	15.06077	15.06061	1.02017	1.49554	1.47900
100	11.35994	11.35993	11.35969	1.84343	2.84445	2.82777
105	8.11561	8.11561	8.11537	3.12330	5.17297	5.15981
110	5.43723	5.43724	5.43691	4.96902	9.03395	9.02806
115	3.38474	3.38475	3.38418	7.44048	13.86529	13.86170
120	1.94359	1.94359	1.94296	10.52345	18.80693	18.80320

RMSE

Table 5: Columns 2 - 6 represent analytical, KW and Q-FFT European call prices, Q-FFT European put prices, KW and Q-FFT Bermudan put prices

⁸These parameters were chosen by Kellezi and Webber so that the density function of the NIG process has the same first four moments as the density function from the VG process.

5. Conclusion

A model is developed that can be used to price path dependent options on Lévy processes. The model combines two existing models: the Fourier transform and quadrature option pricing models, retaining nice features of both models. The model is very general in terms of the underlying processes, with the only constraint being that the log stock price process has to have a closed form conditional characteristic function. The model is also very general in terms of the type of exotic option being priced. It is easy to adjust the model to price other path dependent options such as barrier options, Asian options etc. The model can easily handle infinite activity Lévy processes since a characteristic function based approach is used thus avoiding any problems other lattice based approaches encounter when using the Lévy measure in a neighbourhood of zero.

There are many possible avenues of future research. The accuracy and speed of the model can be improved by using more refined quadrature routines and by constructing trees that are bounded. However I am more interested in applying the model to Lévy processes that incorporate stochastic volatility as a continuous time stochastic process or as a regime switching volatility model. The model in this paper can also be extended to incorporate “piecewise Lévy processes” where, for example, the density function has increments that are stationary on each time interval $[t, t_1]$, $[t_1, t_2]$ and $[t_2, t + \Delta t]$, but the density function can change from one interval to the next, thereby increasing calibration performance with respect to option prices over both strike price and maturity.

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