A REMARK ON A CONJECTURE OF BORWEIN AND CHOI

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Abstract. We prove the remaining case of a conjecture of Borwein and Choi concerning an estimate on the square of the number of solutions to \( n = x^2 + Ny^2 \) for a squarefree integer \( N \).

1. Introduction

We consider the positive definite quadratic form \( Q(x, y) = x^2 + Ny^2 \) for a squarefree integer \( N \). Let \( r_{2,N}(n) \) denote the number of solutions to \( n = Q(x, y) \) (counting signs and order). In this note, we estimate

\[
\sum_{n \leq x} r_{2,N}(n)^2.
\]

A positive squarefree integer \( N \) is called solvable (or more classically “numerus idoneus”) if \( x^2 + Ny^2 \) has one form per genus. Note that this means the class number of the form class group of discriminant \( -4N \) equals the number of genera, \( 2^t \), where \( t \) is the number of distinct prime factors of \( N \). Concerning \( r_{2,N}(n) \), Borwein and Choi [1] proved the following:

Theorem 1.1. Let \( N \) be a solvable squarefree integer. Let \( x > 1 \) and \( \epsilon > 0 \). We have

\[
\sum_{n \leq x} r_{2,N}(n)^2 = \frac{3}{N} \left( \prod_{p \mid 2N} \frac{2p}{p+1} \right) x \log x + \alpha(N)x + O(N^{\frac{3}{4} + \epsilon} x^{\frac{3}{4} + \epsilon})
\]

where the product is over all primes dividing \( 2N \) and

\[
\alpha(N) = -1 + 2\gamma + \sum_{p \mid 2N} \frac{\log p}{p+1} + \frac{2L'(1, \chi_{-4N})}{L(1, \chi_{-4N})} - \frac{12}{\pi^2} \zeta'(2)
\]

where \( \gamma \) is the Euler-Mascheroni constant and \( L(1, \chi_{-4N}) \) is the L-function corresponding to the quadratic character mod \( -4N \).

Based on this result, Borwein and Choi posed the following:

Conjecture 1.2. For any squarefree \( N \),

\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p \mid 2N} \frac{2p}{p+1} \right) x \log x
\]

The main result in [10] was the following.

Theorem 1.3. Let \( Q(x, y) = x^2 + Ny^2 \) for a squarefree integer \( N \) with \( -N \not\equiv 1 \mod 4 \). Let \( r_{2,N}(n) \) denote the number of solutions to \( n = Q(x, y) \) (counting signs and order). Then

\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p \mid 2N} \frac{2p}{p+1} \right) x \log x.
\]

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In this note, we settle the conjecture in the remaining case, namely

**Theorem 1.4.** For \(-N \equiv 1 \mod 4\), we have
\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p|2N} \frac{2p}{p+1} \right) x \log x.
\]

2. **Preliminaries**

Let \(Q(x, y) = ax^2 + bxy + cy^2\) denote a positive definite integral quadratic form with discriminant \(D = b^2 - 4ac\) and \(gcd(a, b, c) = 1\). Given \(Q\), let \(\kappa\) be the largest positive integer with \(D/\kappa^2\) an integer congruent to 0 or 1 modulo 4. We call \(\kappa\) the *conductor* of \(Q\) and set \(d = D/\kappa^2\). Let \(r(Q, n)\) be the number of representations of the integer \(n\) by the form \(Q\). We now relate \(r(Q, n)\) to counting the number of integral ideals of norm \(n\) in a given class in a generalized ideal class group.

Given \(D = \kappa^2d\) we consider ideals in \(\mathcal{O}_K\) where \(K = \mathbb{Q}(\sqrt{d})\). Let \(I_{\kappa}\) be the group of fractional ideals of \(\mathcal{O}_K\) which are quotients of ideals coprime to \(\kappa\) and \(P_{\kappa}\) be the subgroup of fractional ideals which are quotients of principal ideals \((\alpha)\) \(\in\) \(I_{\kappa}\) where \(\alpha \in \mathbb{Z} + \kappa\mathcal{O}\). Then set \(CL_{\kappa}(K) = I_{\kappa}/P_{\kappa}\). The elements of \(CL_{\kappa}(K)\) correspond bijectively to proper equivalence classes of positive definite quadratic forms of discriminant \(D = \kappa^2d\). If the proper equivalence class of \(Q\) corresponds to the ideal class \(\epsilon\), then by [3], page 219, we have
\[
r(Q, n) = \sum_{r \mid \kappa} w((\kappa/r)^2d)J(\epsilon_r, n/r^2)
\]
where
\[
w(D) = \begin{cases} 
6 & \text{if } D = -3 \\
4 & \text{if } D = -4 \\
2 & \text{otherwise.}
\end{cases}
\]

Also \(J(\epsilon_r, n)\) is the number of integral ideals of norm \(n\) in the class \(\epsilon_r\) where \(\epsilon_r\) is the image of \(r\) under the natural homomorphism \(CL_{\kappa}(K) \rightarrow CL_{\kappa}/r(K)\). For the form \(Q(x, y) = x^2 + Ny^2\) where \(-N \equiv 1 \mod 4\), the conductor \(\kappa = 2\) and so we have
\[
r_{2,N}(n) = w(-4N)J(\epsilon, n) + w(-N)J(\epsilon_2, n/4) = 2J(\epsilon, n) + w(-N)J(\epsilon_2, n/4)
\]
where \(\epsilon_2\) is the image under \(CL_2(K) \rightarrow CL_1(K)\), that is, \(\epsilon_2\) is a class in the ideal class group of \(K = \mathbb{Q}(\sqrt{-N})\).

We now discuss a classical result of Rankin [11] and Selberg [12] which estimates the size of Fourier coefficients of a modular form. Specifically, if \(f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}\) is a nonzero cusp form of weight \(k\) on \(\Gamma_0(N)\), then
\[
\sum_{n \leq x} |a(n)|^2 = \alpha(f, f)x^k + O(x^{k-\frac{1}{2}})
\]
where \(\alpha > 0\) is an absolute constant and \(\langle f, f \rangle\) is the Petersson scalar product. In particular, if \(f\) is a cusp form of weight 1, then \(\sum_{n \leq x} |a(n)|^2 = O(x)\). One can adapt their result to say the following. Given two cusp forms of weight \(k\) on a suitable congruence subgroup of \(\Gamma = SL_2(\mathbb{Z})\), say \(f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}\) and \(g(z) = \sum_{n=1}^{\infty} b(n)e^{2\pi inz}\), then
\[
\sum_{n \leq x} a(n)b(n)n^{1-k} = Ax + O(x^{\frac{3}{5}})
\]
where \(A\) is a constant. In particular, if \(f\) and \(g\) are cusp forms of weight 1, then \(\sum_{n \leq x} a(n)b(n) = O(x)\).

We conclude this section with a relationship between genus characters of generalized ideal class groups and the poles of the Rankin-Selberg convolution of \(L\)-functions. Recall that a group homomorphism \(\chi : I_2 \to S^1\) is an ideal class character if it is trivial on \(P_2\), i.e.
\[
\chi(\langle a \rangle) = 1
\]
for \(a \equiv 1 \mod 2\). Thus an ideal class character is a character on the generalized class group \(I_2/P_2\). Recall also that a genus character (see Chapter 12, section 5 in [5]) is an ideal class character of order at most two.

Let us also recall the notion of the Rankin-Selberg convolution of two \(L\)-functions. For squarefree \(N\), consider two ideal class characters \(\chi_1, \chi_2\) for \(CL_2(K)\), the generalized ideal class group of \(K = \mathbb{Q}(\sqrt{-N})\) and their associated Hecke \(L\)-series
\[
L_2(s, \chi_1) = \sum_{\langle a, 2 \rangle = 1} \frac{\chi_1(a)}{N(a)^s}
\]
\[
L_2(s, \chi_2) = \sum_{\langle a, 2 \rangle = 1} \frac{\chi_2(a)}{N(a)^s}
\]
which converge absolutely in some right half-plane. We form the convolution \(L\)-series by multiplying the coefficients,
\[
L_2(s, \chi_1 \otimes \chi_2) = \sum_{\langle a, 2 \rangle = 1} \frac{\chi_1(a)\chi_2(a)}{N(a)^s}
\]

The following result describes a relationship between genus characters \(\chi\) and the orders of poles of \(L_2(s, \chi \otimes \chi)\). The proof is similar to that of Proposition 2.4 in [10].

**Proposition 2.1.** Let \(\chi\) be an ideal class character for \(CL_2(K)\), \(-N \equiv 1 \mod 4\), and \(L_2(s, \chi)\) the associated Hecke \(L\)-series. Then \(\chi\) is a genus character if and only if \(L_2(s, \chi \otimes \chi)\) has a double pole at \(s = 1\).

**Remark 2.2.** By Proposition 2.1, if \(\chi\) is a non-genus character, then \(L_2(s, \chi \otimes \chi)\) has at most a simple pole at \(s = 1\).

3. Proof of Theorem 1.4

**Proof.** As the proof is similar to that of Theorem 1.3 in [10], we sketch the relevant details. If \(-N \equiv 1 \mod 4\), then the discriminant of \(K = \mathbb{Q}(\sqrt{-N})\) is \(-N\). We also assume that \(t\) is the number of distinct prime factors of \(N\) and so the discriminant \(-N\) also has \(t\) distinct prime factors. For \(K = \mathbb{Q}(\sqrt{-N})\), consider the zeta function
\[
\zeta_K(s, 2) = \sum_{\langle a, 2 \rangle = 1} \frac{1}{N(a)^s}
\]
where the sum is over those ideals \(a\) of \(O_K\) prime to 2. We now split up \(\zeta_K(s, 2)\), according to the classes \(c_i\) of the generalized ideal class group \(CL_2(K)\), into the partial zeta functions (see page 161 of [7]).
\[ \zeta_c(s) = \sum_{a \in \mathfrak{c}^i} \frac{1}{N(a)^s} \]

so that \( \zeta_K(s, 2) = \sum_{i=0}^{h_2-1} \zeta_c(s) \) where \( h_2 \) is the order of \( \text{CL}_2(K) \).

Let \( \mathfrak{c} \) be the ideal class in \( \text{CL}_2(K) \) which corresponds to the proper equivalence class of \( Q(x, y) = x^2 + Ny^2 \). Now let \( \chi \) be an ideal class character of \( \text{CL}_2(K) \) and consider the Hecke L-series for \( \chi \), namely

\[ L_2(s, \chi) = \sum_{(a, 2) = 1} \frac{\chi(a)}{N(a)^s}. \]

We may now rewrite the Hecke L-series as

\[ L_2(s, \chi) = \sum_{i=0}^{h_2-1} \chi(\mathfrak{c}_i) \zeta_c(s). \]

And so summing over all ideal class characters of \( \text{CL}_2(K) \), we have

\[ \sum_{\chi} \overline{\chi}(\mathfrak{c}) L_2(s, \chi) = \sum_{i=0}^{h_2-1} \zeta_c(s) \left( \sum_{\chi} \overline{\chi}(\mathfrak{c}_i) \chi(\mathfrak{c}_i) \right). \]

The inner sum is nonzero precisely when \( \mathfrak{c} = \mathfrak{c}_i \). Thus we have

\[ \zeta_c(s) = \frac{1}{h_2} \sum_{\chi} \overline{\chi}(\mathfrak{c}) L_2(s, \chi) \]

and so

\[ \zeta_c(s) = \frac{1}{h_2} (L_2(s, \chi_0) + \overline{\chi}_1(\mathfrak{c}) L_2(s, \chi_1) + \cdots + \overline{\chi}_{h_2-1}(\mathfrak{c}) L_2(s, \chi_{h_2-1})). \]

As \( \chi_0 \) is the trivial character, \( L_2(s, \chi_0) = \zeta_K(s, 2) \). Comparing \( n^{th} \) coefficients, we have

\[ J(\mathfrak{c}, n) = \frac{1}{h_2} (a_n + b_1(n) + \cdots + b_{h_2-1}(n)). \]

where \( a_n \) is the number of integral ideals of \( \mathcal{O}_K \) prime to 2 and of norm \( n \) and the \( b_i \)'s are coefficients of weight 1 cusp forms (see [2]). Recall we also have

\[ r_{2, N}(n) = 2J(\mathfrak{c}, n) + w(-N)J(\mathfrak{c}_2, n/4) \]

and so

\[ r_{2, N}(n) = \frac{2}{h_2} \left( a_n + b_1(n) + \cdots + b_{h_2-1}(n) \right) + w(-N)J(\mathfrak{c}_2, n/4). \]

Thus

\[ \sum_{n \leq x} r_{2, N}(n)^2 = \frac{4}{h_2^2} \left( \sum_{n \leq x} a_n^2 + \sum_{n \leq x} b_1(n)^2 + 2 \sum_{n \leq x} a_n b_1(n) + \sum_{i \neq j, n \leq x} b_i(n) b_j(n) \right) + \]

\[ \frac{4}{h_2} \sum_{n \leq x} \left( a_n + b_1(n) + \cdots + b_{h_2-1}(n) \right) w(-N)J(\mathfrak{c}_2, n/4) + \sum_{n \leq x} w(-N)^2 J(\mathfrak{c}_2, n/4)^2. \]
Assume \(-N \equiv 1 \mod 8\). Applying the main theorem in [6] to the Dirichlet series \(\sum_{n=1}^{\infty} \frac{a_n^2}{n^s}\), we obtain

\[
\sum_{n \leq x} a_n^2 \sim Ax \log x
\]

where \(A = \frac{1}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p \mid N} \frac{p}{p+1}\). As \(-N\) has \(t\) distinct prime factors, we have \(2^t\) genus characters for \(CL(K)\) where \(K = \mathbb{Q}(\sqrt{-N})\). By [7] (see Theorem 1, page 127), we have \(2^t\) genus characters for \(CL_2(K)\). We must now assume that the first \(2^t - 1\) terms arise from L-functions associated to genus characters.

By Proposition 2.1 and an application of Perron’s formula, we obtain

\[
\sum_{n \leq x} b_i(n)^2 \sim Ax \log x.
\]

As this estimate holds for each \(i\) such that \(1 \leq i \leq 2^t - 1\), the term \(Ax \log x\) appears \(2^t\) times in the estimate of \(\sum_{n \leq x} r_{2,N}(n)^2\). By Remark 2.2 and the Rankin-Selberg estimate, the remaining terms are all \(O(x)\). Thus

\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{4}{h^2} \left( \frac{2}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p \mid N} \frac{p}{p+1} \right) x \log x.
\]

By [4], we have \(L(1, \chi_{-N}) = \frac{h\pi}{\sqrt{N}}\) where \(h\) is the class number of \(K\) and \(h_2 = h\). Thus

\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p \mid 2N} \frac{2p}{p+1} \right) x \log x.
\]

For \(-N \equiv 5 \mod 8\), we have \(h_2 = 3h\) and again by [6],

\[
\sum_{n \leq x} a_n^2 \sim \left( \frac{9}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p \mid N} \frac{p}{p+1} \right) x \log x.
\]

Thus

\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p \mid 2N} \frac{2p}{p+1} \right) x \log x.
\]

\(\square\)

**Remark 3.1.** We would like to mention another approach which confirms Theorems 1.3 and 1.4. Let \(Q \in \mathbb{Z}^{2 \times 2}\) be a non-singular symmetric matrix with even diagonal entries and \(q(x) = \frac{1}{2} Q [x] = \frac{1}{2} x^T Q x\), \(x \in \mathbb{Z}^2\), the associated quadratic form in two variables. Let \(r(Q, n)\) denote the number of representations of \(n\) by the quadratic form \(Q\). Now consider the theta function

\[
\theta_Q(z) = \sum_{x \in \mathbb{Z}^2} e^{\pi i z Q [x]}.
\]

The Dirichlet series associated with the automorphic form \(\theta_Q\) is
\[(4\pi)^{-1/2} \zeta_Q\left(\frac{1}{2} + s\right)\]

where

\[\zeta_Q(s) = \sum_{n=1}^{\infty} \frac{r(Q, n)}{n^s} = \sum_{\mathfrak{x} \in \mathbb{Z}^2 \setminus \{0\}} q(\mathfrak{x})^{-s}\]

for \(\Re(s) > 1\). A careful and involved application of the Rankin-Selberg method to the above Dirichlet series (see Theorems 2.1 and 5.1 in [8] and Theorem 5.2 in [9]) combined with a Tauberian argument yields the following (see Theorem 6.1 in [8])

\[\sum_{n \leq x} r(Q, n)^2 \sim A_Q x \log x\]

where

\[A_Q = 12 \frac{A(q)}{q} \prod_{p \mid q} \left(1 + \frac{1}{p}\right)^{-1}.\]

Here \(q = \det Q\) and \(A(q)\) denotes the multiplicative function defined by

\[A(p^e) = 2 + \left(1 - \frac{1}{2}\right)(e - 1)\]

where \(p\) is an odd prime, \(e \geq 1\), and

\[A(2^e) = \begin{cases} 1 & \text{if } e \leq 1, \\ 2 & \text{if } e = 2, \\ e - 1 & \text{if } e \geq 3. \end{cases}\]

Let us now turn to our situation. Consider \(q(\mathfrak{x}) = x^2 + Ny^2 = \frac{1}{2}x^T Qx\) where \(Q = \begin{pmatrix} 2 & 0 \\ 0 & 2N \end{pmatrix}\), \(N\) squarefree. Thus \(q = 4N\). Suppose \(N\) has \(t\) distinct prime factors. Then \(A(4N) = 2^{t+1}\) and so

\[A_Q = \frac{3}{N} 2^{t+1} \prod_{p \mid 2N} \left(1 + \frac{1}{p}\right)^{-1} = \frac{3}{N} \prod_{p \mid 2N} \frac{2p}{p + 1}.\]

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References


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